

# Analysis of Time-Periodic Nonlinear Dynamical Systems Undergoing Bifurcations

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**Abstract.** In this study a new procedure for analysis of nonlinear dynamical systems with periodically varying parameters under critical conditions is presented through an application of the Liapunov–Floquet (L–F) transformation. The L–F transformation is obtained by computing the state transition matrix associated with the linear part of the problem. The elements of the state transition matrix are expressed in terms of Chebyshev polynomials in time  $t$  which is suitable for algebraic manipulations. Application of Floquet theory and the eigen-analysis of the state transition matrix at the end of one principal period provides the L–F transformation matrix in terms of the Chebyshev polynomials. Since this is a periodic matrix, the L–F transformation matrix has a Fourier representation. It is well known that such a transformation converts a linear periodic system into a linear time-invariant one. When applied to quasi-linear equations with periodic coefficients, a dynamically similar system is obtained whose linear part is time-invariant and the nonlinear part consists of coefficients which are periodic. Due to this property of the L–F transformation, a periodic orbit in original coordinates will have a fixed point representation in the transformed coordinates. In this study, the bifurcation analysis of the transformed equations, obtained after the application of the L–F transformation, is conducted by employing *time-dependent center manifold reduction* and *time-dependent normal form theory*. The above procedures are analogous to existing methods that are employed in the study of bifurcations of autonomous systems. For the two physical examples considered, the three generic codimension one bifurcations namely, Hopf, flip and fold bifurcations are analyzed. In the first example, the primary bifurcations of a parametrically excited single degree of freedom pendulum is studied. As a second example, a double inverted pendulum subjected to a periodic loading which undergoes Hopf or flip bifurcation is analyzed. The methodology is semi-analytic in nature and provides quantitative measure of stability when compared to point mappings method. Furthermore, the technique is applicable also to those systems where the periodic term of the linear part does not contain a small parameter which is certainly not the case with perturbation or averaging methods. The conclusions of the study are substantiated by numerical simulations. It is believed that analysis of this nature has been reported for the first time for this class of systems.

**Key words:** Nonlinear dynamic systems, parametric excitation, bifurcation, time-periodic systems, critical cases.

## 1. Introduction

Many engineering systems of practical importance are represented by nonlinear differential equations with periodically varying parameters. It is well known that such parametrically excited systems can give rise to periodic, aperiodic and even chaotic solutions depending upon the value of bifurcation parameters of the system. *A priori* knowledge of the range of the bifurcation parameters must be established so that the asymptotic convergence of the system can be attained by a suitable choice of control system design. Therefore, the study of stability and bifurcation phenomena of periodic system is an essential first step for the design of stable and controllable systems and structures. In general, the methods employed in the study of bifurcations of dynamical systems are local such as the methods of *normal forms* and *center manifold reduction*. Even though the bifurcation phenomena of autonomous systems has been treated extensively via the above mentioned methods, not much work has been

reported in the case of non-autonomous systems. Although the method for computing normal forms for periodic systems has been known for some years now, it requires the computation of a special periodic transformation known as the Liapunov–Floquet (L–F) transformation [1]. It is well known that such a L–F transformation can be used to convert the quasi-linear periodic system to a vector field whose linear part is time-invariant and the nonlinear part consists of time-varying coefficients which are periodic. The resulting dynamically similar equations in the transformed space are amenable to the application of the periodic *normal form theory* and *time-dependent center manifold reduction*. In this paper, the stability of periodic systems undergoing bifurcations has been studied using the methods of *time-dependent normal forms* (TDNF) and *time-dependent center manifold reduction* via the L–F transformation.

Traditionally, the qualitative response of periodic systems have been studied using numerical algorithms [2], perturbation techniques [3], averaging methods [4] and point mapping methods [5, 6]. For a periodic system, the trajectory in the state space is one of dense everywhere and it is extremely difficult to obtain a general structure of the motion through a purely numerical procedure. On the other hand, the application of perturbation and averaging methods to such class of systems have their own limitations due to the fact that they can only be applied to systems where the coefficients of periodic terms can be expressed in terms of a small parameter. Therefore, the analysis using such methods is restricted to rather a smaller part of the parameter space of the system. Further, the procedures are not suitable for large systems.

An alternate method of analysis is provided by the technique called *point mapping*. The idea was introduced by Poincaré [7] and later developed by Birkoff [8], Bernoussou [9] and Arnold [1]. In this approach the continuous-time periodic system is reformulated as discrete-time events by defining a point mapping called the Poincaré map. Thus the original non-autonomous differential system is replaced by a set of difference equations which do not explicitly depend on time. However, in order to obtain the corresponding difference equations, one must construct an exact or an approximate solution of a system of nonlinear differential equations. Most often, this amounts to solving the problem entirely in its original differential form. For example, in [5] an approximate analytical point mapping is obtained based on a Runge–Kutta type numerical search scheme assuming the solutions with integer multiples of principal period  $T$ . However, it is to be noted that this formulation defines the state of the system only at integer values of the principal period  $T$  (or use the idea of shooting for periodic and quasi-periodic solutions) and thus is suitable only for those situations where the bifurcation of periodic solutions to other possible periodic motions occur. Another limitation of such a procedure is the large size of the computer memory required and could become very critical as the size of the system becomes larger and larger. Similar remarks hold for the work by Lindtner *et al.* [6] where an approximate form of Poincaré mapping has been constructed via a numerical technique.

Recently Sinha and his coauthors [10, 11] have been successful in obtaining the state transition matrices of general linear periodic systems numerically as well as symbolically by using an efficient algorithm. In their work the solutions of linear periodic systems are expressed in terms of the shifted Chebyshev polynomials of the first kind and hence the elements of the state transition matrix (STM) are basically expressed in terms of powers of time  $t$ . Applying the Floquet theory and using the eigen-analysis of STM at the end of one period, the L–F transformation matrix can be obtained in a form suitable for algebraic manipulations. The development of a procedure for computing these transformation matrices has given a clear edge in dealing with a wide range of problems associated with periodically varying systems. For

example, an analytical investigation of the quantitative behavior of the quasi-linear periodic systems in stable/unstable manifolds via the L–F transformation and time-dependent normal form theory has been reported in [12]. In yet another study [13], the control methodology of periodic systems via time-invariant methods is presented. In this paper, a quantitative method for stability analysis of nonlinear dynamical systems with periodic coefficients under critical conditions has been presented using both time-dependent normal form and center manifold theories. Since the L–F transformation makes the linear part time-invariant, the stability of a periodic orbit of the original system transpires itself into the stability of a fixed point in the transformed space. The first example deals with a parametrically forced simple pendulum for which the degeneracies associated with flip and fold bifurcations have been considered. The case of a pair of purely imaginary roots is also discussed. As a second example, a double inverted pendulum subjected to periodic loading is selected. The Hopf bifurcation in a double inverted pendulum subjected to a tangential autonomous load has been studied by Sethna *et al.* [15] and thereafter many researchers have contributed on various bifurcation aspects of such an autonomous system. However, when the double pendulum is subjected to a periodic load, the system becomes non-autonomous. Periodic bifurcations of such a pendulum have been reported in [5] by the method of point mappings. In this paper, the dynamics of this four dimensional system undergoing a single Hopf bifurcation or a single flip bifurcation is investigated. The results of such analyses are verified by using numerical simulations.

## 2. Statement of the Problem

The stability and bifurcation problem associated with nonlinear time-periodic dynamical systems can be studied by expanding the equation in Taylor series about a particular known solution. The resulting equation can be represented by a quasi-linear differential equation consisting of a linear part and monomial type nonlinear functions with periodic coefficients. Such an equation can be written in the form

$$\dot{x} = A(\lambda, t)x + f_2(x, \lambda, t) + f_3(x, \lambda, t) + \dots + f_k(x, \lambda, t) + O(|x|^{k+1}, t) \quad (1)$$

where  $\lambda$  is the bifurcation parameter, the  $n \times n$  matrix  $A(\lambda, t)$  and the  $n \times 1$  nonlinear terms  $f_k(x, \lambda, t)$  are  $T$ -periodic functions of time  $t$ . It is to be noted that the nonlinear terms  $f_k(\cdot)$  in (1) represent homogeneous monomials in  $x_i$  of order  $k$ . The linear stability of equation (1) can be discussed using the well-known Floquet theory and the Liapunov’s first theorem [16].

Since the matrix  $A(\lambda, t)$  depends on the bifurcation parameter  $\lambda$ , for some isolated values of the parameter  $\lambda = \lambda_c$ , the Floquet multipliers of the system may fall on the unit circle. In these situations, the problem of bifurcation arise and stability of the system cannot be determined just by the Liapunov’s first theorem. For the system at hand three types of ‘simple’ degeneracies can occur: One real multiplier becomes either 1 (fold bifurcation) or  $-1$  (flip or period doubling bifurcation), or a pair of complex conjugate multipliers with modulus equal to unity (Hopf bifurcation). In this study, such generic one parameter (codimension one) bifurcations of equation (1) are considered through an application of the L–F transformation. For  $\lambda = \lambda_c$ , a transformation of the form

$$x = L(t)z, \quad L(t) = L(t + T) \quad (2)$$

reduces equation (1) to

$$\begin{aligned} \dot{z} = & L^{-1}(t)\{A(\lambda, t)L - \dot{L}\}z + L^{-1}(t)\{f_2(z, t) \\ & + f_3(z, t) + \dots + f_k(z, t) + O(|z|^{k+1}, t)\} \end{aligned}$$

or

$$\dot{z} = Cz + L^{-1}(t)\{f_2(z, t) + f_3(z, t) + \cdots + f_k(z, t) + O(|z|^{k+1}, t)\} \quad (3)$$

where  $C = L^{-1}(t)\{A, (\lambda, t)L - \dot{L}\}$  is a constant matrix and  $L^{-1}(t)$  is the inverse of the L-F transformation matrix  $L(t)$ .  $L(t)$  and  $C$ , in general, are complex. For a real representation, one may use the  $2T$  periodic L-F transformation matrix  $Q(t)$  such that  $x(t) = Q(t)y(t)$  yielding

$$\dot{z} = Rz + Q^{-1}(t)\{f_2(z, t) + f_3(z, t) + \cdots + f_k(z, t) + O(|z|^{k+1}, t)\} \quad (4)$$

where  $R$  is a real  $n \times n$  matrix. Since for generic codimension 1 bifurcations, the fourth order terms in equations (3) or (4) do not affect the local stability behavior [6], the higher order terms in equations (3) and (4) can be omitted. Thus, equations (3) and (4) can be approximated as

$$\begin{aligned} \dot{z} &= Cz + g_2(z, t) + g_3(z, t), \quad g_k(z, t) = g(z, t + T) \\ \dot{z} &= Rz + g_2(z, t) + g_3(z, t), \quad g_k(z, t) = g_k(z, t + 2T), \quad k = 2, 3, \end{aligned} \quad (5)$$

where  $g_2$  and  $g_3$  are vector monomials of order 2 and 3, respectively. Equation (5) is now in a form which is suitable for time-dependent center manifold reduction. Making use of the time-dependent normal form theory to the reduced set of center manifold equations, the nonlinearities can be simplified further to the simplest possible form. Stability questions of such normal form equations can be discussed via analytical methods in many cases. But first, some necessary background material is presented for completeness.

## 2.1. MATHEMATICAL BACKGROUND

### 2.1.1. Computation of L-F Transformation Matrix via Chebyshev Polynomials

It has been shown by Sinha and Wu [10, 11] (also see Joseph *et al.* [14]) that the STMs of linear periodic systems can be obtained in terms of the shifted Chebyshev polynomials of the first kind. The technique is efficient and since the STM is basically expressed in terms of powers of  $t$ , it is suitable for algebraic manipulations as well. In fact, if the dimension is small, the STM can be expressed in a closed form as an explicit function of system parameters as shown by Sinha and Juneja [11] for the case of Mathieu equation. In this approach each element of the STM,  $\Phi(t)$ , is expressed in terms of the shifted Chebyshev polynomials of the first kind.

Once the  $n \times n \Phi(t)$ , the STM of linear part of equation (1) has been computed using the method of Chebyshev expansions, it can be written as the product of two  $n \times n$  matrices as

$$\Phi(t) = L(t)e^{Ct} \quad (6)$$

where  $L(t)$  is a  $T$  periodic matrix and  $C$  is a constant matrix. Since  $\Phi(0) = I$ , equation (6) yields  $L(0) = L(T) = I$ . Hence, the Floquet Transition Matrix (FTM)  $\Phi(T)$ , defined as STM evaluated at the end of one principal period, can be written as

$$\Phi(T) = e^{CT}. \quad (7)$$

By performing an eigen-analysis on the FTM,  $C$  can be computed easily. Then the  $T$  periodic L-F transformation matrix is

$$L(t) = \Phi(t)e^{-Ct}. \quad (8)$$

In order to evaluate a  $2T$  periodic L–F transformation matrix,  $Q(t)$  which yields a real constant matrix  $R$ , first we note that (Coddington and Levinson [17])

$$\Phi(2T) = \Phi^2(T) = e^{CT} e^{C^*T} = e^{2RT} \quad (9)$$

where  $C^*$  is the conjugate matrix of  $C$ ,  $R = (C + C^*)/2$  and the  $2T$  L–F transformation matrix can be represented as

$$\begin{aligned} Q(t) &= \Phi(t)e^{-Rt}, \quad 0 \leq t \leq T \\ Q(\tau + T) &= \Phi(\tau)Q(T)e^{-R\tau}; \quad T \leq (T + \tau) \leq 2T; \quad 0 \leq \tau \leq T. \end{aligned} \quad (10)$$

It should be noted that  $Q(t) = Q(t + 2T)$ .

If one is interested in finding  $\Phi^{-1}(t)$ , then there are two avenues.  $\Phi(t)$  can possibly be inverted through a symbolic software like MACSYMA/MATHEMATICA/MAPLE, which is still not an easy task, or one can first find the state transition matrix  $\Psi(t)$  of the adjoint system

$$\dot{w}(t) = -A^T(t)w(t) \quad (11)$$

and use the following relationship (Yakubovich and Starzhinskii [18]),

$$\Phi^{-1}(t) = \Psi^T(t). \quad (12)$$

The computation of  $\Phi^{-1}(t)$  is essential in determining  $L^{-1}(t)$  or  $Q^{-1}(t)$ . For example, the inverse  $T$ -periodic L–F transformation matrix can be evaluated utilizing the properties of the adjoint system as shown below.

$$L^{-1}(t) = [\Phi(t)e^{-Ct}]^{-1} = e^{Ct}\Phi^{-1}(t) = e^{Ct}\Psi^T(t). \quad (13)$$

Such an approximation of L–F transformations has been found to be extremely convergent and since it is periodic, the elements  $L_{ij}(t)$  or  $Q_{ij}(t)$  have the truncated Fourier representation

$$L_{ij}(t) \approx \sum_{n=-q}^q c_n \exp(i2\pi nt/T), \quad i = \sqrt{-1} \quad (14)$$

or

$$Q_{ij}(t) \approx \frac{a_0}{2} + \sum_{n=1}^q a_n \cos \frac{\pi nt}{T} + \sum_{n=1}^q b_n \sin \frac{\pi nt}{T}. \quad (15)$$

Since complex matrix  $L(t)$  (or the real matrix  $Q(t)$ ) can be computed as a function of  $t$ , all algebraic manipulations involving this matrix in equations (3) and (4) can be done in symbolic form.  $L_{ij}^{-1}(t)$  and  $Q_{ij}^{-1}(t)$  have similar Fourier representations. It is important that the L–F transformation matrices and their inverses must be calculated with a high degree of accuracy in order to guarantee a reasonably accurate system dynamics. Therefore, one must be careful in computation of the STM,  $\Phi(t)$ . It has been shown by Sinha and Wu [11] that a 15 to 18 terms Chebyshev polynomial expansion provides extremely accurate representations of  $\Phi(t)$  or  $L(t)$  even for relatively large systems such as  $20 \times 20$ . A convergence study has been reported by Joseph *et al.* [14].

### 2.1.2. Time-Dependent Normal Form for Time-Periodic Systems

The fact that  $A(t)$  is time-dependent in equation (1), a direct application of *normal form theory* is not possible. Using the transformation

$$x(t) = Q(t)z(t) \quad (16)$$

equation (5) takes the form

$$\dot{z} = Rz + Q^{-1}(t)\{f_2(z, t) + f_3(z, t)\} \quad (17)$$

where  $R$  is an  $n \times n$  constant matrix and the nonlinear terms of order four and higher have been neglected. The form of equation (17) is amenable to direct application of the method of *time-dependent normal forms* (TDNF) for equations with periodic coefficients as shown by Arnold [1].

The equation (17) in its Jordan canonical form can be written as

$$\dot{y} = Jy + w_2(y, t) + w_3(y, t) \quad (18)$$

where  $J$  is the Jordan form of matrix  $R$  and  $w_k(y, t)$ ,  $k = 2, 3$  are  $2T$  periodic functions and contain homogeneous monomials of  $y_i$  of order 2 and 3. Using a sequence of near identity transformations of the form

$$y = v + h_r(v, t) \quad (19)$$

where  $h_r(v, t)$  is a formal power series in  $v$  of degree  $r$  with periodic coefficients having the principal period  $2T$ , equation (18) can be reduced to its simplest form

$$\dot{v} = Jv + w_2(v, t) + w_3(v, t). \quad (20)$$

It is important to note that the  $w_2(v, t)$  and  $w_3(v, t)$  contains only a finite number of Fourier harmonics. This is due to the fact that the solution of the homological equation depends on the resonance condition relating the eigenvalues of  $J$  and the Fourier frequencies of  $w_r(v, t)$  [1]. It should be pointed out that the solution of the time-dependent homological equation requires the solution of a large set of linear algebraic equations even for a  $2 \times 2$  system. For example, if for such a system, the L-F transformation matrix  $Q(t)$  is represented by a fifteen term complex Fourier expansion and let us say that the degree of the monomials  $r = 3$ , then one needs to solve  $(2 \times 124)$  equations in blocks of 31.

### 2.1.3. Center Manifold Reduction for Time-Periodic Systems

In situations where some of the eigenvalues of equation (18) are critical, the stability of equation (18) can be discussed in the center manifold via the *time-periodic center manifold theorem*. Application of the normal form procedure to the reduced set of equations in the center manifold is found to retain the stability characteristics of the original  $n$  dimensional system. In the following, a theorem due to Malkin [19] has been utilized to develop a practical method for finding the center manifold relations for the time-periodic systems.

Let us assume that equation (18) has  $n_1$  eigenvalues that are critical and  $n_2$  eigenvalues that have negative real parts. Therefore, equation (18) may be rewritten in the form

$$\begin{Bmatrix} \dot{y}_c \\ \dot{y}_s \end{Bmatrix} = \begin{bmatrix} J_c & 0 \\ 0 & J_s \end{bmatrix} \begin{Bmatrix} y_c \\ y_s \end{Bmatrix} + \begin{Bmatrix} w_{c2} \\ w_{s2} \end{Bmatrix} + \begin{Bmatrix} w_{c3} \\ w_{s3} \end{Bmatrix} \quad (21)$$

where the subscripts  $c$  and  $s$  represent the critical and stable vectors, respectively. According to the *center manifold theorem*, there exists a relation

$$y_s = h(y_c, t) \tag{22}$$

such that  $h(y_c, t)$  is of the form

$$h(y_c, t) = \sum_{m_t} B_s^{(m_1 \dots m_{n_1})}(t) y_1^{m_1} \dots y_{n_1}^{m_{n_1}}, \quad m_1 + \dots + m_{n_1} \geq 1 \tag{23}$$

where  $B_s^{m_1 \dots m_{n_1}}(t)$  are periodic coefficients with period  $2T$  and the superscript  $m_1 \dots m_{n_1}$  is for the identification of the periodic coefficient of the particular monomial type nonlinear term. The relation  $y_s$  given by equation (22) can be obtained as the formal solutions of the equations

$$\frac{\partial h}{\partial t} + \sum_{i=1}^{n_1} \frac{\partial h}{\partial y_c} (J_c y_c + W_c) = J_s y_s + W_s \tag{24}$$

where  $W_c = w_{c2} + w_{c3}$  and  $W_s = w_{s2} + w_{s3}$  are nonlinear vector monomials of the critical and stable states of the system, respectively. It is important to note that the resulting solutions will be meaningful only if the coefficients  $B_s^{m_1 \dots m_{n_1}}(t)$  are also periodic. Although there exists an infinite number of expansion similar to equation (23) which have finite coefficients and also satisfy equation (24), there is only one with periodic coefficients. This result is first reported by Malkin [19].

As a result of substitution of equation (23) in equation (24), a set of differential equations in terms of the unknown coefficients  $B_s^{m_1 \dots m_{n_1}}(t)$  are obtained in a form

$$\frac{dB_s^{(m_1 \dots m_{n_1})}}{dt} - \lambda_j B_s^{(m_1 \dots m_{n_1})} = C_s^{(m_1 \dots m_{n_1})} \tag{25}$$

where  $\lambda_j, j = 1, 2, \dots, n_2$  are the eigenvalues of the stable part of the system and  $C_s^{(m_1 \dots m_{n_1})}$  are the known integral rational functions of the periodic coefficients of the right hand side of equation (21). The coefficients  $B_s^{(m_1 \dots m_{n_1})}$  can be obtained by formally solving the above set of differential equations. For this purpose,  $B_s^{(m_1 \dots m_{n_1})}$  is assumed in the form of a finite Fourier expansion as

$$B_s^{(m_1 \dots m_{n_1})}(t) = p_0 + \sum_{n=1}^1 p_n \cos\left(\frac{2\pi n t}{\tau}\right) + \sum_{n=1}^l q_n \sin\left(\frac{2\pi n t}{\tau}\right) \tag{26}$$

where  $\tau = 2T$ . Substituting equation (26) in equation (25) and equating like terms on both sides of the equation, a set of algebraic equations in terms of the unknown coefficients  $p_n$  and  $q_n$  are obtained. The constants  $p_n$  and  $q_n$  can be computed by solving these algebraic equations and therefore the coefficients  $B_s^{(m_1 \dots m_{n_1})}$  can be determined in the form of equation (26). Substitution of equation (22) in equation (21) clearly decouples the stable and critical states and hence, the problem reduces to the investigation of stability of a  $n_1$  dimensional system in the center manifold. The resulting system of  $n_1$  periodic equations are of the form

$$\dot{y}_c = J_c y_c + W_c^* \tag{27}$$

where vector  $W_c^*$  contains nonlinear monomials which are functions of  $y_c$  only.

### 3. Applications

EXAMPLE 1. As an example of a single degree of freedom system, the bifurcations of a parametrically forced simple pendulum is considered. It is assumed that the support of the pendulum is subjected to a sinusoidal motion with frequency  $\omega$ . The general equation of motion for the pendulum is given by [5]

$$\ddot{x} + \delta \dot{x} + (\alpha^2 - \beta \sin \omega \tau) \sin x = 0 \tag{28}$$

where  $\delta, \alpha^2, \beta$  and  $\omega$  are the parameters of the system. Expanding  $\sin x$  in Taylor series and truncating the higher order terms beyond the cubic powers, the above equation can be rewritten in state space form as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\alpha^2 - \beta \sin \omega \tau) & -\delta \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -[(\alpha^2 - \beta \sin \omega \tau)/6]x_1^3 \end{Bmatrix} \tag{29}$$

where  $\{x_1 \ x_2\}^T = \{x \ \dot{x}\}^T$ .

#### (i) A Pair of Purely Imaginary Roots (Codimension Zero Case)

When the parameter  $\delta = 0$ , it can be shown that equation (29) can either be marginally stable or unstable, thus forming a periodic Hamiltonian system. Selecting the remaining parameters to be  $\alpha^2 = 0.1, \beta = 1.635$  and  $\omega = 2$ , it is found that the Floquet multipliers associated with the linear part of equation (29) are complex and are on the unit circle. Normalizing with  $\omega t' = 2\pi t$ , and following the Chebyshev expansion method [10], the STM of equation (29) can be obtained as a power series in  $t$ . Pursuing the steps described in Section 2.1.1, the  $2T$  periodic L-F transformation matrix  $Q(t)$  can be computed for the chosen parameter set. Applying the transformation  $x = Q(t)z$ , equation (29) is transformed to

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + Q^{-1}(t) \begin{Bmatrix} 0 \\ E(z, t) \end{Bmatrix} \tag{30}$$

where  $R_{11} = -2.26927, R_{12} = -0.52530, R_{21} = 9.88854$  and  $R_{22} = 2.26927$  are the elements of the real matrix  $R, Q^{-1}(t)$  is the inverse of the L-F transformation matrix  $Q(t), E(z, t) = -(\alpha^2 \mu - \beta \mu \sin 2\pi t)(Q_{11} z_1 + Q_{12} z_2)^3/6, \mu = (2\pi/\omega)^2$  and  $Q_{ij}$  are the elements of  $Q(t)$ . Here  $(\dot{\cdot})$  represents the differentiation with respect to  $t'$ . Because of the transformation, the two Floquet multipliers of equation (29) on the unit circle transform to a purely imaginary pair in equation (30). Further, it is to be noted that the nonlinear part of equation (30) can be expressed in terms of Fourier expansions due to the periodic nature of the Liapunov-Floquet transformation. Therefore, in canonical form, equation (30) can be written as

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \begin{Bmatrix} f_{11}(t, \tau)y_1^3 + f_{12}(t, \tau)y_1^2 y_2 + f_{13}(t, \tau)y_1 y_2^2 + f_{14}(t, \tau)y_2^3 \\ f_{21}(t, \tau)y_1^3 + f_{22}(t, \tau)y_1^2 y_2 + f_{23}(t, \tau)y_1 y_2^2 + f_{24}(t, \tau)y_2^3 \end{Bmatrix} \tag{31}$$

where  $\tau = 2T$ ,  $\{\lambda_1, \lambda_2\} = \{0.2118i, -0.2118i\}$  are the eigenvalues of  $R$ , and the periodic coefficients  $f_{ij}(t, \tau)$ ;  $i, j = 1, \dots, 4$  are expressed as

$$f_{ij}(t, \tau) = a_0^{ij} + \sum_{n=1}^l a_n^{ij} \cos\left(\frac{2\pi n t}{\tau}\right) + \sum_{n=1}^l b_n^{ij} \sin\left(\frac{2\pi n t}{\tau}\right). \tag{32}$$

After experimenting with various sets of system parameters, it was observed that  $l = 15$  to 18 provided accurate representations of functions  $f_{ij}(t, \tau)$ . It is also consistent with the number of Fourier terms taken in the representation of the L-F transformation  $Q(t)$ .

Since the dimension of the system under consideration and the dimension of the center manifold are the same for this case, further reduction through the application of center manifold theory is not possible. Therefore, only the TDNF procedure is applied to simplify the nonlinearities. For this purpose, consider a near-identity nonlinear transformation

$$\begin{aligned} y_1 &= u + g_{11}(t, \tau)u^3 + g_{12}(t, \tau)u^2v + g_{13}(t, \tau)uv^2 + g_{14}(t, \tau)v^3 \\ y_2 &= v + g_{21}(t, \tau)u^3 + g_{22}(t, \tau)uv^2 + g_{23}(t, \tau)uv^2 + g_{24}(t, \tau)v^3 \end{aligned} \tag{33}$$

where the periodic coefficients  $g_{ij}(t, \tau)$ ;  $i, j = 1, \dots, 4$  are once again of the form given by equation (32) but with unknown constants  $\tilde{a}_n$  and  $\tilde{b}_n$ . Substituting equation (33) in equation (30) and solving the resulting homological equation, the unknown constants  $\tilde{a}_n$  and  $\tilde{b}_n$  can be evaluated for all the coefficients except for the coefficients  $f_{12}(t, \tau)$  and  $f_{23}(t, \tau)$  which correspond to the resonance terms. The resulting equation takes the form

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} + \begin{Bmatrix} f_{12}(t, \tau)u^2v \\ f_{23}(t, \tau)uv^2 \end{Bmatrix}. \tag{34}$$

Since the eigenvalues  $\{\lambda_1, \lambda_2\} = \{0.2118i, -0.2118i\}$  are irrational, equation (34) is similar to the normal form obtained for the autonomous case. However, when the eigenvalues are rational, it should be noted that additional resonant terms involving Fourier harmonic terms will also appear [23]. The stability and dynamical behavior of equation (34) can be studied using some of the procedures described by Bruno [20] and Hale and Kocak [21]. However, first we observe that by multiplying  $\dot{u}$  and  $v$  and  $\dot{v}$  by  $u$  and adding, a linear differential equation in  $(uv)$  can be obtained. Therefore, an analytical solution of equation (34) can be found as

$$-(uv)^{-1} = \int_0^t a(\zeta, \tau) d\zeta \tag{35}$$

where  $a(\zeta, \tau)$  is a complex Fourier function of  $f_{12}$  and  $f_{23}$ . The differential equation (34) can be decoupled by substituting the solution (35) into equation (34). The resulting linear differential equation in  $u$  can be shown to be of the form

$$\dot{u} = \left[ -i\omega_0 + \varepsilon f_{12}(t, \tau) / - \int_0^t a(\zeta, \tau) d\zeta \right] u = \left[ -i\omega_0 + \varepsilon \sum_{n=-q}^q c_n \exp(i2\pi n t / \tau) \right] u \tag{36}$$

where  $c_n$  are complex constants of Fourier expansion of the periodic terms and hence

$$u = e^{[-i\omega_0 + \varepsilon \sum_{n=-q}^q c_n \exp(i2\pi n t / \tau)]t} u_0. \tag{37}$$

In a similar manner,  $v$  can also be computed. It should be noted that the stability of these solutions entirely depend on the real part of the constant  $c_0$  of the function  $c_n \exp(i2\pi nt/\tau)$ . The solution is stable or unstable depending on whether the constant has a negative real part or a positive real part, respectively. When the real part is zero, the solutions are closed orbits and behave like centers which turns out to be the case here. Similar results have been shown by Bruno [20] in a different fashion. In order to identify the fixed point  $(0, 0)$  correctly, only the constant terms of the periodic expansion are retained in equation (34) and the transformation  $u = r \cos \theta$ ,  $v = r \sin \theta$  is applied to yield

$$\frac{dr}{d\theta} = -r \tan \theta + ir^3 \{(7.329 + 0.938i) \cos^2 \theta + (4.392 - 0.058i) \sin^2 \theta\}; \quad i = \sqrt{-1}. \quad (38)$$

The above equation is of the form

$$\frac{dr}{d\theta} = r f_1(\theta) + r^3 f_2(\theta) \quad (39)$$

and the mean value of  $f_1(\theta)$  over period  $\tau$  decides the behavior of the equilibrium point [21]. In the present case the mean value is zero and hence  $r$  is a constant. Therefore, the fixed point  $(0, 0)$  of equation (34) is a center and hence the resulting dynamics of the original equation (29) can only be periodic or quasi-periodic. This result is also expected due to the fact that this is a periodic Hamiltonian system. No Hopf bifurcation can take place for  $\delta = 0$ . The dynamics is verified via numerical simulation in Figure 1a, wherein the trajectories never close but approach arbitrarily close, representing a quasi-periodic motion around a torus. A Poincaré plot of the motion shown in Figure 1b indicates that the Poincaré points repeat every 15 principal periods but never coincide. Since, for this case, the ratio between the natural and the excitation frequency is commensurable, such a behavior is expected and clearly represents a quasi-periodic solution. Incidentally, it is worthwhile to mention that the same problem was studied in [5] by point mappings method. In there, it was reported that for the same parameter set, there exists a stable  $2T$  periodic solution which is contrary to the results obtained here.

### (ii) Flip Bifurcation

For  $\delta = 0.31623$  {damped case},  $\alpha^2 = 0.1$ ,  $\beta = 1.75381$ ,  $\omega = 2$  one of the Floquet multipliers is found to be  $-0.370294$  (stable multiplier) and the other is approximately equal to  $-1$  ( $-0.999996$ , critical) which corresponds to a flip bifurcation. Following the steps described in the above section, once again, a Jordan canonical form similar to equation (31) can be obtained with  $\{\lambda_1, \lambda_2\} = \{-0.99346, 0\}$ . Since one of the eigenvalues is critical ( $\lambda_2 = 0$ ), the dimension of the center manifold is one. Noting that the nonlinear terms in equation (31) are cubic, consider a center manifold relation of the form

$$y_1 = B_s^3(t)y_2^3 \quad (40)$$

where  $B_s^3(t)$  is the periodic coefficient assumed in the Fourier form of equation (26). Following the steps given in Section 2.1.3, the differential equation for the coefficient is given by

$$\frac{dB_s^3(t)}{dt} - (-0.99346)B_s^3(t) = f_{14}(t, \tau). \quad (41)$$

Solving the above differential equation for the unknown constants  $p_n$  and  $q_n$ , in a manner similar to solving the homological equation, one obtains  $B_s^3(t)$ . The computed periodic center

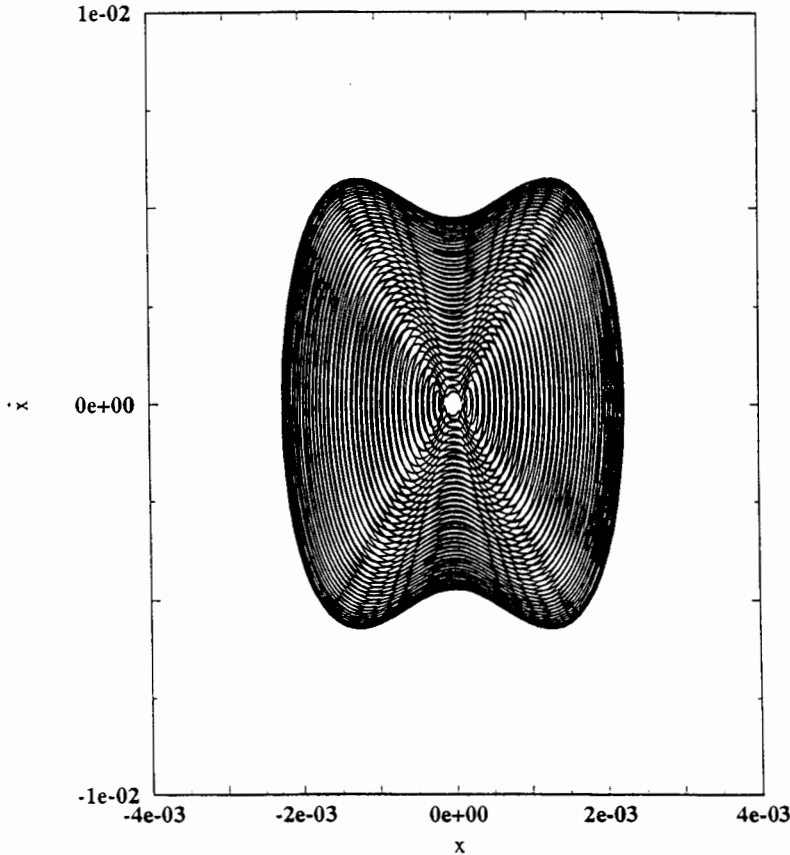
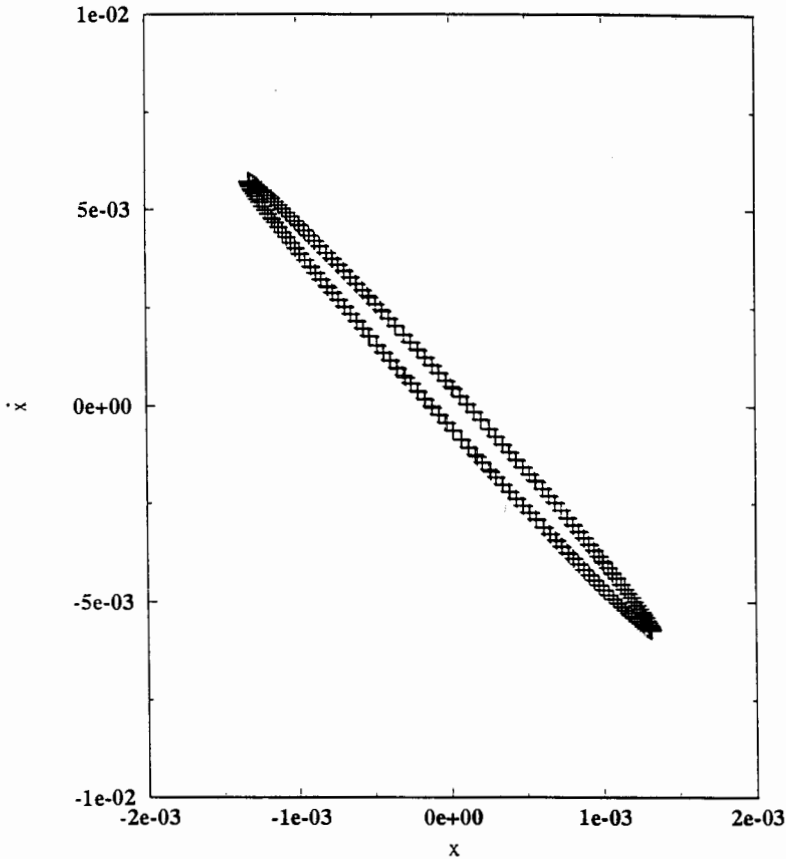


Fig. 1a. Phase plane plot – parametrically excited simple pendulum.  $\alpha^2 = 0.1$ ,  $\beta = 1.635$ ,  $\delta = 0$ .

manifold coefficient for this case is provided in Appendix A. Although all computation were done in double precision, only first eight significant digits of the constants of Fourier coefficients are reported in appendices for sake of brevity. Substitution of equation (40) in equation (31) yields the center manifold equation in  $y_1$ . Keeping only the cubic terms, we obtain

$$\dot{y}_2 = f_{24}(t, \tau)y_2^3 + O(y_2^5). \quad (42)$$

Since the coefficient of the cubic term on the right hand side of equation (42) is periodic, if the constant term (the mean value) is negative, the fixed point is stable; otherwise unstable. It is found that the mean value  $a_0 = 0.02264$  and therefore, the instability of the fixed point implies that the resulting period 2 motion is unstable. A phase plane plot and the Poincaré map corresponding to this parameter set clearly show the period doubling behavior in Figure 2. It is also observed that the two Poincaré points corresponding to the  $2T$  periodic solution are slowly drifting away from the periodic orbit indicating instability. Further, an approximate analytical form of the  $2T$  periodic motion can be obtained via the Harmonic Balance method applied to equation (29). Expanding equation (29) about this solution yields a linear equation with  $2T$  periodic coefficients. It is found that the Floquet multipliers corresponding to this linearized equation are  $(-0.0638, -5.79565)$  and therefore, the periodic motion is indeed unstable.



*Fig. 1b.* Poincaré plot – parametrically excited simple pendulum.  $\alpha^2 = 0.1$ ,  $\beta = 1.635$ ,  $\delta = 0$ .

### (iii) *Fold Bifurcation*

For the parameter set  $\delta = 0.40497$  (damped case),  $\alpha^2 = 4.1$ ,  $\beta = 4.1269$ ,  $\omega = 2$ , it can be verified that one of the Floquet multipliers lies within the unit circle and the other is approximately equal to 1 ( $= 0.999487$ ) which corresponds to a fold bifurcation. Therefore, the structure of the fold bifurcation in the transformed space has a single zero eigenvalue and an eigenvalue with a negative real part. Hence, the reduction procedures are similar to that of the flip bifurcation and the reduced center manifold equation is again of the form of equation (42). The periodic center manifold coefficient, for this case, is provided in Appendix A. For reasons of brevity, they are not repeated here. However, the fold bifurcation results in a  $T$ -periodic solution whose stability characteristics are extremely sensitive to change in parameters. Again, the state  $y_2$  is found to be very slowly diverging since the constant  $a_0$  of the periodic coefficient  $f_{24}(t, \tau)$  is 0.0047. The phase plane plot and the Poincaré map for this parameter set is provided in Figure 3. It is observed that the Poincaré point of the  $T$ -periodic solution of this case is drifting away from the orbit, representing an unstable orbital stability behavior of the solution.

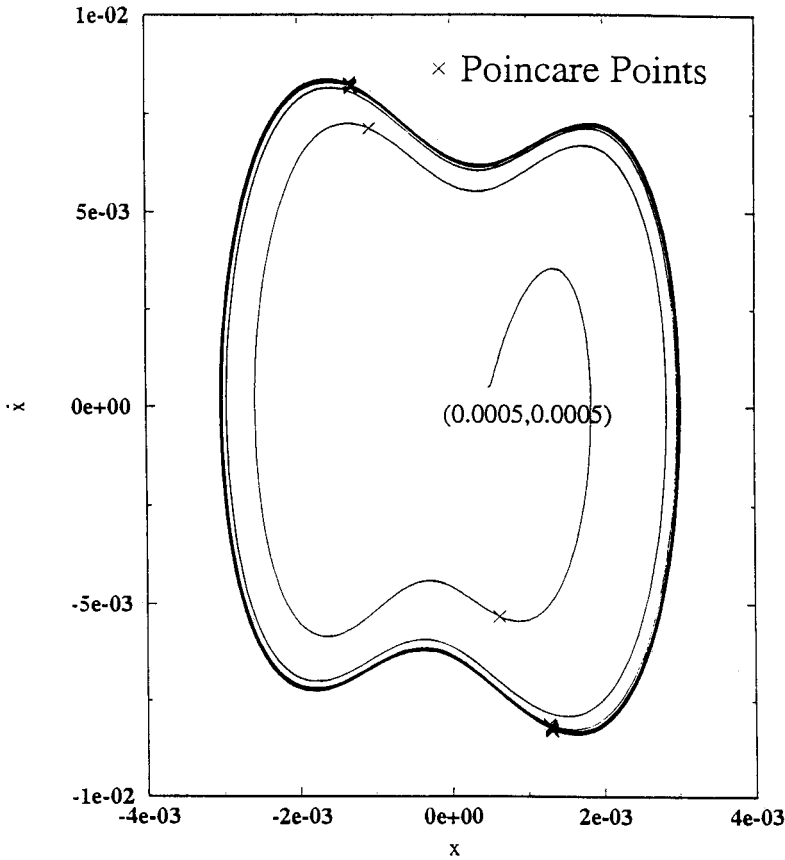


Fig. 2. Phase plane plot – parametrically excited simple pendulum under flip bifurcation.  $\alpha^2 = 0.1, \beta = 1.73581, \delta = 0.31623$ .

(iv) *Limitation of Traditional Averaging in Bifurcation Studies*

The traditional averaging procedure has been widely employed in the study of parametrically excited systems. However, in absence of a small parameter, the averaged dynamics fails to retain the bifurcational characteristics of the original system. In this section, this fact is demonstrated by applying the traditional averaging technique to the problem considered above. For this purpose, let  $\omega t' = 2\pi t$  and rewrite equation (28) as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha_1^2 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\delta_1 x_2 - \beta_1 \sin 2\pi t(x_1 - x_1^3/6) + \alpha_1^2 x_1^3/6 \end{Bmatrix} \quad (43)$$

where  $\mu = 2\pi/\omega, \alpha_1 = \mu\alpha, \delta_1 = \mu\delta, \text{ and } \beta_1 = \mu^2\beta$ . Applying the transformation given by

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \cos \alpha_1 t & \sin \alpha_1 t / \alpha_1 \\ -\alpha_1 \sin \alpha_1 t & \cos \alpha_1 t \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} \quad (44)$$

to equation (43) and time-averaging over the principal period  $t_1 (= 1 \text{ here})$ , one obtains

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{Bmatrix} f_1(\tilde{z}) \\ f_2(\tilde{z}) \end{Bmatrix} \quad (45)$$

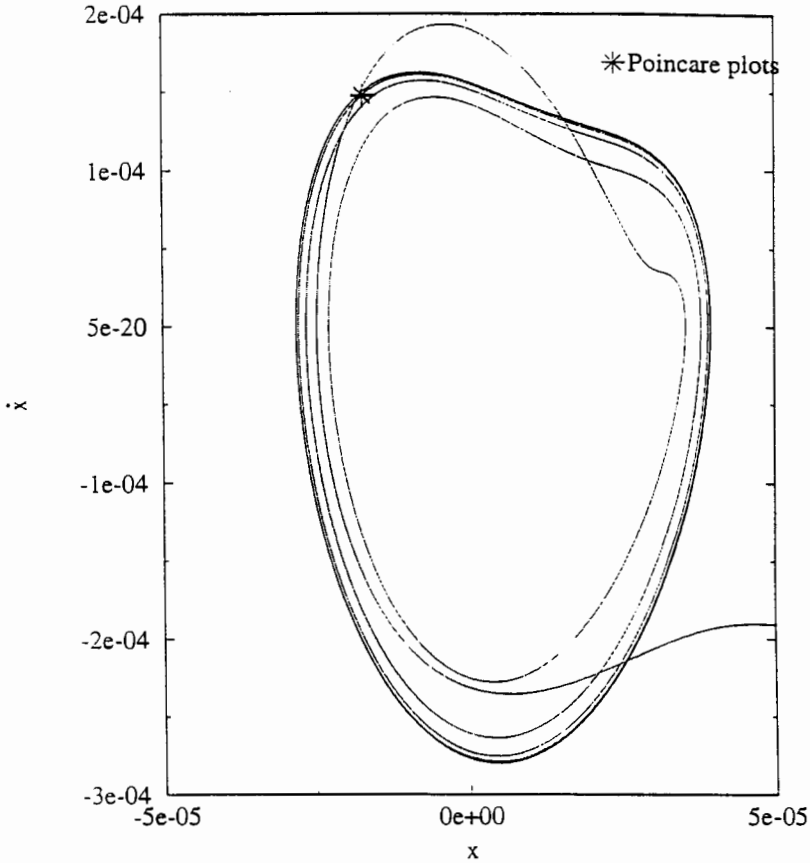


Fig. 3. Phase plane plot – parametrically excited simple pendulum under fold bifurcation.  $\alpha^2 = 4.1$ ,  $\beta = 4.1269$ ,  $\delta = 0.404969$ .

where  $f_1(\tilde{z})$  and  $f_2(\tilde{z})$  are the monomials of degree three in  $\tilde{z}$  and the constants  $a_{11}$ ,  $a_{12}$ ,  $a_{21}$  and  $a_{22}$  obtained as a result of time-averaging over  $t_1 = 1$  are given as

$$\begin{aligned}
 a_{11} &= 0.25[\delta_1(1 - 2\alpha_1) + \beta_1\pi \sin 2\alpha_1/S_1]/\alpha_1, \\
 a_{12} &= a_{21} = 0.25[(\beta_1\pi + \delta_1 S_1)(1 - \cos 2\alpha_1)]/(\alpha_1^2 S_1), \\
 a_{22} &= -0.25[\delta_1(1 + 2\alpha_1) + \beta_1\pi \sin 2\alpha_1/S_1]\alpha_1 \quad \text{and} \quad S_1 = \pi^2 - \alpha_1^2.
 \end{aligned}$$

The stability and bifurcation of the original problem should have similar characteristics as the averaged system. The Floquet multipliers of the linear part of equation (28) and the eigenvalues of the linear part of the equation (45) are computed for various sets of parameters and are provided in Table 1 for a comparison. It is very clear from Table 1 that the results of the averaging procedure are incorrect for most of the cases considered. In general, the averaging technique is expected to provide good results only when  $\alpha_1^2 \gg \beta_1$ .

**EXAMPLE 2.** In this example, the stability of a two mass inverted pendulum subjected to non-conservative periodic load undergoing Hopf and flip bifurcations is discussed. The nonlinear equations of motion of the system shown in Figure 4, are of the form (cf. [22] for the autonomous case)

Table 1. A comparison of eigenvalues by Floquet method and averaging method.

$\alpha^2$	$\beta$	$\delta$	Floquet multipliers	Eigenvalues after applying L-F transformation	Eigenvalues (averaging)	
1	0.1	1.2	0	$-0.2803 \pm 0.9599 * i$ on unit circle	$\pm 1.2867 * i$ on imag. axis	$\pm 1.7665$ unstable
2	0.1	1.635	0	$-0.9777 \pm 0.2102 * i$ on unit circle	$\pm 0.2118 * i$ on imag. axis	$\pm 2.4068$ unstable
3	0.1	2.0	0	$-0.3211, -3.1140$ unstable	$\pm 1.1359$ unstable	$\pm 2.9442$ unstable
1	0.1	1.5	0.32	$-0.3981 \pm 0.4602 * i$ stable	$-0.4967 \pm 0.8575 * i$ stable	2.1303, -3.1238 unstable
2	0.11	1.754	0.32	$-0.3703, -0.9999$ critical	$-0.9935, 0$ critical	$-0.34974, 2.5039$ unstable
3	0.1	2.0	0.32	$-0.2056, -1.8010$ unstable	$-1.5818, 0.5884$ unstable	$-3.9598, 2.8598$ unstable
1	4.1	0.2	0.41	$0.5288 \pm 0.4602 * i$ stable	$-0.6361 \pm 0.0449 * i$ stable	$-0.6300, -0.6427$ stable
2	4.1	4.127	0.41	0.2802, 1.0002 critical	$-1.2723, 0$ critical	$-0.6540, -0.6183$ stable
3	4.1	5.0	0.41	0.2157, 1.2993 unstable	$-1.5341, 0.2618$ unstable	$-0.6594, -0.6129$ stable

$$\begin{aligned}
 \ddot{\phi}_1 = & -0.5(B_1 + 2B_2)\dot{\phi}_1 + B_2\dot{\phi}_2 + 0.5\bar{k}(\bar{p} - 3)\phi_1 + 0.5\bar{k}(2 - \bar{p})\phi_2 \\
 & - 0.5(\dot{\phi}_1^2 + \dot{\phi}_2^2)(\phi_1 - \phi_2) - (\bar{p}\bar{k}/12)\{(\phi_1 - \gamma\phi_2)^3 - (1 - \gamma)^3\phi_2^3\} \\
 & - ((\phi_1 - \phi_2)^2/4)\{\bar{k}(\bar{p} - 4)\phi_1 + \bar{k}(3 + \bar{p}(\gamma - 2))\phi_2 \\
 & - (B_1 + 3B_2)\dot{\phi}_1 + 3B_2\dot{\phi}_2\} \tag{46}
 \end{aligned}$$

$$\begin{aligned}
 \ddot{\phi}_2 = & 0.5(B_1 + 4B_2)\dot{\phi}_1 - 2B_2\dot{\phi}_2 + 0.5(5 - \bar{p})\bar{k}\phi_1 + \{(\bar{p}(1.5 - \gamma) - 2)\bar{k}\}\phi_2 \\
 & + 0.5(\phi_1 - \phi_2)(3\dot{\phi}_1^2 + \dot{\phi}_2^2) + (\bar{p}\bar{k}/12)\{(\phi_1 - \gamma\phi_2)^3 - 3(1 - \gamma)^3\phi_2^3\} \\
 & + ((\phi_1 - \phi_2)^2/4)\{\bar{k}(2\bar{p} - 7)\phi_1 + \bar{k}(5 + \bar{p}(\gamma - 3))\phi_2 \\
 & - (2B_1 + 5B_2)\dot{\phi}_1 + 5B_2\dot{\phi}_2\} \tag{47}
 \end{aligned}$$

where  $m$  is the mass,  $l$  is the length of the links of the pendulum,  $\phi_1$  and  $\phi_2$  the displacement angles,  $\dot{\phi}_1$  and  $\dot{\phi}_2$  are the corresponding rates,  $\gamma$  ( $0 \leq \gamma \leq 1$ ) is the load-direction parameter and  $P = P_1 + P_2 \cos \omega\tau$ . Other symbols appearing in equations (46) and (47) are defined as  $b_1$  and  $b_2 =$  damping parameters,  $B_1 = b_1/ml^2$ ,  $B_2 = b_2/ml^2$ ,  $\bar{p} = Pl/k$ ,  $\bar{k} = k/ml^2$ ,  $k =$  stiffness parameter,  $P_1 =$  magnitude of static load,  $P_2 =$  amplitude of the dynamic periodic

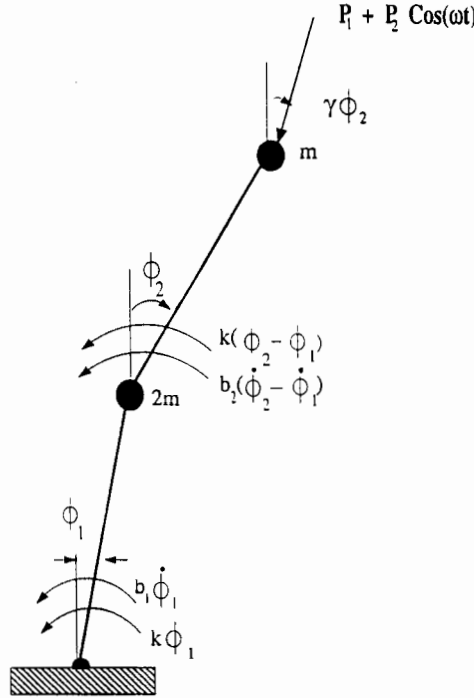


Fig. 4. A double inverted pendulum subjected to a periodic follower force.

load. Equations (46) and (47) are rewritten in the state-space form as

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5\bar{k}(\bar{p} - 3) & 0.5\bar{k}(2 - \bar{p}) & -0.5(B_1 + 2B_2) & B_2 \\ 0.5\bar{k}(5 - \bar{p}) & \bar{k}[\bar{p}(1.5 - \gamma) - 2] & 0.5(B_1 + 4B_2) & -2B_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \quad (48)$$

$$+ \begin{Bmatrix} 0 \\ 0 \\ [-0.5(y_3^2 + y_4^2)(y_1 - y_2) - (\bar{p}\bar{k})[(y_1 - \gamma y_2)^3 - (1 - \gamma)^3 y_2^3]/12 - \\ 0.25(y_1 - y_2)^2[\bar{k}(\bar{p} - 4)y_1 + \bar{k}(3 + \bar{p}(\gamma - 2))y_2 - (B_1 + 3B_2)y_3 + 3B_2 y_4] \\ [0.5(y_1 - y_2)(3y_3^2 + y_4^2) + \bar{p}\bar{k}[(y_1 - \gamma y_2)^3 - 3(1 - \gamma)^3 y_2^3]/12 + \\ 0.25(y_1 - y_2)^2[\bar{k}(2\bar{p} - 7)y_1 + \bar{k}(5 + \bar{p}(\gamma - 3))y_2 - (2B_1 + 5B_2)y_3 + 5B_2 y_4] \end{Bmatrix}$$

where  $\{y_1, y_2, y_3, y_4\} = \{\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2\}$ . In the following, the dynamics of a primary single Hopf and a single flip bifurcation of the above four dimensional system is discussed via the center manifold principle by reducing the problem to a two and a single dimension, respectively.

(i) Hopf Bifurcation

For the parameter set,  $\bar{k} = 2.0$ ,  $B_1 = B_2 = 0.016$ ,  $P_1 l/k = 0.5$ ,  $P_2 l/k = 0.966$ ,  $\gamma = 0.8$ ,  $\omega = 1.0$ , equation (48) yields a pair of complex Floquet multipliers with modulus one

which corresponds to a single Hopf bifurcation. After normalizing the time with  $\omega t' = 2\pi t$ , the L–F transformation corresponding to equation (48) is computed. The application of this transformation to equation (48) leads to the following dynamically equivalent Jordan canonical form.

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} = \begin{bmatrix} -0.17791 + 1.14613i & 0 & 0 & 0 \\ 0 & -0.17791 - 1.14613i & 0 & 0 \\ 0 & 0 & 0.3381i & 0 \\ 0 & 0 & 0 & -0.3391i \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + \begin{pmatrix} \sum a_q(t)y_1^{m_1}y_2^{m_2}y_3^{m_3}y_4^{m_4} \\ \sum b_q(t)y_1^{m_1}y_2^{m_2}y_3^{m_3}y_4^{m_4} \\ \sum c_q(t)y_1^{m_1}y_2^{m_2}y_3^{m_3}y_4^{m_4} \\ \sum d_q(t)y_1^{m_1}y_2^{m_2}y_3^{m_3}y_4^{m_4} \end{pmatrix}, \quad \sum_{i=1}^4 m_i = 3 \quad (49)$$

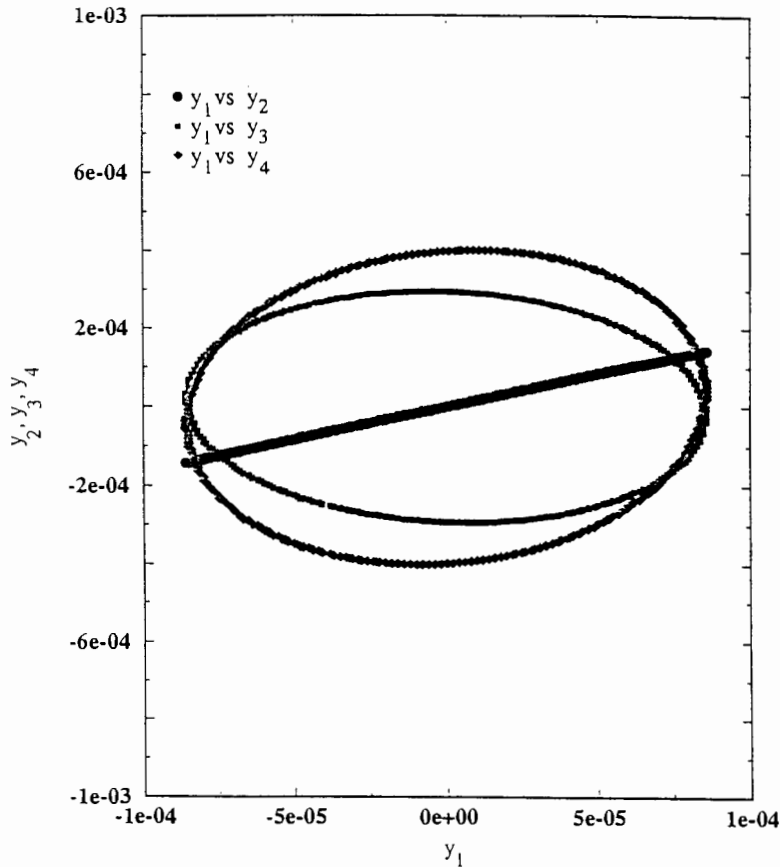
where  $a_q(t)$ ,  $b_q(t)$ ,  $c_q(t)$  and  $d_q(t)$  are the complex vector periodic coefficients consisting of 31 elements (this corresponds to the number of Fourier terms taking in the expansion of L–F transformation matrix) with period  $2T$  corresponding to all possible monomials of order 3 in  $y_1, y_2, y_3$  and  $y_4$ . Note that two of the eigenvalues in equation (49) are purely imaginary which is to be expected in this case. The center manifold relations for this problem are assumed in the form

$$\begin{aligned} y_1 &= B_{11}^3(t)y_3^3 + B_{12}^3(t)y_3^2y_4 + B_{13}^3(t)y_3y_4^2 + B_{14}^3(t)y_4^3 \\ y_2 &= B_{21}^3(t)y_3^3 + B_{22}^3(t)y_3^2y_4 + B_{23}^3(t)y_3y_4^2 + B_{24}^3(t)y_4^3 \end{aligned} \quad (50)$$

where  $B_{ij}^3(t)$ ,  $i = 1, 2$  and  $j = 1, 2, 3, 4$  are unknown periodic coefficients with period  $2T$ . Note that in the above equation the states corresponding to stable eigenvalues are expressed in terms of the states corresponding to the critical eigenvalues.

Substituting equation (50) in equation (49), eight ordinary differential equations in  $B_{ij}^3(t)$ ,  $i = 1, 2$  and  $j = 1, 2, 3, 4$ , similar to equation (25) are obtained. The periodic coefficients appearing on the right hand side of these differential equations are nothing but the known periodic coefficients corresponding to the cubic nonlinear terms appearing in equation (49). The unknown periodic coefficients  $B_{ij}^3(t)$ ,  $i = 1, 2$  and  $j = 1, 2, 3, 4$  can be obtained by formally solving these differential equations. In order to obtain a particular solution,  $B_{ij}^3(t)$  is assumed in the form of equation (26) with unknown constant coefficients and the like terms on both sides of the equations are equated to obtain a set of linear algebraic equations in terms of the unknowns  $p_n$  and  $q_n$ . The computation of all the unknowns of the  $B_{ij}^3(t)$ 's requires the solution of a set of  $8 \times 31$  linear algebraic equations. These algebraic equations can be solved such that each of the  $B_{ij}^3(t)$ ,  $i = 1, 2$  and  $j = 1, 2, 3, 4$  can be obtained as Fourier series expansions. Noting that the problem under consideration consists of only cubic nonlinearities, it is not necessary to solve for all the periodic coefficients  $B_{ij}^3(t)$ ,  $i = 1, 2$  and  $j = 1, 2, 3, 4$  in the center manifold relation equation (50). Instead, it suffices to compute only one coefficient per relation. This is due to the fact that the substitution of equation (50) in equation (49) gives rise to nonlinearities that are of powers greater than three which are not critical in the stability characteristics of that equation. Therefore, the simplification does not affect the final outcome of the result. Henceforth, for this case, only coefficients  $B_{11}^3(t)$  and  $B_{21}^3(t)$  are computed.

Substitution of center manifold relations (50) in equation (49) results in the differential equations for the critical states  $y_3$  and  $y_4$  which contain nonlinearities of cubic and higher



*Fig. 5a.* Phase plane plot – double inverted pendulum under Hopf bifurcation.

orders. Since the higher order terms do not affect the stability characteristics [6], the terms of order higher than 3 are neglected. The equations thus obtained are similar to equation (31) representing a Hopf bifurcation behavior in a two dimensional system with cubic nonlinearity. Following the procedure outlined earlier in the study of a single degree of freedom system, the application of time-dependent normal forms to these equations provide a simplified nonlinear equation similar to equation (34). The behavior of the fixed point of the resulting equation is found to be a center by employing the similar methods outlined in the beginning of Section 3. For brevity, the calculations are not reported here. Following similar arguments as those presented for example 1, the motion resulting from the Hopf bifurcation is found to be quasi-periodic and bounded. It can readily be seen that the Poincaré plots provided in Figures 5a and 5b also confirm this result.

### (ii) *Flip Bifurcation*

Consider the parameters  $\bar{k} = 2.0$ ,  $B_1 = B_2 = 0.0175$ ,  $P_1 = 0$ ,  $P_2 = 0.5$ ,  $\gamma = 0.895$ ,  $\omega = 1.1$  such that one of the Floquet multipliers of equation (48) becomes  $-0.999938$  ( $\approx -1$ ) and the system undergoes a flip bifurcation. After the transformation, the following Jordan canonical

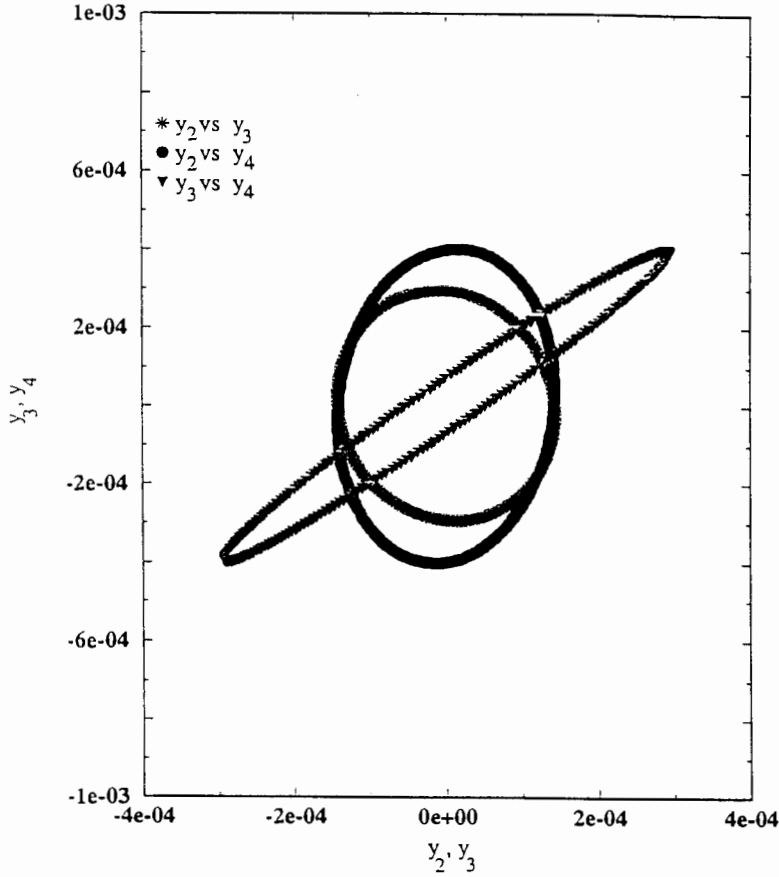


Fig. 5b. Phase plane plot – double inverted pendulum under Hopf bifurcation.

form is obtained

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} = \begin{bmatrix} -0.1685 + 0.9557i & 0 & 0 & 0 \\ 0 & -0.1685 - 0.9957i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0137 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + \left\{ \begin{array}{l} \sum a_q(t) y_1^{m_1} y_2^{m_2} y_3^{m_3} y_4^{m_4} \\ \sum b_q(t) y_1^{m_1} y_2^{m_2} y_3^{m_3} y_4^{m_4} \\ \sum c_q(t) y_1^{m_1} y_2^{m_2} y_3^{m_3} y_4^{m_4} \\ \sum d_q(t) y_1^{m_1} y_2^{m_2} y_3^{m_3} y_4^{m_4} \end{array} \right\}, \quad \sum_{i=1}^4 m_i = 3 \quad (51)$$

where  $a_q(t)$ ,  $b_q(t)$ ,  $c_q(t)$  and  $d_q(t)$  once again are vector periodic coefficients with period  $2T$ . It is observed that the eigenvalue corresponding to the third state is zero and the remaining eigenvalues have negative real parts. The center manifold relations for this case can be assumed in the form

$$y_1 = B_{11}^3(t)y_3^3; \quad y_2 = B_{21}^3(t)y_3^3; \quad B_{31}^3(t)y_3^3 \quad (52)$$

where  $B_{11}^3(t)$ ,  $B_{21}^3(t)$  and  $B_{31}^3(t)$  are unknown coefficients with period  $2T$ . These can once again be determined by solving the differential equations in  $B_{11}^3(t)$ ,  $B_{21}^3(t)$  and  $B_{31}^3(t)$  as

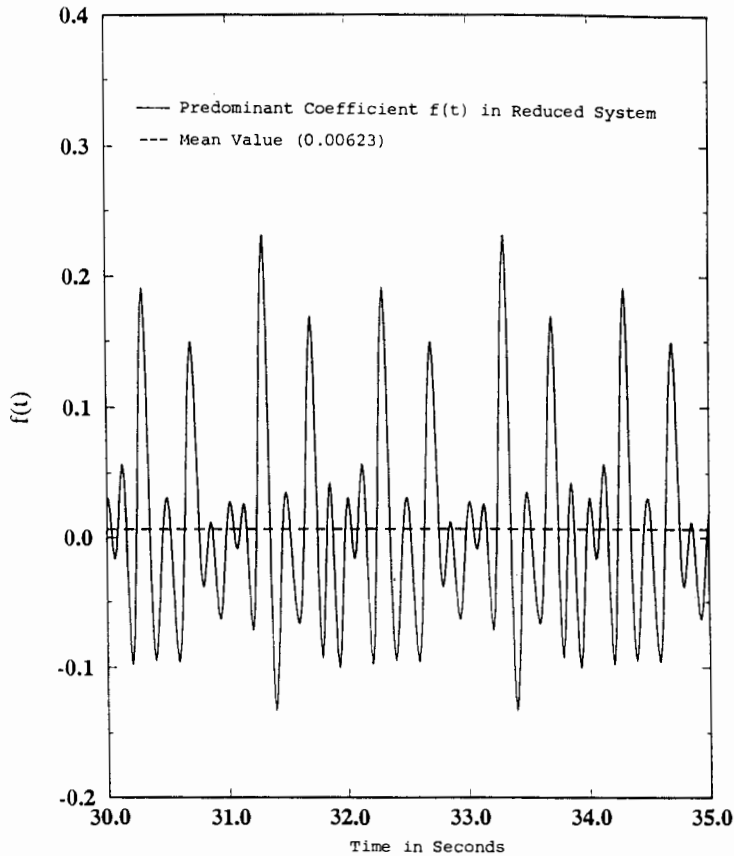


Fig. 6. Periodic coefficient of the reduced system – double inverted pendulum under flip bifurcation.

described earlier in this section. Substituting equation (52) into equation (51) and neglecting the higher order terms beyond the cubics, the one dimensional center manifold equation is found to be of the form of equation (42). Since the mean value of the periodic coefficient of the reduced center manifold equation for this case is positive (shown in Figure 6), the fixed point is unstable and hence the corresponding  $2T$  periodic orbits in the original coordinates are unstable. The Poincaré points for each of the coordinate set (2 points per set) are plotted in Figure 7. Note that the arrows in Figure 7 indicate the direction of drift of the Poincaré points moving away from the periodic orbits confirming their instability. The simulation is performed for about 150 periods and therefore, each point in Figure 7 has about 150 points. In a similar way, the fold bifurcation of this system can also be studied. However, the results are not included for brevity.

#### 4. Discussion and Conclusions

In this paper, a general approach for stability analysis of nonlinear dynamical systems with periodically varying coefficients undergoing various bifurcations is presented. The approach is based on the application of the well-known periodic L-F transformation matrix and the time-dependent simplification principles such as time-dependent normal forms and center manifold reduction theories. First, following the recent development by Sinha and his associates [10–

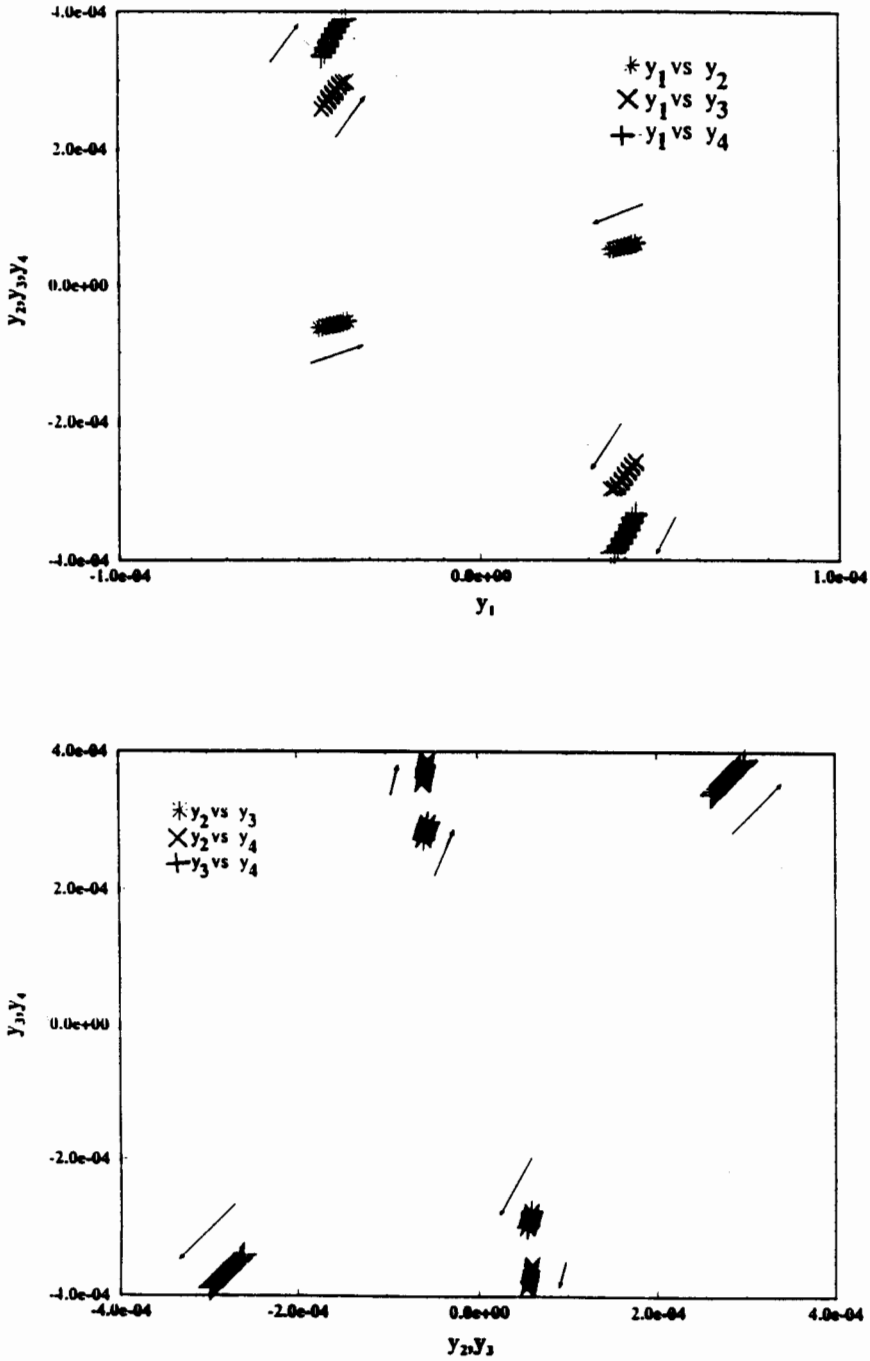


Fig. 7. Poincaré plots – double inverted pendulum under flip bifurcation.

14], a procedure for the computation of L–F transformation matrices associated with general linear periodic systems is outlined. Application of the L–F transformation to the variational time periodic system yields a similar dynamical system with a time-invariant linear part. Therefore, the resulting system is amenable to analysis via the *time-dependent normal form*

and *center manifold theories* as suggested by Arnold [1] and Malkin [19], respectively. For the two examples considered, the stability is examined when the system undergoes flip, fold and Hopf bifurcations. By applying the L–F transformation and the time-dependent reduction principles, it is shown that the dynamics of the higher dimensional nonlinear periodic system can be studied in a lower dimensional center manifold. For example, the four dimensional double inverted pendulum undergoing a single Hopf bifurcation or a single flip bifurcation has been investigated in a two or single dimensional center manifold, respectively. The conclusions obtained from the reduced equations are verified using numerical simulations (viz. Figures 1 through 4 for a single degree of freedom system and Figures 5 and 7 for a two-degrees of freedom system) including the Poincaré mapping techniques.

Similar studies can be carried out using the averaging procedure [4] and the method of point mappings [5, 6]. While averaging procedure does provide the qualitative nature of the bifurcations of time-periodic systems with small parameters, more often it does not represent the correct behavior of the solution as indicated from the entries in Table 1. The Poincaré mapping approach presented in [5] and [6] certainly provides an alternative in the study of bifurcations of such systems. Nevertheless, such methods are not free from computational difficulties. Further, the construction of approximate Poincaré map as suggested by Lindtner *et al.* [6] can be really involved for large systems. On the other hand, the procedure followed in this paper is straightforward and analogous to the existing methods in the study of bifurcations of autonomous systems. The main advantage is that the procedure based on L–F transformation allows one to transform back to the original coordinates by means of the inverse L–F transformation matrix while the approach in [5] does not have this advantage. Such a step may be essential if one wants to deal with control problems of nonlinear periodic systems. The procedure presented here is equally applicable in the study of secondary bifurcations of periodic systems, however, such results are not included in the present study. They will be reported elsewhere.

A major outcome of the procedures outlined in this paper is in the design of linear/nonlinear control systems for periodically varying systems with critical modes. Already, such procedures have been found to be useful in the case of autonomous systems and the results corresponding to the periodic systems will be reported in future.

## Appendix A

### *Center manifold relation for the single degree of freedom system for flip bifurcation*

$$y_1 = \{0.36599189 + 0.10060650 * \cos 2\pi t - 0.03919308 * \cos 4\pi t \\ + 0.00205263 * \cos 6\pi t + 0.00217666 * \cos 8\pi t - 0.00029856 * \cos 10\pi t \\ - 0.00001626 * \cos 12\pi t - 0.05887959 * \sin 2\pi t \\ - 0.01999508 * \sin 4\pi t + 0.01257054 * \sin 6\pi t - 0.00152234 * \sin 8\pi t \\ - 0.00023195 * \sin 10\pi t + 0.00003418 * \sin 12\pi t\}y_2^3.$$

### *Center manifold relation for the single degree of freedom system for fold bifurcation*

$$y_1 = \{0.00380799 - 0.00014216 * \cos 2\pi t + 0.00028701 * \cos 4\pi t$$

$$\begin{aligned}
& - 0.00010421 * \cos 6\pi t - 0.00002787 * \cos 10\pi t - 0.00095701 * \sin 2\pi t \\
& + 0.00033305 * \sin 4\pi t - 0.00024266 * \sin 6\pi t + 0.00005221 * \sin 8\pi t \\
& - 0.000012184 * \sin 10\pi t + 0.00001987 * \sin 12\pi t \} y_2^3.
\end{aligned}$$

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