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ANALYSIS OF QUASILINEAR DYNAMICAL SYSTEMS WITH PERIODIC COEFFICIENTS VIA LIAPUNOV–FLOQUET TRANSFORMATION

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Abstract—In this paper, a new analysis technique in the study of general quasilinear systems with periodically varying parameters is presented. The method is based on the fact that all quasilinear periodic systems can be replaced by similar systems whose linear parts are time-invariant, via the well-known Liapunov–Floquet (L–F) transformation. A general technique for the computation of the L–F transformation matrices is outlined. In this technique, the state vector and the periodic matrix of the linear system equations are expanded in terms of the shifted Chebyshev polynomials over the principal period. Such an expansion reduces the original problem to a set of linear algebraic equations from which the state transition matrix can be constructed over the period as an explicit function of time. Application of Floquet theory and use of symbolic software yields the L–F transformation matrix in a form suitable for algebraic manipulations. Once the transformation has been applied, the solution of the resulting system is obtained through an application of the *time-dependent normal form theory*. The method is suitable for both numerical and symbolic computations and in some cases approximate closed form solutions can be obtained. Two simple examples of quasilinear periodic systems—namely, a commutative system with quadratic non-linearity and a Mathieu equation with cubic non-linearity—are used to demonstrate the effectiveness of the method. For verification, results obtained from the proposed technique are compared with the numerical solutions computed using a standard Runge–Kutta type algorithm. It is shown that the present technique is applicable to systems where the periodic matrix does not contain a small parameter, which is not the case with averaging and perturbation procedures. It can also be used even for those systems for which the generating solutions do not exist in the classical sense.

1. INTRODUCTION

The study of systems governed by a set of ordinary differential equations with periodic coefficients is of great importance in diverse branches of science and engineering. Floquet [1] discovered the fundamental form of the solution of linear periodic systems which is widely used nowadays for stability analysis of such systems. Since then, numerous researchers have worked on the stability and dynamical analysis of linear periodic systems. A comprehensive and systematic exposition of these results has been provided by Yakubovich and Starzhinskii [2]. Hill's method [3], classical methods such as averaging and perturbation techniques [4, 5] and Floquet Theory [6] are some of the most commonly used mathematical methods in the analysis of such systems. Hill's approach is only suitable for obtaining the stability boundaries. On the other hand, the classical methods have their own limitations due to the fact that they can only be applied to systems where the periodic coefficients can be expressed in terms of a small parameter. Also, they are suitable for relatively smaller systems. Therefore, Floquet analysis coupled with a numerical integration code has served as the main tool in various applications [7, 8]. Recently, Sinha and Wu [9] have proposed an efficient technique for computing the State Transition Matrix (STM) and the stability analysis of linear periodic systems through an application of the Chebyshev polynomials in conjunction with the Floquet theory. Further, it has also been shown that the technique can be applied in the symbolic form [10] and in the study of control systems subjected to parametric periodic loading [11].

Due to the nature of existence of multiple solutions in non-linear equations, the linearized equations alone fail to provide answers to many questions associated with the non-linear systems. Therefore, one must investigate the perturbed motion about a particular solution. Such equations were first obtained by Liapunov [12] and Poincaré [13] while studying the stability of periodic motions associated with non-linear autonomous systems. These variational equations are quasilinear and contain periodic coefficients, since they represent the dynamics in the neighborhood of a known periodic motion. Averaging [5] and perturbation techniques [6] have been found applicable to such quasilinear systems with a tacit assumption that a generating solution exists and the periodic terms as well as the non-linearity can be expressed in terms of a suitable small parameter. On the other hand, standard numerical techniques such as shooting methods [14] have also been found applicable for calculating branch points, etc. However, such methods are certainly not free from numerical instability difficulties. In an alternate method of analysis known as the *point mappings* technique [15, 16], the original non-autonomous differential equation is replaced by a set of difference equations which do not depend explicitly on time. In principle, these difference equations are easier to analyze; however, one faces serious computational difficulties in the application of this technique to real engineering problems, even if the dimensions are small. Also, in order to obtain the corresponding difference equations, one must construct an exact or an approximate solution of a system of non-linear time-varying equations. Since one must frequently settle for an approximate representation of the solution, the point mappings obtained based on this solution are also approximate and may not portray the correct behavior of the original system.

Another viable approach is to use the Liapunov–Floquet Theorem, which allows transformation of the quasilinear periodic systems into a new set of similar equations whose linear parts are time-invariant. However, at present there are no methods available to compute this transformation matrix [17, 18] for a general periodic system. For certain special class of linear systems, it is possible to obtain the Liapunov–Floquet transformations as indicated in [19, 20]. In order to determine such a transformation for a general periodic system, one must compute the STM as an explicit function of time. Recently, Sinha and his co-authors [9–11, 21] have been successful in obtaining the STMs of general linear periodic systems numerically as well as symbolically by using an efficient algorithm. In their work the solutions of linear periodic systems are expressed in terms of the shifted Chebyshev polynomials of the first kind and hence the elements of the state transition matrix are basically expressed in terms of powers of time t. The subsequent development by Pandiyan *et al.* [22] based on the above work includes a computational procedure through which the Liapunov–Floquet (L–F) transformation can be obtained in a form suitable for algebraic manipulations. The inverse of the L–F transformation can also be computed by a similar procedure considering the adjoint system equation.

The development of a procedure for computing these transformation matrices has given a clear edge in dealing with a wide range of problems associated with periodically varying systems. In this paper, a quantitative analysis of quasilinear dynamical systems with periodic coefficients has been presented through an application of the Liapunov–Floquet (L–F) transformation. It is shown that the original quasilinear periodic system can be transformed to a dynamically similar form in which the linear part is time-invariant. The analysis of the transformed equations has been carried out through the use of *normal form theory*. The solutions thus obtained are mapped back to the original coordinates by applying the inverse L–F transformations and compared with the numerical results obtained by a Runge–Kutta type algorithm. In order to demonstrate the effectiveness of the proposed analysis procedure, two examples have been studied in detail. In the first example, a commutative system with quadratic non-linearity is considered for which the L–F transformation matrix can be computed in an exact closed form by simple methods. The second example consists of non-linear Mathieu equation, the L–F transformation of which has been computed using the Chebyshev polynomials as described in Section 3. The solutions of these examples have been obtained in stable and center manifolds for some typical sets of system parameters. It has been shown that the proposed technique is applicable to a wide class of problems including the situations where the generating solutions do not exist and/or the parameter multiplying the linear periodic terms are no longer small. It is also shown that in many cases it is possible to obtain approximate analytical solutions which compare extremely well with the numerical solutions. The results obtained by the traditional averaging method are also presented for comparison purposes. The authors believe that the solutions of this nature for quasilinear periodic systems have been presented for the very first time.

2. DYNAMICAL EQUATIONS OF TIME-PERIODIC SYSTEMS

In many engineering applications, time-periodic dynamic systems are represented by differential equations of the form given by

$$\dot{x} = A(t)x + f_2(x,t) + f_3(x,t) + \cdots + f_k(x,t) + O(|x|^{k+1}, t) \tag{1}$$

where x is an $n \times l$ vector and the $n \times n$ matrix $A(t)$ and $n \times l$ non-linear vectors $f_k(x,t)$ are T-periodic functions of time t. Such equations arise naturally and/or in the study of stability and bifurcation of periodic orbits which may represent steady-state solutions of autonomous or nonautonomous systems. It is to be noted that the non-linear terms $f_k(\cdot)$ in (1) represent homogeneous monomials in x_i of order k. For later use, we rewrite the above equation as

$$\dot{x} = A(t)x + F(x,t) \tag{2}$$

where $F(\cdot)$ is appropriately defined in terms of $f_k(\cdot)$s. The linear part of (1) is, of course, given by

$$\dot{x} = A(t)x; \qquad A(t) = A(t+T). \tag{3}$$

The stability and response of (3) can be discussed using the well-known Floquet theory.

3. COMPUTATION OF L–F TRANSFORMATION MATRIX VIA CHEBYSHEV POLYNOMIALS

It has been shown by Sinha and co-workers [9–11, 21] that the STMs of linear periodic systems can be obtained in terms of the shifted Chebyshev polynomials of the first kind. The technique is efficient (see [23]) and since the STM is expressed as an explicit function of time t, it is suitable for algebraic manipulations as well. In fact, if the dimension is small, the STM can be expressed in a closed form as an explicit function of system parameters as shown by Sinha and Juneja [21] for the case of Mathieu equation.

In order to compute the L–F transformation matrix $L(t)$, one needs to find the STM $\Phi(t)$ associated with the linear system given by equation (3). If $A(t)$ in equation (3) is commutative, then $L(t)$ can be computed as

$$L(t) = e^{B_T(t)} \tag{4}$$

where $B_T(t)$ can be obtained in a closed form as shown in reference [20]. For a non-commutative $A(t)$, first, the Chebyshev polynomial expansion technique is used to compute the STM $\Phi(t)$. This requires a set of solutions of (3) with n initial conditions: $x_i(0) = (1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, 0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)$. By defining the resulting set of Chebyshev coefficient vector $\bar{b}_i$s of the n solutions in a matrix form, the STM can be written as

$$\Phi(t) = \hat{S}(t)\bar{B} \tag{5}$$

where $\bar{B} = [\bar{b}_1 \bar{b}_2 \bar{b}_3 \ldots\ldots \bar{b}_n]$ and $\Phi(0) = I$. It has to be noted that the STM is valid only for $0 \le t \le T$ since the shifted Chebyshev polynomials of the first kind are defined over the interval $[0, T]$. When $t > T$, the STM can be evaluated using Floquet theory [6] as

$$\Phi(t) = [\Phi(\psi)][\Phi(T)]^n \tag{6}$$

where $t = nT + \psi$, $\psi \in [0, T]$, and $n = 1, 2, 3, \ldots$. Once $\Phi(t)$ is known, the T-periodic complex matrix $L(t)$ or the $2T$-periodic real matrix $Q(t)$ can be computed in the following way [24]. Since $\Phi(0) = I$, $L(0) = L(T) = I$, the Floquet Transition Matrix (FTM) $\Phi(T)$ can be written as

$$\Phi(T) = e^{CT} \tag{7}$$

where C is a $n \times n$ constant complex matrix. By performing an eigen-analysis on $\Phi(T)$, the matrix C can be computed easily. Then the T-periodic, L–F transformation matrix is

$$L(t) = \Phi(t)e^{-Ct}. \tag{8}$$

In order to evaluate the $2T$-periodic real. L–F transformation matrix $Q(t)$, first we note that (cf. [6]):

$$\Phi(2T) = \Phi^2(T) = e^{CT}e^{C^*T} = e^{2RT} \tag{9}$$

where C^* is the conjugate matrix of C, the $n \times n$ constant real matrix $R = [C + C^*]/2$ and the $2T$ periodic L–F matrix can be represented as

$$Q(t) = \Phi(t)e^{-Rt}; \quad 0 \leq t \leq T$$
$$Q(\tau + T) = \Phi(\tau)Q(T)e^{-R\tau}; \quad T \leq (T + \tau) \leq 2T; \quad 0 \leq \tau \leq T. \tag{10}$$

It should be noted that $Q(t) = Q(t + 2T)$.

If one is interested in finding $L^{-1}(t)$ or $Q^{-1}(t)$, then there are two avenues. $L(t)$ and $Q(t)$ may possibly be inverted through the use of a symbolic software like MACSYMA/MATHEMATICA/MAPLE. However, this is neither realistic nor convenient at this point in time. The other approach is to first find the STM $\Psi(t)$ of the adjoint system

$$\dot{w} = -A^T(t)w \tag{11}$$

and use the following relationship (cf. ref. [2]),

$$\Phi^{-1}(t) = \Psi^T(t). \tag{12}$$

The computation of $\Phi^{-1}(t)$ is critical in determining $L^{-1}(t)$ or $Q^{-1}(t)$. For example, the inverse T-periodic L–F transformation matrix $L^{-1}(t)$ can be evaluated utilizing the properties of the adjoint system as shown below:

$$L^{-1}(t) = [\Phi(t)e^{-Ct}]^{-1} = e^{Ct}\Phi^{-1}(t) = e^{Ct}\Psi^T(t). \tag{13}$$

The technique described here has been successfully employed to compute the STMs for relatively large systems [11]. Such an approximation of L–F transformations has been found to be extremely convergent and since it is periodic with period T, the elements $L_{ij}(t)$ or $Q_{ij}(t)$ have the truncated Fourier representation

$$L_{ij}(t) \approx \sum_{n=-q}^{q} c_n \exp(i2\pi nt/T), \quad i = \sqrt{-1} \tag{14}$$

or

$$Q_{ij}(t) \approx \frac{a_0}{2} + \sum_{n=1}^{q} a_n \cos\frac{\pi nt}{T} + \sum_{n=1}^{q} b_n \sin\frac{\pi nt}{T}. \tag{15}$$

Since complex matrix $L(t)$ (or the real matrix $Q(t)$) can be computed as a function of t, all algebraic manipulations involving this matrix in (3) can be done in symbolic form using MATHEMATICA or MACSYMA. $L_{ij}^{-1}(t)$ and $Q_{ij}^{-1}(t)$ have similar Fourier representations. Computations of $Q(t)$ and $Q^{-1}(t)$ have been reported by Sinha and Joseph [25] in the context of control problems associated with linear periodic systems. In this study only the real L–F transformation $Q(t)$ has been used to make it more appealing to the engineering community. A 15–18 term Chebyshev expansion has been found to yield extremely accurate representation of $Q(t)$. The accuracy of $Q(t)$ is directly dependent upon the convergence and accuracy of the STM $\Phi(t)$ itself. A convergence study in the computation of $\Phi(t)$ has been reported by Joseph *et al.* [11].

4. ANALYSIS OF QUASILINEAR PERIODIC SYSTEMS

In this section, solution techniques applicable to the quasilinear equation (2) are discussed. The fact that $A(t)$ is time dependent in (2), a direct application of *normal form theory* is not possible. Therefore, first, the $2T$ periodic L–F transformation matrix $Q(t)$ is used such that

$$x(t) = Q(t)z(t) \tag{16}$$

transforms (2) to a vector field where the linear part is time-invariant. The resulting equation is of the form

$$\dot{z} = Rz + Q^{-1}F(Qz, t) \tag{17}$$

where R is an $n \times n$ constant matrix. The form of (17) is amenable to direct application of the method of *time-dependent normal forms* (TDNF) as shown by Arnold [26].

Equation (17) in its Jordan canonical form can be written as

$$\dot{y} = Jy + w_2(y,t) + \cdots + w_k(y,t) + \cdots O(|y|^{k+1},t) \tag{18}$$

where J is the Jordan form of matrix R and w_ks are T-periodic functions and contain homogeneous monomials of y_i of order k. In order to remove the non-linear terms $w_r(y,t)$ for $r = 2,\ldots,k$, in a successive fashion, one applies a sequence of transformations of the form

$$y = u + h_r(u,t) \tag{19}$$

where $h_r(u,t)$ is a formal power series in u of degree r with periodic coefficients having the principal period T. Application of the above change of coordinates to equation (18) yields

$$\dot{u} = Ju - \{(D_y h_r)Ju - Jh_r + \partial h_r/\partial t\} + w_r(u,t) + O(|u|^{r+1};t) \tag{20}$$

where $(D_y h_r = [\partial h_r/\partial y])$ such that the homological equation

$$(D_y h_r)Ju - Jh_r + \partial h_r/\partial t = w_r(u,t) \tag{21}$$

is satisfied. Since for periodic systems, the monomials $w_r(u,t)$ and the unknown non-linear transformation $h_r(u,t)$ can be expressed in finite Fourier form, the solvability condition for a given degree of non-linearity can be obtained as [24]

$$h_{r,j,\sigma} = a_{r,j,\sigma}/(i\sigma\Omega + m\cdot\Lambda - \lambda_j) \tag{22}$$

where $h_{r,j,v}$, and $a_{r,j,v}$ are the Fourier coefficients of expansions of $h_r(u,t)$ and $w_r(u,t)$, respectively. Further, $\Omega = \pi/T$, integer σ takes values between $-q$ and q, $q = 1, 2, \ldots$ and $\Lambda \equiv (\lambda_1, \lambda_2, \ldots \lambda_n)$ are the eigenvalues of matrix J. It is clear that when

$$(i\sigma\Omega + m\cdot\Lambda - \lambda_j) \neq 0\,\forall\,\sigma, \tag{23}$$

equation (17) can be reduced to a linear form. Otherwise, the corresponding resonant terms will stay in the reduced equation so that (18) takes the simplest non-linear form as

$$\dot{u} = Ju + \sum_{r=2}^{k} w_r(u,t) + \cdots O(|u|^{k+1},t). \tag{24}$$

It is important to note that the $w_r(u,t)$ contains only a finite number q of Fourier harmonics. This is due to the fact that the solution of the homological equation (21) depends on the resonance condition relating the eigenvalues of J and the Fourier frequencies of $w_r(u,t)$. It should be pointed out that the solution of the time-dependent homological equation requires the solution of a large set of linear algebraic equations even for a 2×2 system. For example, if for such a system, the L–F transformation matrix $Q(t)$ is represented by a 15 term complex Fourier expansion and let us say that the degree of the monomials $r = 3$, then one needs to solve (2×124) equations in the blocks of 31!

In an attempt to minimize the computational efforts without any significant compromise in the quality of solutions, only the constant terms in w_r [cf. equation (18)] are retained and the periodically varying terms may be neglected. The argument for such an approximation can be made on the basis of the work reported by Rosenblat and Cohen [27, 28]. Since the resulting approximate equation is autonomous, the analysis can be carried out via the *time-independent normal form (TINF) theory*. Neglecting the terms containing the periodic variations in (18) yields

$$\dot{\tilde{y}} = J\tilde{y} + \sum_{r=2}^{k} \tilde{w}_r(\tilde{y}) + O(|\tilde{y}|^{k+1}) \tag{25}$$

which is autonomous and $\tilde{y}$ and $\tilde{w}(\tilde{y})$ are the approximate states and non-linear monomial functions, respectively. Once again a transformation of the form given in (19) is considered, except that h_r is no longer a function of time t. Substitution of such a transformation in (25) provides an autonomous homological equation and resonant conditions similar to (23) with $v\Omega = 0$. The simplest normal form when some of the resonant conditions are not satisfied for this case is given as

$$\dot{u} = Ju + \sum_{r=2}^{k} \tilde{w}_r(u) + O(|u|^{k+1}). \tag{26}$$

It should be noted that (26) contains only a few non-linear terms $\tilde{w}(\tilde{y})$ as compared to those appearing in (25). For more details and classifications of normal forms corresponding to different cases, one can refer to Arrowsmith and Place [24] and/or Bruno [29]. The advantage in computational effort in the time-independent form can be seen clearly if one considers a similar 2×2 system with cubic monomials, where the number of algebraic equations to be solved is just (2×4) in comparison to (2×124) sets for the time-dependent form.

In many instances, the resulting normal form equations after the application of either TDNF or TINF theory can be analyzed analytically and the stability characteristics can be examined. Usually, some algebraic manipulations and/or transformation to polar coordinates render closed form solutions for simple non-linear equations. Therefore, the solutions of the original quasilinear time-periodic equations can be obtained in the original coordinates by substituting back all the transformations, including the L–F transformation.

5. APPLICATIONS

To demonstrate the applicability and effectiveness of the suggested approach, two examples are considered. In the first example a commutative system with quadratic non-linearity is analyzed. In this case, the L–F transformation is obtained in a closed form and various solutions are computed using the techniques described above. For this problem, it is possible to obtain closed form solutions in terms of the parameters of the system. As a second example, a non-linear Mathieu equation is considered for which the L–F transformation cannot be obtained in a closed form and one must resort to the computational algorithm discussed in Section 3. Although these examples are simple, they bring out the key points clearly and show the superiority of the proposed methods over the classical methods such as averaging, perturbation, etc. Whereas the results of the proposed technique, based on L–F transformation and normal forms, provide reasonably good solutions even for moderately large parameters multiplying the non-linear terms, the traditional averaging procedure is applicable only when the parameters multiplying the periodic terms and the non-linear terms are both small.

Example 1. *A commutative system with quadratic non-linearity*

Consider a commutative system with quadratic non-linearity as shown below:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} -1 + \alpha\cos^2 t & 1 - \alpha\sin t\cos t \\ -1 - \alpha\sin t\cos t & -1 + \alpha\sin^2 t \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \varepsilon \begin{Bmatrix} x_1^2 \\ x_2^2 \end{Bmatrix} \tag{27}$$

where α is a control parameter and ε, the nonlinearity parameter, is not necessarily small. The STM $\mathbf{\Phi}(t)$ of this system is known to be [30]

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{(\alpha-1)t}\cos t & e^{-t}\sin t \\ -e^{(\alpha-1)t}\sin t & e^{-t}\cos t \end{bmatrix}. \tag{28}$$

Following the procedure outlined in Lukes [20], the L–F transformation matrix $Q(t)$ of the system is found as

$$Q(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}. \tag{29}$$

Applying the transformation $x = Q(t)z$, equation (27) takes the form

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{bmatrix} \alpha - 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \varepsilon Q^{-1}(t) \begin{Bmatrix} [z_1\cos t + z_2\sin t]^2 \\ [-z_1\sin t + z_2\cos t]^2 \end{Bmatrix} \tag{30}$$

where $Q^{-1}(t)$ is the inverse of the L–F matrix $Q(t)$. By using the time-dependent near-identity transformation of the form

$$z_1 = u + p_1(t)u^2 + p_2(t)uv + p_3(t)v^2$$

$$z_2 = v + q_1(t)u^2 + q_2(t)uv + q_3(t)v^2. \tag{31}$$

Equation (30) can be reduced to the following linear decoupled form

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} (\alpha - 1) & 0 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}. \tag{32}$$

The time-dependent coefficients $p_i(t)$ and $q_i(t)$ of (31) are given in Appendix A. The above equation yields

$$u(t) = u(0)e^{(\alpha - 1)t}$$

$$v(t) = v(0)e^{-t} \tag{33}$$

where $u(0)$ and $v(0)$ are the initial values. Since $u(t)$ and $v(t)$ are known, $z_1(t)$ and $z_2(t)$ can be obtained from (31) and $\boldsymbol{x}(t)$ is immediately determined from (16).

In an attempt to obtain a solution of (27) via TINF theory, as discussed in Section 4, the variations of the periodic coefficients of the non-linear terms of (30) are neglected. For this case, the resulting system turns out to be simply the linear part of that equation itself and hence the solution is once again given by (33). When the traditional averaging procedure is directly applied to (27), it is found that the system always reduces to a critical quasilinear system of the form

$$\begin{Bmatrix} \dot{\bar{z}}_1 \\ \dot{\bar{z}}_2 \end{Bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix} + \varepsilon \begin{Bmatrix} p_1\bar{z}_1^2 + p_2\bar{z}_1\bar{z}_2 + p_3\bar{z}_2^2 \\ q_1\bar{z}_1^2 + q_2\bar{z}_1\bar{z}_2 + q_3\bar{z}_2^2 \end{Bmatrix} \tag{34}$$

where p_i and q_i, $i = 1,2,3,\ldots$ are constants depending upon α. It is known [30] that the commutative system given by (27) with $\varepsilon \equiv 0$ is asymptotically stable for the parameter range $0 < \alpha < 1$, whereas (34) exhibits an unstable behavior! Thus the averaging technique is unsuitable for this problem. The solutions of this system in stable and center manifolds are computed using the time-dependent normal form (TDNF), the time-independent normal form (TINF) and the traditional averaging methods. These results are compared with the corresponding numerical solutions in Figs 1 and 2, respectively. The solutions based on TDNF and TINF are observed to possess uniform convergence in the stable manifold. When $\alpha = 1$, (30) has a critical (zero) eigenvalue and a real eigenvalue. Also, the periodic coefficients of non-linear terms have zero mean. Therefore, the solvability

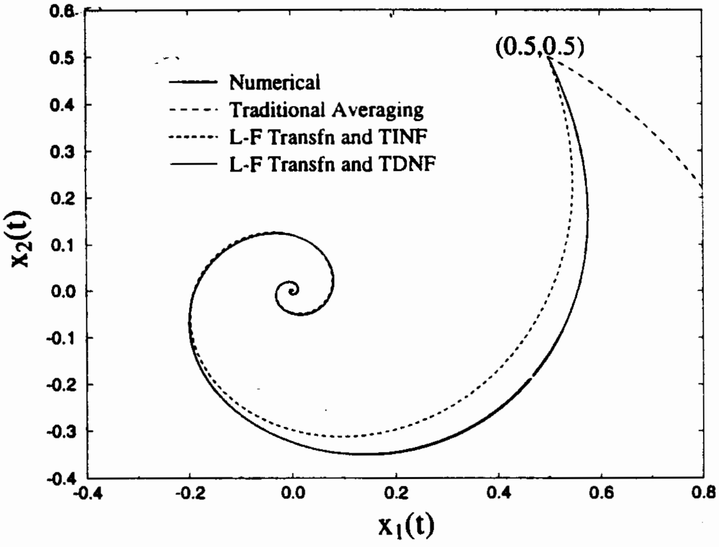

Fig. 1. Comparison of solution of commutative system in stable manifold ($\alpha = 0.7$, $\varepsilon = 0.3$).

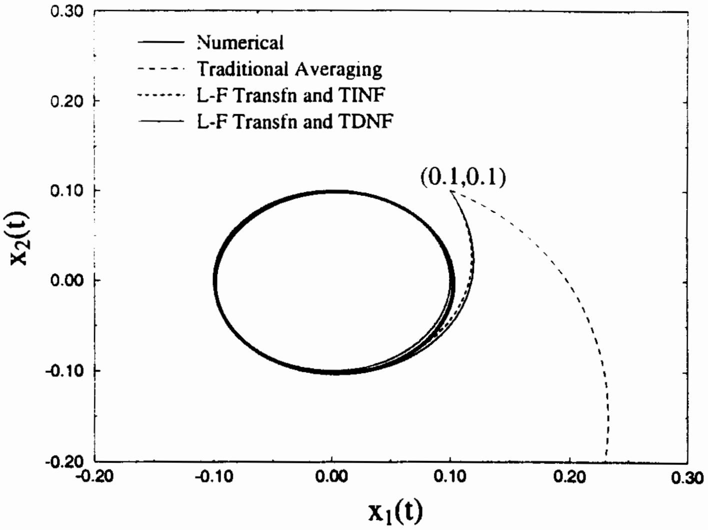

Fig. 2. Comparison of solution of commutative system in center manifold ($\alpha = 1.0$, $\varepsilon = 0.3$).

condition (23) is completely satisfied and hence the TDNF reduces this system to a linear form. This solution is periodic and compares well with the numerical result (cf. Fig. 2). Similarly, the TINF solution reaches a periodic orbit and stays there. The nature of the stability of this solution can easily be determined by constructing the center manifold for (30) in the form $z_2 = f(t)z_1^2$, where $f(t)$ is a periodic function similar to $p_i(t)$ (cf. Appendix A). Therefore, (30) can be decoupled and the solution be shown to be indeed a stable periodic orbit.

Example 2. Mathieu equation with cubic non-linearity

Consider the Mathieu equation with cubic non-linearity in the form

$$\ddot{x} + \delta\dot{x} + (\alpha + \beta\cos\omega t)x + \varepsilon x^3 = 0 \tag{35}$$

where δ, α, β, ω and ε are the parameters of the system. In the state space form the above equation is rewritten as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\alpha + \beta\cos\omega t) & -\delta \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -\varepsilon x_1^3 \end{Bmatrix} \tag{36}$$

where $\{\dot{x}_1, \dot{x}_2\}^T = \{x, \dot{x}\}^T$. Following the steps described in Section 3, the $2T$ periodic real L–F transformation matrix $Q(t)$ can be computed for a given parameter set. Applying the transformation $x = Q(t)z$, (36) is transformed to

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + Q^{-1}(t) \begin{Bmatrix} 0 \\ -\varepsilon(Q_{11}z_1 + Q_{12}z_2)^3 \end{Bmatrix} \tag{37}$$

where R_{ij} are the elements of the real matrix R and Q_{ij} are the elements of the L–F transformation matrix $Q(t)$. It is to be noted that the non-linear part of (37) can be expressed in Fourier form due to the periodic nature of the Liapunov–Floquet transformation. Equation (37) can be written in the canonical form as

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \varepsilon \begin{Bmatrix} f_{11}(t,\tau)y_1^3 + f_{12}(t,\tau)y_1^2 y_2 + f_{13}(t,\tau)y_1 y_2^2 + f_{14}(t,\tau)y_2^3 \\ f_{21}(t,\tau)y_1^3 + f_{22}(t,\tau)y_1^2 y_2 + f_{23}(t,\tau)y_1 y_2^2 + f_{24}(t,\tau)y_2^3 \end{Bmatrix} \tag{38}$$

where $\tau = 2T$, $\Lambda = \{\lambda_1, \lambda_2\}$ are the eigenvalues of R and the periodic coefficients $f_{ij}(t,\tau)$; $i, j = 1, \ldots, 4$ are expressed as

$$f_{ij}(t,\tau) = a_0^{ij} + \sum_{n=1}^{l} a_n^{ij}\cos\left(\frac{2\pi nt}{\tau}\right) + \sum_{n=1}^{l} b_n^{ij}\sin\left(\frac{2\pi nt}{\tau}\right). \tag{39}$$

After experimenting with various sets of system parameters, it was observed that $l = 15$ provided quite accurate representations of functions $f_{ij}(t, \tau)$. This has been reported earlier by Pandiyan *et al.* [22]. It is also consistent with the number of Fourier terms taken in the representation of the L–F transformation $Q(t)$.

In order to obtain a solution of equation (38) using TDNF, consider a near-identity non-linear transformation

$$
\begin{aligned}
y_1 &= u + g_{11}(t,\tau)u^3 + g_{12}(t,\tau)u^2v + g_{13}(t,\tau)uv^2 + g_{14}(t,\tau)v^3 \\
y_2 &= v + g_{21}(t,\tau)u^3 + g_{22}(t,\tau)uv^2 + g_{23}(t,\tau)uv^2 + g_{24}(t,\tau)v^3
\end{aligned}
\tag{40}
$$

where the periodic coefficients $g_{ij}(t,\tau)$; $i,j = 1\ldots, 4$ are once again of the form given by (39) but with unknown constants $\tilde{a}_n$ and $\tilde{b}_n$. Substituting (40) into (38) and solving the resulting homological equation as described earlier, the unknown constants $\tilde{a}_n$ and $\tilde{b}_n$ can be evaluated. In situations when there is no resonance, the Fourier series assumed for $g_{ij}(t,\tau)$ and its derivative are found to be convergent [26]. On the other hand, if resonance takes place, the unknown constants of the corresponding periodic coefficient cannot be determined and some non-linear terms remain even after the normal form reduction. As long as there is no resonance, all non-linear terms are eliminated and the reduced normal form is just the linear part of (38). Therefore, the solution of the non-linear Mathieu equation in the original coordinates can be obtained by substituting back all the intermediate transformations. Even when some of the non-linear terms remain due to resonance, the resulting equation can still be used to provide many useful conclusions about the stability and dynamical behavior of the system. Such procedures are described by Bruno [29] and Hale and Kocak [31].

In order to obtain an approximate solution of (38) via the TINF theory, the variations of the periodic coefficients of non-linear terms are neglected in comparison with their predominant means. This approximation results in an equation of the form

$$
\begin{Bmatrix} \dot{\tilde{y}}_1 \\ \dot{\tilde{y}}_2 \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{Bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{Bmatrix} + \begin{Bmatrix} a_3\tilde{y}_1^3 + a_4\tilde{y}_1^2\tilde{y}_2 + a_5\tilde{y}_1\tilde{y}_2^2 + a_6\tilde{y}_2^3 \\ b_3\tilde{y}_1^3 + b_4\tilde{y}_1^2\tilde{y}_2 + b_5\tilde{y}_1\tilde{y}_2^2 + b_6\tilde{y}_2^3 \end{Bmatrix}
\tag{41}
$$

where $\tilde{y}_1$ and $\tilde{y}_2$ are the approximate states and $a_j, b_j, j = 3,\ldots,6$ are the means of the periodic coefficients of (38). To apply the TINF theory, consider a non-linear near-identity transformation of the form

$$
\begin{aligned}
\tilde{y}_1 &= u + p_1u^3 + p_2u^2v + p_3uv^2 + p_4v^3 \\
\tilde{y}_2 &= v + q_1u^3 + q_2u^2v + q_3uv^2 + q_4v^3
\end{aligned}
\tag{42}
$$

where p_i and q_i are unknown constants. Substituting (42) into (41) and solving the resulting autonomous homological equation, (41) can be reduced to a linear form in most of the situations except when resonances due to non-linearity occur. The approximate solutions for this case can also be obtained in a fashion similar to the procedure discussed above. It is also observed that as long as the system has eigenvalues that are distinct with negative or positive real parts, one could reduce the system to a linear form completely.

At this point an application of the traditional averaging procedure to (35) is briefly discussed. Assuming a generating solution of the form (cf. ref. [4])

$$
x(t) = z_1(t)\cos\omega_0 t + \frac{z_2(t)}{\omega_0}\sin\omega_0 t
\tag{43}
$$

where $\omega_0^2 = \alpha$ and z_i, $i = 1, 2$ are the slowly varying coefficients of the solution. Using this solution in (35) and averaging over the principal period $2\pi/\omega$, one gets a set of quasilinear autonomous differential equations with cubic non-linearity in the averaged coefficients $\bar{z}$ as

$$
\begin{Bmatrix} \dot{\bar{z}}_1 \\ \dot{\bar{z}}_2 \end{Bmatrix} = \begin{bmatrix} R_{11}^0 & R_{12}^0 \\ R_{21}^0 & R_{22}^0 \end{bmatrix} \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{Bmatrix} + \varepsilon \begin{Bmatrix} \bar{a}_3\bar{z}_1^3 + \bar{a}_4\bar{z}_1^2\bar{z}_2 + \bar{a}_5\bar{z}_1\bar{z}_2^2 + \bar{a}_6\bar{z}_2^3 \\ \bar{b}_3\bar{z}_1^3 + \bar{b}_4\bar{z}_1^2\bar{z}_2 + \bar{b}_5\bar{z}_1\bar{z}_2^2 + \bar{b}_6\bar{z}_2^3 \end{Bmatrix}
\tag{44}
$$

where $\bar{a}_i$ and $\bar{b}_i$, $i = 3,\ldots, 6$ are the constants depending on the parameters of the system and R_{ij}^0 are the elements of the constant matrix R^0. However, the eigenvalues of the matrix R^0

will be quite different from those appearing in (41). The solution of the above equation can be discussed via the TINF theory.

Case studies

In the following, several case studies are presented by selecting various sets of parameters α, β, δ and ε in the Mathieu equation given by (35). The parameter ω is selected as 2π in all cases of this study.

Case 1: equation with eigenvalues in stable manifold ($\alpha \neq 0$):
Parameter set 1: $\alpha = 10.0$; $\beta = 0.2$; $\delta = 1.8974$; $\varepsilon = 0.3$

Note that the parameters β and ε multiplying the periodic and the non-linear terms, respectively, are selected small. The suggested approaches including the *traditional averaging* are applied to this set and the results, along with the numerical solution, are presented in Fig. 3. It is understandable that all the methods predict the behavior of the system correctly due to the smallness of the parameters β as well as ε. For brevity, the coefficients of near-identity transformations for the normal forms are not recorded in this paper. The details of the Fourier coefficients $g_{ij}(t, \tau)$ used in the near-identity transformations of the TDNF [cf. equation (40)] and the constant coefficients p_i and q_i, $i = 1, \ldots, 4$ [cf. equation (42)] used in the near-identity transformations of the TINF for this case can be found in [32].

Parameter set 2: $\alpha = 0.5$; $\beta = 4.0$; $\delta = 0.4243$; $\varepsilon = 0.3$

For this case β, the parameter multiplying the periodic term is selected to be eight times larger than α. From Fig. 4, it is observed that except for the averaging method, all other techniques predict similar trajectories which finally approach the fixed point (0, 0).

Case 2: *equation without generating solution* ($\alpha = 0$)

Note that when $\alpha = 0$ the fundamental frequency ω_0 of the autonomous part of (35) is zero and without a generating solution the averaging approach cannot be applied.

Parameter set 1: $\alpha = 0$; $\beta = 4.0$; $\delta = 0.4243$; $\varepsilon = 0.3$

For this set, the solutions based on TDNF and TINF theories have been found to predict the dynamic behavior quite precisely in comparison with the numerical solution, as shown in Fig. 5.

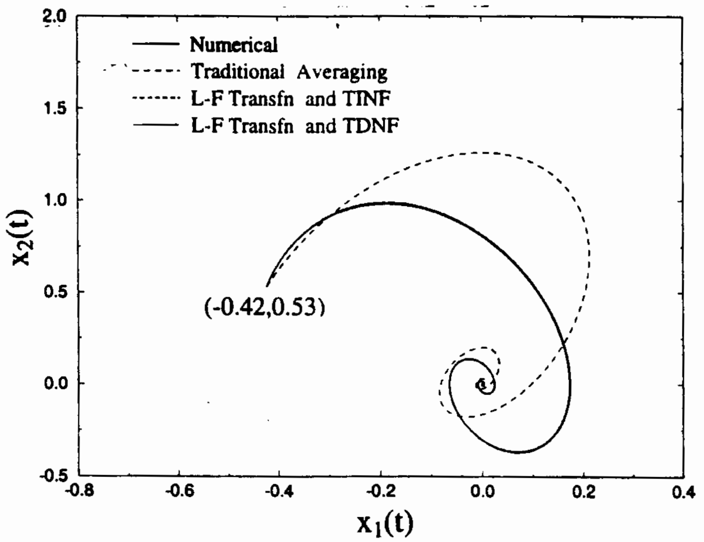

Fig. 3. Comparison of solution of Mathieu's equation with cubic non-linearity ($\alpha = 10.0$; $\beta = 0.2$; $\delta = 1.8974$; $\varepsilon = 0.3$).

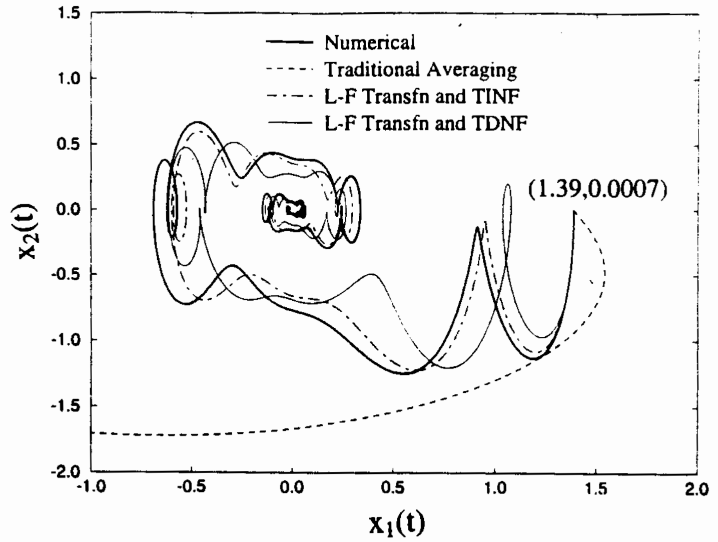

Fig. 4. Comparison of solution of Mathieu's equation with cubic non-linearity ($\alpha = 0.5$; $\beta = 4.0$; $\delta = 0.4243$; $\varepsilon = 0.3$).

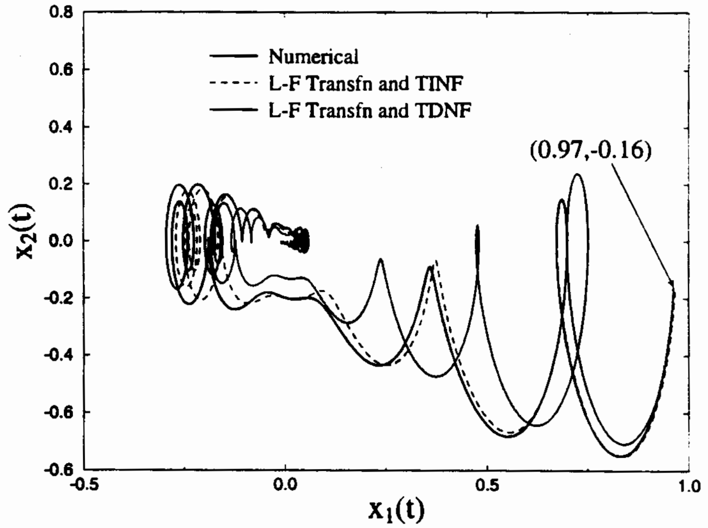

Fig. 5. Comparison of solution of Mathieu's equation with cubic non-linearity ($\alpha = 0$; $\beta = 4.0$; $\delta = 0.4243$; $\varepsilon = 0.3$).

Parameter set 2: $\alpha = 0$; $\beta = 4.0$; $\delta = 0.4243$; $\varepsilon = 3.0$

This set shows that the behavior of the system is well predicted by the suggested techniques when compared with the numerical solution even though the non-linearity parameter ε is 10 times larger than the value used in set 1. The comparison is shown in Fig. 6.

The coefficients of the near-identity transformations for the TDNF as well as the TINF methods are once again omitted for brevity.

Case 3: *equation with eigenvalues in center manifold*

The application of the suggested techniques to the special case when the resonance condition prevails due to the presence of a pair of purely imaginary roots is shown here. For this case, by applying normal forms, all the non-resonant terms are annihilated; however, homological equations corresponding to the resonant terms cannot be resolved. Therefore, some of the non-linear terms of third degree stay in the reduced equation and hence, in general, a closed form solution is not possible. Under such circumstances it may be possible

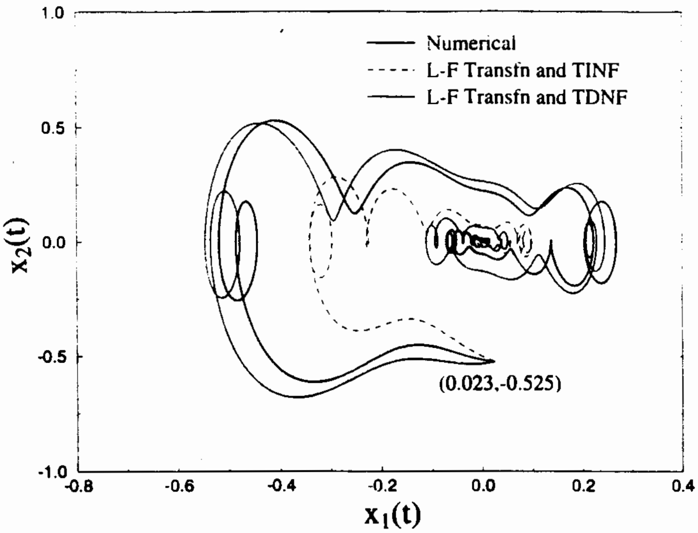

Fig. 6. Comparison of solution of Mathieu's equation with cubic non-linearity ($\alpha = 0$; $\beta = 4.0$; $\delta = 0.4243$; $\varepsilon = 3.0$).

to neglect the periodic variations of the non-linear coefficients and still retain the stability behavior of the system. In the following, a case study is provided to demonstrate the applications in similar situations.

Parameter set: $\alpha = 4.0$; $\beta = 7.468$; $\delta = 0$; $\varepsilon = 0.3$

For this set of parameters, applications of the L–F transformation, the Jordan canonical transformation and the near-identity transformation [cf. equation (40)] to (36) results in an equation of the form

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} -i\omega_0 & 0 \\ 0 & i\omega_0 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} + \varepsilon \begin{Bmatrix} f_{12}(t,\tau)u^2 v \\ f_{23}(t,\tau)uv^2 \end{Bmatrix} \tag{45}$$

where $f_{12}(t,\tau)$ and $f_{23}(t,\tau)$ are complexified periodic functions corresponding to the resonant terms (as shown in Appendix B) and $\pm i\omega_0 = \pm i0.816$ are the eigenvalues of the system. Multiplying $\dot{u}$ by v and $\dot{v}$ by u and adding, a linear differential equation in (uv) can be obtained. Therefore, the analytical solution of equation (45) can be found in (uv), which is of the form

$$uv = -1\bigg/\int_0^t a(\chi,\tau)\mathrm{d}\chi \tag{46}$$

where $a(t,\tau)$ is a complex Fourier function. It has been shown by Bruno [29] that this kind of situation corresponds to a non-structurally stable focus or a center whose existence and character is totally determined by the non-linear terms of the normal form. However, as pointed out earlier, the stability of equation (45) can be characterized from the autonomous part alone. Therefore, dropping all the periodic terms in $f_{12}(t,\tau)$ and $f_{23}(t,\tau)$ and using the transformation $u = r\cos\theta$, $v = r\sin\theta$, one obtains

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{-0.0125r^3\cos^3\theta\sin\theta + ir(\omega_0\cos 2\theta - 0.0125r^2\cos\theta\sin^3\theta)}{\dfrac{-i\omega_0}{2}\sin 2\theta}. \tag{47}$$

The above equation is of the form

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = rf_1(\theta) + r^3 f_2(\theta) \tag{48}$$

and only the mean value of $f_1(\theta)$ over period T decides the behavior of the equilibrium point [29]. In the present case the mean value is zero and hence r is found to be a constant. Thus

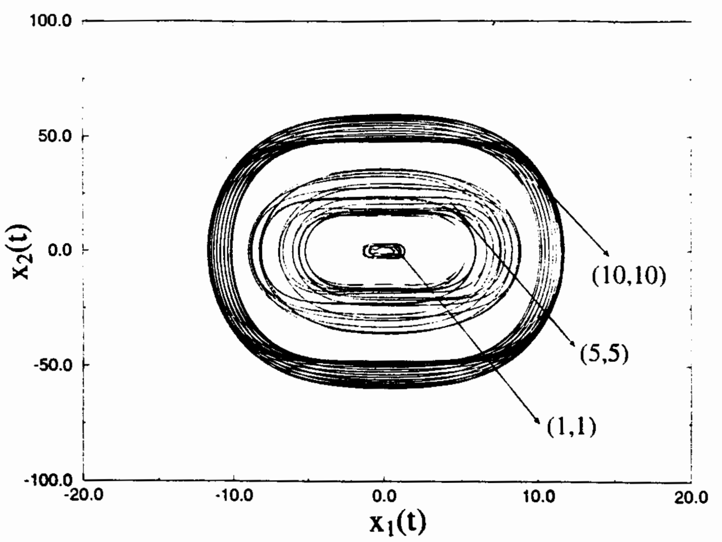

Fig. 7. Phase plot of Mathieu's equation in center manifold ($\alpha = 4.0$; $\beta = 7.468$; $\delta = 0$; $\varepsilon = 0.3$).

the fixed point (0, 0) is a center which agrees with the numerical results of the original Mathieu equation, as shown in Fig. 7.

A similar conclusion can also be drawn without reducing the equation to the normal form, as shown in the following. After the application of the L–F transformation to (36) and neglecting the periodic variations of the coefficients, one obtains

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{bmatrix} 0 & -\beta_1 \\ \beta_1 & 0 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \begin{Bmatrix} 0.0245 y_1^2 y_2 + 0.00587 y_2^3 \\ -0.0995 y_1^3 - 0.0245 y_1 y_2^2 \end{Bmatrix} \tag{49}$$

where $\beta_1 = 0.816$. It is not possible to reduce (49) any further in the real domain and this form itself is the normal form for the case considered here. Using $y_1 = r\cos\theta$ and $y_2 = r\sin\theta$, (49) transforms to

$$\frac{dr}{d\theta} = \frac{-(0.0234\sin 2\theta + 0.0069\sin 4\theta)r^3}{0.816 - 0.013r^2\sin^2 2\theta - r^2(0.00567\sin^4\theta + 0.0995\cos^4\theta)}. \tag{50}$$

At this point, one can use the transformation

$$r = \rho + a(\theta)\rho^3 \tag{51}$$

to obtain

$$\frac{d\rho}{d\theta} = -\rho^3\left[0.0234\sin(2\theta) + 0.0069\sin(4\theta) + \frac{da(\theta)}{d\theta}\right] + O(|\rho^4|). \tag{52}$$

Selecting

$$\frac{da(\theta)}{d\theta} = -0.0234\sin(2\theta) - 0.0069\sin(4\theta), \tag{53}$$

$dr/d\theta$ is identically transformed to $d\rho/d\theta = 0$. Then using Lemma 11.3 together with the *Center Theorem of Liapunov*, as given in [31], one can see that the fixed point (0, 0) is a center and every solution in this neighborhood is periodic. Solving (49) for other fixed points, one finds two more fixed points, namely (0, 11.8) and (0, −11.8) and these points are also found to have the same behavior as that of the fixed point (0, 0). The global behavior of (49) is studied via numerical integration, including all these fixed points and the results are shown in Fig. 8. When these numerical solutions are transformed back to the original coordinates via the L–F transformation, it is found that the solutions indeed behave like a center, as indicated by Fig. 9. Considering the approximate nature of this analysis, when compared to the corresponding numerical solution in the original coordinates (cf. Fig. 7), it is observed that the dynamic patterns match remarkably well.

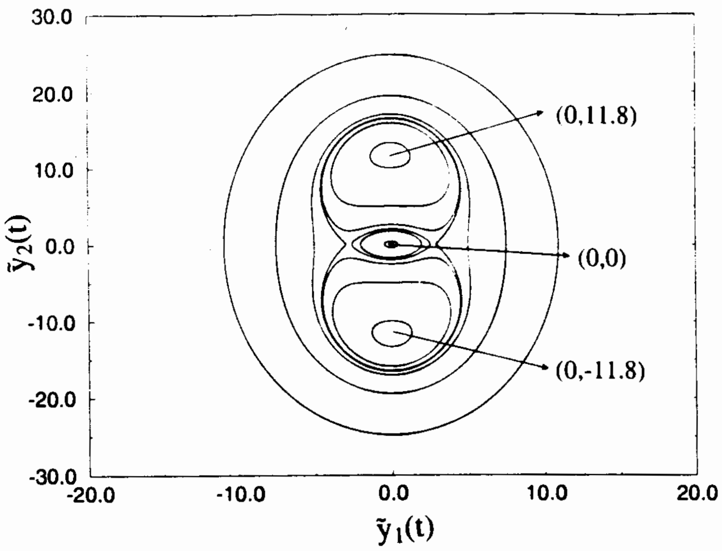

Fig. 8. Global dynamics in the transformed plane ($\alpha = 4.0$; $\beta = 7.468$; $\delta = 0$; $\varepsilon = 0.3$).

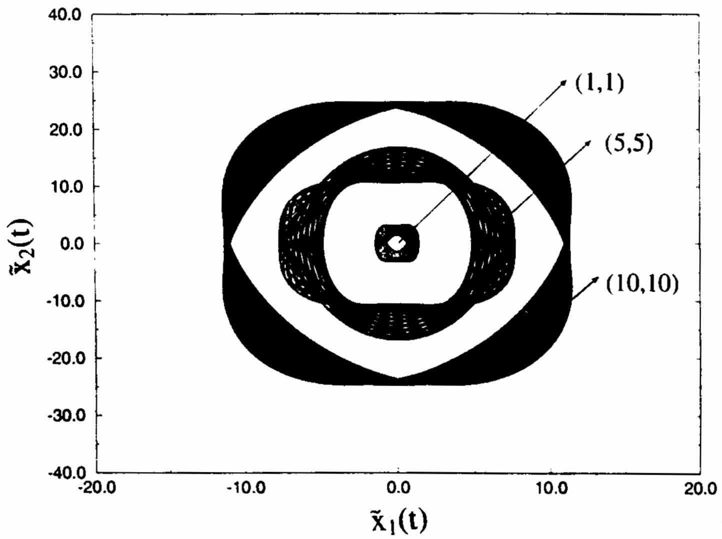

Fig. 9. Phase plot of Mathieu's equation in center manifold ($\alpha = 4.0$; $\beta = 7.468$; $\delta = 0$; $\varepsilon = 0.3$).

6. CONCLUSIONS

In this paper, new methods for the analysis of general quasilinear dynamic systems with periodically varying coefficients are developed. The strategy is based on the fact that the L–F transformation matrices can be used to transform all such systems into a similar forms whose linear parts are time-invariant. The resulting dynamical equations are then analyzed using the *normal form theory*.

The L–F transformation matrices have been computed in terms of shifted Chebyshev polynomials. Such computations are byproducts of state transition matrices obtained earlier in references [9–11, 21–23]. On the basis of the convergence study reported by Joseph *et al.* [11] and our experience with the various case studies reported here, a 15 term Chebyshev polynomial representation of the L–F transformation matrices provides excellent accuracy. This transformation converts the linear part of the quasilinear periodic dynamical equation to a time-invariant form regardless of the magnitude of the parameters associated with the periodic matrix. Further, the analysis is carried out through an application of *time-dependent normal form theory*. Simplified results using the *autonomous normal form theory* have also been recorded. The results obtained from the *traditional* *averaging* method and a Runge–Kutta numerical scheme have also been reported for comparison.

From the examples studied in Section 5, the proposed methods provide extremely convergent solutions and correct behaviors of the dynamics in the stable manifold when compared to the numerical results. When the dynamics are in the center manifold, the normal form reduction still retains the original dynamic characteristics and the stability behavior can be studied analytically. It is to be noted that the averaging technique provides comparable results only for those cases when both the periodic matrix and the non-linearity contain a small parameter. In many instances, it is found that the solution obtained via the proposed methods matches the numerical solutions even when the parameters multiplying the non-linear terms are relatively large (cf. Fig. 6). It is also shown that for some cases, like the commutative systems, the present methods can even provide closed form symbolic solutions as well. It is observed that the solutions obtained via *time-dependent normal forms* show slower behavior than those obtained from *time-independent normal forms*; however, both solutions are found to be slower than the numerical solutions. Nevertheless, they depict the correct behavior of the system in stable, center and unstable manifolds. The reason for such a slow behavior can be attributed to different pointing directions of the averaged drift of such methods and the projection of the true motion [26].

To conclude, the authors would like to state that for the first time the computation and application of Liapunov–Floquet transformation in the analysis of non-linear periodic systems has been demonstrated. It has been shown that the proposed analysis techniques are virtually free from the small parameter limitations and the question of the existence of the so-called "*generating solution*" does not arise, which is certainly not the case with the classical methods such as averaging, perturbation, etc. It is anticipated that the suggested analysis techniques would serve as efficient tools in the study of bifurcations of non-linear systems with periodically varying coefficients. As demonstrated in [25], the controllers for periodically varying systems can be effectively designed using the time-invariant methods through the application of the Liapunov–Floquet transformation. This idea can be easily extended to the control of non-linear systems with periodic coefficients following the developments outlined in this study. These investigations will be reported in due course.

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REFERENCES

1. G. Floquet, Sur les equations differentials lineaires a coefficients periodiques. *Annales Scientifiques de l'Ecole Normale Superieue* Series 2, Vol. 12 (1883).
2. V. A. Yakubovich and V. M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients*, Parts I and II. John Wiley, New York (1975).
3. K. G. Lindh and P. W. Likins, Infinite determinant methods for stability analysis of periodic-coefficient differential equations. *AIAA J.* **8**, 680–686 (1970).
4. J. A. Sanders and F. Verhulst, *Averaging Methods in Nonlinear Dynamical Systems*. Springer, New York (1985).
5. A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*. John Wiley, New York (1979).
6. E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*. McGraw-Hill, New York (1955).
7. D. A. Peters and K. H. Hohenemser, Application of the Floquet transition matrix to problems of lifting rotor stability. *J. Am. Helicopter Soc.* **16**, 25–33 (1971).
8. P. Friedmann, C. E. Hammond and T. H. Woo, Efficient numerical treatment of periodic systems with application to stability problems. *Int. J. Num. Meth. Engng* **11**, 1117–1136 (1977).
9. S. C. Sinha and D.-H. Wu, An efficient computational scheme for the analysis of periodic systems. *J. Sound Vibr.* **151**, 91–117 (1991).
10. S. C. Sinha, D.-H. Wu, V. Juneja and P. Joseph, Analysis of dynamic systems with periodically varying parameters via Chebyshev polynomials. *J. Vibration Acoust., Trans. ASME* **115**, 96–102 (1993).
11. P. Joseph, R. Pandiyan and S. C. Sinha, Optimal control of mechanical systems subjected to periodic loading via Chebyshev polynomials. *Optimal Control Applic. Methods* **14**, 75–90 (1993).
12. A. M. Liapunov, Sur une serie relative a la theorie des equations differentielles lineaires a coefficients periodiques. *Compt. Rend.* **123**, 1248–1252 (1896).
13. H. Poincaré, *Les Methodes Nouvelles de la Mecanique Celeste* (3 Vols). Gauthier-Villars, Paris (1899).
14. R. Seydel, Numerical computation of periodic orbits that bifurcate from stationary solutions of ordinary differential equation. *Appl. Math. Comput.* **9**, 257–271 (1981).
15. J. Bernussou, *Point Mapping Stability*. Pergamon Press, Oxford (1977).
16. R. S. Guttalu and H. Flashner, Analysis of dynamical systems by truncated point mappings and cell mapping. In *Nonlinear Dynamics in Engineering Systems*, IUTAM Symposium, Stuttgart, Germany, 1989. Springer, Berlin (1990).

17. F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*. Springer, Berlin (1990).
18. R. Bellman, *Nonlinear Analysis* (Vol. 1). Academic Press, New York (1970).
19. M.-Y. Wu, Transformation of a linear time-varying system into a linear time-invariant system. *Int. J. Control* **27**, 589–602 (1978).
20. D. L. Lukes, *Differential Equations: Classical to Controlled*. Academic Press, New York (1982).
21. S. C. Sinha and V. Juneja, An approximate analytical solution for systems with periodic coefficients via symbolic computation. *AIAA/ASME/ASCE/AHS/ASC 32nd Structures, Structural Dynamics and Materials Conference*, 8–10 April 1990. A Collection of Papers, Part I, pp. 790–797 (1991).
22. R. Pandiyan, J. S. Bibb and S. C. Sinha, Liapunov–Floquet transformation: computation and application to periodic systems. In *Proceedings of the Symposium on Dynamics and Vibration of Time-varying Systems and Structures*, S. C. Sinha and R. M. Evan-Iwanowski (eds), 14th Biennial Conference on Mechanical Vibration and Noise, Albuquerque, 19–22 September, Vol. DE-56, pp. 337–348 (1993).
23. D.-H. Wu and S. C. Sinha, A new approach in the linear analysis of systems with periodic coefficients for applications in the rotorcraft dynamics. *Aeronaut. J.* **98** (971), 9–16 (1994).
24. D. K. Arrowsmith and C. M. Place, *An Introduction to Dynamical Systems*. Cambridge University Press, Cambridge (1990).
25. S. C. Sinha and P. Joseph, Control of general dynamic systems with periodically varying parameters via Liapunov–Floquet transformation. *J. Dynamic Syst., Measurement, Control, Trans. ASME* (to appear) (1994).
26. V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer, New York (1988).
27. S. Rosenblat and D. S. Cohen, Periodically perturbed bifurcation—I. Simple bifurcation. *Studies appl. Mathematics* **63**, 1–23 (1980).
28. S. Rosenblat and D. S. Cohen, Periodically perturbed bifurcation—II. Hopf bifurcation. *Studies appl. Mathematics* **65**, 95–112 (1981).
29. A. D. Bruno, *Local Methods in Nonlinear Differential Equations*. Springer, Berlin (1989).
30. R. H. Mohler, *Nonlinear Systems, Volume 1: Dynamics and Control*. Prentice-Hall, New York (1991).
31. J. Hale and H. Kocak, *Dynamics and Bifurcations*. Springer, New York (1991).
32. S. C. Sinha and R. Pandiyan, Analysis of quasilinear dynamical systems with periodic coefficients via Liapunov–Floquet transformation. Report No. SD93-102, Nonlinear Systems Research Laboratory, Department of Mechanical Engineering, Auburn, University, Auburn U.S.A. (1993).

APPENDIX A

Coefficients of near-identity transformations of time-dependent normal form for the commutative system:

$$p_1(t) = \left[\frac{0.75\alpha\varepsilon}{2 - 2\alpha + \alpha^2}\right]\cos t + \left[\frac{-\varepsilon(1 - 0.25\alpha)}{10 - 2\alpha + \alpha^2}\right]\cos 3t + \left[\frac{0.75\varepsilon(2 - \alpha)}{2 - 2\alpha + \alpha^2}\right]\sin t + \left[\frac{0.25\varepsilon(2 + \alpha)}{10 - 2\alpha + \alpha^2}\right]\sin 3t$$

$$p_2(t) = \varepsilon[-0.5\cos t - 0.1\cos 3t - 0.2\sin 3t]$$

$$p_3(t) = \left[\frac{0.25\alpha\varepsilon}{2 + 2\alpha + \alpha^2}\right]\cos t + \left[\frac{\varepsilon(1 + 0.25\alpha)}{10 + 2\alpha + \alpha^2}\right]\cos 3t + \left[\frac{0.25\varepsilon(2 + \alpha)}{2 + 2\alpha + \alpha^2}\right]\sin t + \left[\frac{-0.25\varepsilon(2 - \alpha)}{10 + 2\alpha + \alpha^2}\right]\sin 3t$$

and

$$q_1(t) = \left[\frac{-0.5\varepsilon(1 - \alpha)}{2 - 4\alpha + 4\alpha^2}\right]\cos t + \left[\frac{-0.5\varepsilon(1 + \alpha)}{10 - 4\alpha + 4\alpha^2}\right]\cos 3t + \left[\frac{0.5\alpha\varepsilon}{2 - 4\alpha - 4\alpha^2}\right]\sin t + \left[\frac{0.5\varepsilon(1 + \alpha)}{10 - 4\alpha + 4\alpha^2}\right]\sin 3t$$

$$q_2(t) = \left[\frac{0.5\varepsilon(2 - \alpha)}{2 - 2\alpha + \alpha^2}\right]\cos t + 0.167\varepsilon\cos 3t + \left[\frac{0.5\alpha\varepsilon}{2 - 2\alpha + \alpha^2}\right]\sin t - 0.056\varepsilon(2 + \alpha)\sin 3t$$

$$q_3(t) = \varepsilon[-0.75\cos t + 0.05\cos 3t + 0.1\sin 3t].$$

APPENDIX B

The non-linear coefficients $f_{12}(t, \tau)$ and $f_{23}(t, \tau)$ of equation (45). In the following, $\rho = \pi t$, $C_{n\rho} \equiv \text{Cosine}[n\rho]$, $S_{n\rho} \equiv \text{Sine}[n\rho]$ and $i = \sqrt{(-1)}$:

$$\begin{aligned} f_{12}(t, \tau) = \{ &-0.01248 - 0.00271C_{4\rho} + 0.01222C_{8\rho} + 0.00269C_{12\rho} - 0.01594S_{\rho} - 0.00475S_{3\rho} \\ &- 0.00725S_{5\rho} - 0.0105S_{7\rho} + 0.00548S_{9\rho} - 0.00048S_{11\rho} + 0.00214S_{13\rho} \\ &+ 0.00061S_{15\rho}\} + i*\{0.02215C_{\rho} + 0.04170C_{3\rho} - 0.01839C_{5\rho} + 0.01488C_{7\rho} \\ &- 0.02128C_{9\rho} - 0.00425C_{11\rho} - 0.00791C_{13\rho} - 0.00388C_{15\rho} + 0.04838S_{4\rho} \\ &+ 0.03152S_{8\rho} + 0.00550S_{12\rho}\} \end{aligned}$$

$$\begin{aligned} f_{23}(t, \tau) = i*\{ &-0.01248 - 0.00271C_{4\rho} + 0.01222C_{8\rho} + 0.00269C_{12\rho} - 0.01595S_{\rho} \\ &- 0.00475S_{3\rho} - 0.00725S_{5\rho} - 0.01053S_{7\rho} + 0.00548S_{9\rho} - 0.00048S_{11\rho} \\ &+ 0.00214S_{13\rho} + 0.00061S_{15\rho}\} + \{-0.00349C_{\rho} - 0.00829C_{3\rho} + 0.00840C_{5\rho} \\ &+ 0.00495C_{7\rho} - 0.00113C_{9\rho} + 0.00076C_{11\rho} - 0.00073C_{13\rho} - 0.00009C_{15\rho} \\ &- 0.01265S_{4\rho} + 0.00409S_{8\rho} + 0.00128S_{12\rho}\}. \end{aligned}$$