Analysis of Dynamic Systems With Periodically Varying Parameters Via Chebyshev Polynomials

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In this paper a general method for the analysis of multidimensional second-order dynamic systems with periodically varying parameters is presented. The state vector and the periodic matrices appearing in the equations are expanded in Chebyshev polynomials over the principal period and the original differential problem is reduced to a set of linear algebraic equations. The technique is suitable for constructing either numerical or approximate analytical solutions. As an illustrative example, approximate analytical expressions for the Floquet characteristic exponents of Mathieu's equation are obtained. Stability charts are drawn to compare the results of the proposed method with those obtained by Runge-Kutta and perturbation methods. Numerical solutions for the flap-lag motion of a three-bladed helicopter rotor are constructed in the next example. The numerical accuracy and efficiency of the proposed technique is compared with standard numerical codes based on Runge-Kutta, Adams-Moulton, and Gear algorithms. The results obtained in both the examples indicate that the suggested approach is extremely accurate and is by far the most efficient one.

1 Introduction

The study of systems governed by a set of ordinary linear or nonlinear differential equations with period coefficients is of great importance in diverse branches of science and engineering. The investigation of stability and response prediction are the two most significant dynamic problems associated with such systems. For stability analysis, one may linearize the equations of motion for small perturbations about a periodic equilibrium position. Linearization of perturbed equations results in a set of ordinary linear differential equations with periodic coefficients. In the past, several methods have been used to investigate the stability of systems with periodic coefficients. These include Perturbation method (Bolotin, 1964; Johnson, 1972; Johnson, 1974; Jordan and Smith, 1977; Nayfeh, 1973; Yakubovitch and Starzhinski, 1975), Hill's method (Bolotin, 1964; Brockett, 1970; Yakubovitch and Starzinski, 1975), Averaging method (Yakubovitch and Starzhinski, 1974; Bolotin, 1964); and Floquet theory with numerical integration (Friedmann et al., 1977; Gaonkar et al., 1981; Gockel, 1972; Hsu and Cheng, 1973; Hsu, 1974; Peters and Hohenemser, 1971; Sinha et al., 1979).

Perturbation method is applicable only if the parameter multiplying the periodic terms is small. The solution vector is expanded as a power series in terms of the small parameter and substituted into the original equation. Equating the coefficients of powers of the small parameter, a set of differential equations is obtained. These differential equations are solved to obtain the desired solution. The modifications of this method include Whitaker's method and the method of multiple scales. But since the condition of the small parameter exists, it limits the applicability of the method. The method requires more knowledge of the system behavior and each region of the system requires a separate analysis. The Averaging method is also restricted to systems with small parameters.

Hill's infinite determinant method is one of the oldest methods, used to compute the transition curve or the boundary. It determines the boundary between the stable and the unstable regions in the parameter space without actually calculating the response vector. To get the combination of the parameter values which correspond to the transition point, the solution is expanded in the form of Fourier series and substituted into the dynamical equation. By equating the coefficients of each trigonometric term equal to zero, infinite number of linear algebraic equations are obtained. The determinant of their coefficients is zero only at transition points. Solving the infinite determinant precisely yields the transition point. In practice a truncated Fourier series is taken instead of the infinite series and the first estimate of the transition point is found. The number of terms is then increased and the new transition point is found. This procedure is repeated several times to find the limit point of the series. The limit point corresponds to the actual transition point. This is a very time consuming process and it does not yield the response vector at any arbitrary point in the parameter space. Further Hill's method is not very convenient of digital computation if one has to deal with systems having large number of degrees of freedom.

Many authors (D'Angelo, 1970; Hsu, 1974; Richards, 1983; Sinha et al., 1979) have tried to determine the stability and response from an approximate system of equations which are

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usually obtained by replacing the elements of periodic coefficients matrix by piecewise constants or linear functions. In a practical application, one can approximate the periodic matrix at most by a series of step functions and compute the transition matrix during one period, which then yields the stability condition, etc. Such a technique was employed by Friedmann et al. (1977) for a numerical evaluation of the transition matrix. Although the method is straightforward, it is only a second order algorithm at the most. For more accurate solutions, it is necessary to apply higher order numerical schemes such as the Runge-Kutta-Gill method or similar algorithms and utilize Floquet theory to establish stability conditions. This approach has been adopted by several authors (Friedmann et al., 1977; Gaonkar et al., 1981; Gockel, 1972) in a variety of stability and response problems. However, since n integration passes are required in the computation of transition matrix for an $n \times n$ system, this approach tends to be expensive in terms of computer time as n becomes large. Therefore a modified fourth order method was suggested in Friedmann et al. (1977), which requires only a single integration pass and thereby reduces the computation time by a reasonable amount. A recent study by Gaonkar et al. (1981) shows that the Hamming's fourth-order predictor-corrector method in a single-pass scheme is a very economical approach.

Very recently, Sinha and Wu (1991) presented a new numerical scheme for the computation of Floquet Transition Matrix associated with a class of periodic systems. In this approach, the solution vector is expanded in terms of the shifted Chebyshev polynomials. The attractive feature of this technique is that it reduces the original differential system to a system of linear algebraic equations from which the solutions in the interval of one period can be obtained very easily. In this paper the technique has been generalized to dynamic systems described by a set of second-order equations with periodically varying mass, damping, and stiffness matrices. Two illustrative examples are presented.

2 Brief Review

2.1 Floquet Theory. Consider a set of linear homogeneous differential equations of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \tag{1}$$

where $\mathbf{A}(t)$ is an $n \times n$ matrix of periodic functions of period T and $\mathbf{x}(t)$ is an $n \times 1$ response vector. According to Floquet Theory (Kaplan, 1962; Richards, 1983), there exist n constants $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ (complex, in general), such that, Eq. (1) has n linearly independent solutions of the form

$$x_i(t) = \exp(\alpha_i t)\nu_i(t) \quad ; i = 1,2,\ldots n \tag{2}$$

where $\nu_i(t)$ is periodic (period T), if $\exp(\alpha_i t)$'s are distinct, else it is a product of a periodic function and a polynomial in time t. The degree of the polynomial in t depends upon the multiplicity of $\exp(\alpha_i t)$. The system is unstable, if there exists an α_i such that,

$$|\exp(\alpha_i T)| = |\lambda_i| > 1 \tag{3}$$

λ_i's are known as the characteristic multipliers and α_i's are known as the characteristic exponents of the system. The solution of the linear Eq. (1) can also be written as

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(t_0). \tag{4}$$

$\Phi(t)$ is commonly known as the State Transition Matrix (STM). It can be shown that λ_i's are the eigenvalues of the Floquet Transition Matrix (FTM), $\Phi(T)$.

2.2 Chebyshev Polynomials of the First Kind. The Chebyshev Polynomials of the first kind are a sequence of orthogonal polynomials over the interval $[-1, 1]$ with respect to the weight function $(1-t^2)^{-1/2}$ and are given by (Fox and Parker, 1968; Luke, 1969)

$$T_n(t) = ((-1)^n 2^n n!/(2n)!)(1-t^2)^{1/2}(d^n/dt^n(1-t^2)^{n-1/2}), \quad n = 0, 1, 2, 3, \ldots \tag{5}$$

The shifted Chebyshev Polynomials of the first kind are orthogonal in the interval $[0, 1]$ with respect to the weight function $w(t) = (t-t^2)^{-1/2}$ and are given by

$$T_n^*(t) = T_n(2t-1); \quad t \in [0, 1]. \tag{6}$$

Any continuous function can be expanded in terms of these polynomials as

Nomenclature

- a, m = parameters in Eq. (22)
- a_n = coefficients in the expansion of known functions of t in terms of Chebyshev polynomials
- $\mathbf{A}(t)$ = $n \times n$ matrix of periodic functions
- $\mathbf{b}_p$ = $nq \times 1$ vector of unknown coefficients of Chebyshev polynomials
- $\mathbf{C}$ = $n \times n$ constant damping matrix
- $\mathbf{C}^*(t)$ = $n \times n$ periodic damping matrix
- $\mathbf{C}_T$ = thrust coefficient
- $\mathbf{C}_{d0}$ = blade profile drag coefficient
- D = the trace of $\Phi(1)$
- F = dimensionless flapping frequency
- $\mathbf{G}$ = $q \times q$ integration matrix
- J_n = Bessel functions of the first kind
- $\mathbf{K}$ = $n \times n$ constant stiffness matrix
- $\mathbf{K}^*(t)$ = $n \times n$ periodic stiffness matrix
- $\mathbf{M}$ = $n \times n$ constant mass matrix
- $\mathbf{M}^*(t)$ = $n \times n$ periodic mass matrix
- q = number of terms in the Chebyshev expansion
- $\mathbf{Q}$ = $q \times q$ operational matrix of product
- R = elastic coupling parameter
- $\mathbf{s}(t)$ = $q \times 1$ vector of Chebyshev polynomials
- $\mathbf{S}(t)$ = $nq \times n$ matrix defined by $\mathbf{I} \otimes \mathbf{s}^T(t)$
- t = time
- T = the period of $\mathbf{A}(t)$
- $T_n(t)$ = Chebyshev polynomials of the first kind
- $T_n^*(t)$ = shifted Chebyshev polynomials of the first kind
- $w(t)$ = the appropriate weight function
- $\mathbf{x}(t)$ = $n \times 1$ response vector, in the state space form
- $\dot{\mathbf{x}}(t)$ = $n \times 1$ velocity vector
- $\mathbf{y}(t)$ = $n \times 1$ the response vector
- Z = stiffness parameter
- α_i = characteristic exponents
- β, θ, ζ = the flapping angle, pitch angle, and lag angle
- $\beta_e, \theta_e, \zeta_e$ = the steady-state value of flapping angle, pitch angle, and lag angle
- γ = Lock number
- $\Delta\beta$, $\Delta\zeta$ = perturbation flapping and lag angle
- ϵ = constant as given in Eq. (11)
- $\theta_\beta, \theta_\zeta$ = pitch-flap and pitch-lag coupling ratio
- λ_i = characteristic multiplier
- μ = advance ratio
- σ = rotor solidity
- ψ = rotor azimuth angle; $\psi = \Omega t$
- $\Phi(t)$ = state transition matrix
- $\Phi(T)$ = Floquet transition matrix
- Ω = rotor angular velocity
- ω_ζ = dimensionless inplane frequency
- $\otimes$ = Kronecker product

$$f(t)=\sum_{n=0}^{\infty} a_n T_n^*(t); \quad t \in [0, 1]. \tag{7}$$

The coefficients a_n are obtained from

$$a_n=\frac{\int_0^1 w(t)f(t)T_n^*(t)dt}{\int_0^1 w(t)T_n^*(t)T_n^*(t)dt}; \quad n=0, 1, 2, \ldots \tag{8}$$

Trigonometric functions like $\sin(kt)$ and $\cos(kt)$ can be expanded (Luke, 1969) in series of shifted Chebyshev polynomials for $t \in [0, 1]$ are given by

$$\sin(kt)=2\cos(k/2)\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(k/2)T_{2n+1}^*(t)$$

$$+\sin(k/2)J_0(k/2)T_0^*(t)+2\sin(k/2)\sum_{n=0}^{\infty}(-1)^n J_{2n}(k/2)T_{2n}^*(t) \tag{9}$$

$$\cos(kt)=2\cos(k/2)\sum_{n=0}^{\infty}(-1)^n J_{2n}(k/2)T_{2n}^*(t)+\cos(k/2)$$

$$\times J_0(k/2)T_0^*(t)+2\sin(k/2)\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(k/2)T_{2n}^*(t) \tag{10}$$

where J's are the Bessel functions of the first kind.

The integration matrix associated with these polynomials is expressed in the form

$$\int_0^t \mathbf{s}(\tau)d\tau=\mathbf{s}^T(t)\mathbf{G} \tag{11}$$

where $\mathbf{s}(t)$ is a $q\times 1$ vector of shifted Chebyshev polynomials of the first kind given by

$$\mathbf{s}^T(t)=\{T_0^*(t)T_1^*(t)T_2^*(t).\ldots.T_{q-1}^*(t)\}. \tag{12}$$

and $\mathbf{G}$ is a $q\times q$ integration matrix given by

$$\mathbf{G}^T=\begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ -1/8 & 0 & 1/8 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ -1/6 & -1/4 & 0 & 1/12 & 0 & \cdot & \cdot & \cdot & 0 \\ 1/6 & 0 & -1/8 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/(4q-4) \\ \dfrac{-(-1)^{q-1}}{2q(q-2)} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dfrac{1}{(4q-8)} & 0 \end{bmatrix} \tag{13}$$

The generating formulae in Eq. (13) are for $q\geq 3$. For $q\leq 2$, the upper left (2×2) block should be used.

The multiplication of any two arbitrary functions given by

$$f(t)=\sum_{n=0}^{\infty} a_n T_n^*(t)\cong \mathbf{s}^T(t)\mathbf{a} \text{ and } g(t)=\sum_{n=0}^{\infty} b_n T_n^*(t)\cong \mathbf{s}^T(t)\mathbf{b}$$

can be obtained as

$$f(t)g(t)\cong \mathbf{s}^T(t)\mathbf{Q}\mathbf{b}, \tag{14}$$

where $\mathbf{Q}$ is a $q\times q$ product matrix defined by

$$\mathbf{Q}=\begin{bmatrix} a_0 & a_1/2 & a_2/2 & \cdot\;\cdot\;\cdot & a_{q-1}/2 \\ a_1 & a_0+a_2/2 & (a_1+a_3)/2 & \cdot\;\cdot\;\cdot & (a_{q-2}+a_q)/2 \\ a_2 & (a_1+a_3)/2 & a_0+a_4/2 & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ a_{q-1} & (a_{q-2}+a_q)/2 & \cdot & \cdot\;\cdot\;\cdot & a_0+a_{2q-2}/2 \end{bmatrix} \tag{15}$$

It may be noted that the relation

$$T_k^*(t)T_r^*(t)=[T_{k+r}^*(t)+T_{|k-r|}^*(t)]/2 \tag{16}$$

is used to obtain the Eq. (15) and polynomials of order $q+1$ and above are neglected.

3 Method of Analysis

Consider the linear system

$$(\mathbf{M}+\mathbf{M}^*(t))\ddot{\mathbf{y}}(t)+(\mathbf{C}+\mathbf{C}^*(t))\dot{\mathbf{y}}(t)+(\mathbf{K}+\mathbf{K}^*(t))\mathbf{y}(t)=0 \tag{17}$$

where $\mathbf{y}(t)$ is an $n\times 1$ vector, $\{y_1y_2\ldots.y_n\}^T$, $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are $n\times n$ mass, damping, and stiffness matrices, respectively. The superscript * indicates the periodic components of the corresponding matrices. Integrating Eq. (17) once gives

$$\int_0^t(\mathbf{M}+\mathbf{M}^*(\eta))\ddot{\mathbf{y}}(\eta)d\eta+\int_0^t(\mathbf{C}+\mathbf{C}^*(\eta))\dot{\mathbf{y}}(\eta)d\eta$$

$$+\int_0^t(\mathbf{K}+\mathbf{K}^*(\eta))\mathbf{y}(\eta)d\eta=\mathbf{M}\dot{\mathbf{y}}(t)-\mathbf{M}\dot{\mathbf{y}}(0)+\mathbf{M}^*(t)\dot{\mathbf{y}}(t)$$

$$-\mathbf{M}^*(0)\dot{\mathbf{y}}(0)-\dot{\mathbf{M}}^*(t)\mathbf{y}(t)+\dot{\mathbf{M}}^*(0)\mathbf{y}(0)$$

$$+\int_0^t\ddot{\mathbf{M}}^*(\eta)\mathbf{y}(\eta)d\eta+\mathbf{C}\mathbf{y}(t)-\mathbf{C}\mathbf{y}(0)+\mathbf{C}^*(t)\mathbf{y}(t)-\mathbf{C}^*(0)\mathbf{y}(0)-$$

$$-\int_0^t\dot{\mathbf{C}}^*(\eta)\mathbf{y}(\eta)d\eta+\int_0^t(\mathbf{K}+\mathbf{K}^*(\eta))\mathbf{y}(\eta)d\eta=0 \tag{18}$$

where integration by parts has been used where necessary. The element of periodic matrices $\mathbf{M}^*$, $\dot{\mathbf{M}}^*$, $\mathbf{C}^*$, $\dot{\mathbf{C}}^*$, and $\mathbf{K}^*$ are now expanded in Chebyshev polynomials with known coefficients. Equation (18) is then integrated in conjuction with a Chebyshev expansion for the state vector $\mathbf{y}(t)$ with unknown coefficients given by

$$\mathbf{y}(t)=\mathbf{S}(t)\mathbf{b}_P \qquad (19)$$

where $\mathbf{S}(t) = \mathbf{I}_n \otimes \mathbf{s}^T(t)$ with $\mathbf{s}^T(t)$ being defined in Eq. (12). Further,

$$\mathbf{y}(t)=\{y_1 y_2 y_3 \ldots y_n\}^T \text{ where } y_i(t)=\sum_{r=0}^{q-1} b_r T_r^*(t) \text{ and}$$

$$\mathbf{b}_P=\{\mathbf{b}^1\mathbf{b}^2\mathbf{b}^3\ldots\mathbf{b}^n\}^T \text{ where } \mathbf{b}^i=\{b_0^i b_1^i b_2^i \ldots b_{q-1}^i\}^T.$$

The symbol $\otimes$ represents the Kronecker product as defined by Bellman (1970). Substitution of these expansions in Eq. (18) and using the definitions of the integration and product matrices given by the Eqs. (13) and (15) results in the following equation for the velocity vector

$$(\mathbf{M}+\mathbf{M}^*(t))\dot{\mathbf{y}}(t)=\mathbf{S}^T(t)[(\mathbf{M}_{P2}-(\mathbf{I}_n\otimes\mathbf{G})\mathbf{M}_{P3}-\mathbf{C}\otimes\mathbf{I}_q-\mathbf{C}_{P1} + (\mathbf{I}_n\otimes\mathbf{G})\mathbf{C}_{P2}-\mathbf{K}\otimes\mathbf{G}-(\mathbf{I}_n\otimes\mathbf{G})\mathbf{K}_{P1})\mathbf{b}_P+(\mathbf{C}^*(0)\otimes\mathbf{I}_q + \mathbf{C}\otimes\mathbf{I}_q-\dot{\mathbf{M}}^*(0)\otimes\mathbf{I}_q)\mathbf{y}_0+(\mathbf{M}\otimes\mathbf{I}_q+\mathbf{M}^*(0)\otimes\mathbf{I}_q)\mathbf{y}_1] \qquad (20)$$

where the subscripted matrices are defined in the Appendix.

Integrating Eq. (20) and once again using the Chebyshev expansions for the state vector and $\mathbf{M}^*$ and using the operational matrices finally results in the following set of algebraic equations for the unknown vector $\mathbf{b}_P$,

$$\mathbf{W}\mathbf{b}_P=\mathbf{f}_P \qquad (21)$$

where $\mathbf{W}=\mathbf{M}\otimes\mathbf{I}_q+\mathbf{M}_{P1}-2(\mathbf{I}_n\otimes\mathbf{G})\mathbf{M}_{P2}+(\mathbf{I}_n\otimes\mathbf{G}^2)\mathbf{M}_{P3} + \mathbf{C}\otimes\mathbf{G}+(\mathbf{I}_n\otimes\mathbf{G})\mathbf{C}_{P1}-(\mathbf{I}_n\otimes\mathbf{G}^2)\mathbf{C}_{P2}+\mathbf{K}\otimes\mathbf{G}^2+(\mathbf{I}_n\otimes\mathbf{G}^2)\mathbf{K}_{P1}$

and $\mathbf{f}_P=(\mathbf{M}\otimes\mathbf{I}_q+\mathbf{M}^*(0)\otimes\mathbf{I}_q-\dot{\mathbf{M}}^*(0)\otimes\mathbf{G}+\mathbf{C}^*(0)\otimes\mathbf{G} + \mathbf{C}\otimes\mathbf{G})\mathbf{y}_0+(\mathbf{M}\otimes\mathbf{G}+\mathbf{M}^*(0)\otimes\mathbf{G})\mathbf{y}_1$

where the subscripted matrices are defined in the Appendix. The velocity vector can then be determined by substituting the known vector $\mathbf{b}_P$ in Eq. (20).

In order to evaluate the FTM a set of n $\mathbf{b}_P$'s is calculated from Eq. (20) with appropriate n initial conditions on $y_i(0)$ and $\dot{y}_i(0)$. Since $\dot{\mathbf{y}}(t)$ is known from Eq. (21), the $(2n\times 2n)$ FTM can be constructed rather easily by evaluating $\mathbf{y}(T)$ and $\dot{\mathbf{y}}(T)$. The initial conditions are chosen such that $\Phi(0)=\mathbf{I}$.

4 Applications

4.1 Mathieu's Equation (*Solution via Symbolic Computation*). The Mathieu's equation is of the form,

$$\ddot{y}(t^*)+a(1+m\cos(t^*))y(t^*)=0. \qquad (22)$$

With $t = t^*/(2\pi)$ Eq. (22) transforms to

$$\ddot{y}(t)+p(1+m\cos(2\pi t))y(t)=0 \qquad (23)$$

where $$p=a(2\pi)^2. \qquad (24)$$

Let $y(t)$ be expanded in terms of the shifted Chebyshev polynomials $\mathbf{s}(t)$ with undetermined coefficients b_i's. Then $y(t)$ can be expressed as

$$y(t)=\mathbf{s}^T(t)\mathbf{b} \qquad (25)$$

where the vector $\mathbf{b}$ is given by

$$\mathbf{b}=\{b_0 b_1 b_2 \ldots . b_{q-1}\}^T. \qquad (26)$$

Integrating Eq. (23) with respect to t yields

$$\dot{y}(t)-\dot{y}(0)+\int_0^t py(\tau)d\tau+\int_0^t pm\cos(2\pi\tau)y(\tau)d\tau=0 \qquad (27)$$

Using Eq. (25) and expanding $\cos(2\pi t)$ in terms of Chebyshev polynomials [c.f. equation (10)] and the operational matrices $\mathbf{G}$ and $\mathbf{Q}$ from Eqs. (13) and (15), respectively, Eq. (27) yields

$$\dot{y}(t)=\dot{y}(0)-p\mathbf{s}^T(t)\mathbf{G}\mathbf{b}-pm\mathbf{s}^T(t)\mathbf{G}\mathbf{Q}\mathbf{b} \qquad (28)$$

Integrating Eq. (28) gives the following set of equations

$$\mathbf{s}^T(t)\mathbf{b}-\mathbf{s}^T(t)\mathbf{y}_0-\mathbf{s}^T(t)\mathbf{G}\dot{\mathbf{y}}_0+p\mathbf{s}^T(t)\mathbf{G}^2\mathbf{b}+pm\mathbf{s}^T(t)\mathbf{G}^2\mathbf{Q}\mathbf{b}=0 \qquad (29)$$

where $\mathbf{y}_0$ and $\dot{\mathbf{y}}_0$ are vectors of dimension $q\times 1$ given by

$$\mathbf{y}_0=\{y(0)\ 0\ 0\ \ldots.\ 0\}^T;$$

$$\dot{\mathbf{y}}_0=\{\dot{y}(0)\ 0\ 0\ \ldots.\ 0\}^T \qquad (30)$$

Cancelling $\mathbf{s}^T(t)$ in Eq. (29) finally results in a set of algebraic equations for the unknown vector $\mathbf{b}$ given by

$$(\mathbf{I}+p\mathbf{G}^2+pm\mathbf{G}^2\mathbf{Q})\mathbf{b}=(\mathbf{y}_0+\mathbf{G}\dot{\mathbf{y}}_0). \qquad (31)$$

The vector $\mathbf{b}$ can be calculated symbolically using MACSYMA from the above set of linear algebraic equations. Then from Eq. (25) the response $y(t)$ can be computed in symbolic form. The velocity can then be obtained from the Eq. (28) by using the known vector $\mathbf{b}$.

By computing y and $\dot{y}$ at the end of the period (i.e., $t=1$), the Floquet Transition Matrix (FTM) $\Phi(1)$ is constructed. The FTM can be written as

$$\Phi(1)=\begin{bmatrix}\Phi_{11}(1) & \Phi_{12}(1)\\ \Phi_{21}(1) & \Phi_{22}(1)\end{bmatrix} \qquad (32)$$

where $\Phi_{11}(1)$ and $\Phi_{21}(1)$ are y and $\dot{y}$ respectively calculated at the end of the period, $t = 1$, for the following initial conditions

$$y(0)=1 \text{ and } \dot{y}(0)=0; \qquad (33)$$

and $\Phi_{12}(1)$ and $\Phi_{22}(1)$ are y and $\dot{y}$, calculated at the end of the period $t = 1$ respectively for the initial conditions

$$y(0)=0,\ \dot{y}(0)=1 \qquad (34)$$

From Floquet Theory it can be shown that the determinant of $\Phi(1)$ is unity and hence the characteristic equation of $\Phi(1)$ is of the form

$$\lambda^2+D\lambda+1=0, \text{ where } D=\text{Trace }[\Phi(1)] \qquad (35)$$

From the characteristic equation it can be seen that the response is stable if and only if $|D|<2$, and unstable if $|D|>2$. $|D|=2$ corresponds to the transition point. The stability chart was constructed for 9 expansion terms, using the analytical results. The analytical expression for the transition curve was obtained from the relationship, $|D(a,m)|=2$.

The boundaries correspond to periodic solutions, and two branches emanate on the "a" axis from each $a_0=n^2/4$, n being an integer. A perturbation analysis of the Eq. (22) is also undertaken as described in Stoker (1950) using expansions of the form given by

$$a=a_0+ma_1+m^2a_2+\ldots$$
$$y(t)=y_0(t)+my_1(t)+m^2y_2(t)+\ldots$$

where m is assumed to be a smaller parameter.

The transition curves as obtained through perturbation method are given by,

$n=1$:

$$a=(1/4)-(1/8)m+(7/128)m^2-(39/2048)m^3+(335/98304)m^4 +(8965/4718592)m^5-(399071/150994944)m^6 +(9004121/4831838208))m^7$$

$$a=(1/4)+(1/8)m+(7/128)m^2+(39/2048)m^3+(335/98304)m^4 -(8965/4718592)m^3-(399071/150994944)m^6 -(9004121/4831838208)m^7$$

$n=2$:

$$a=1-(1/12)m^2+(53/3456)m^4-(18289/4976640)m^6$$
$$a=1+(5/12)m^2+(437/3456)m^4+(18289/4976640)m^6$$

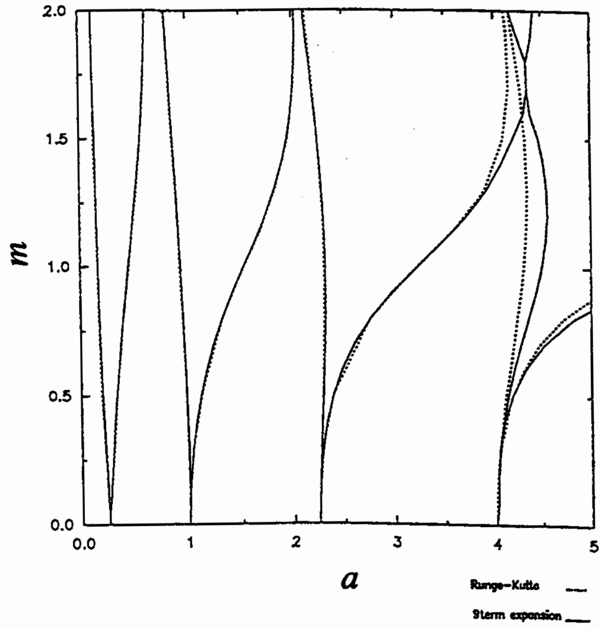

Fig. 1 Stability chart of 9 term (Chebyshev polynomial) vs. Runge-Kutta scheme for Mathieu's equation

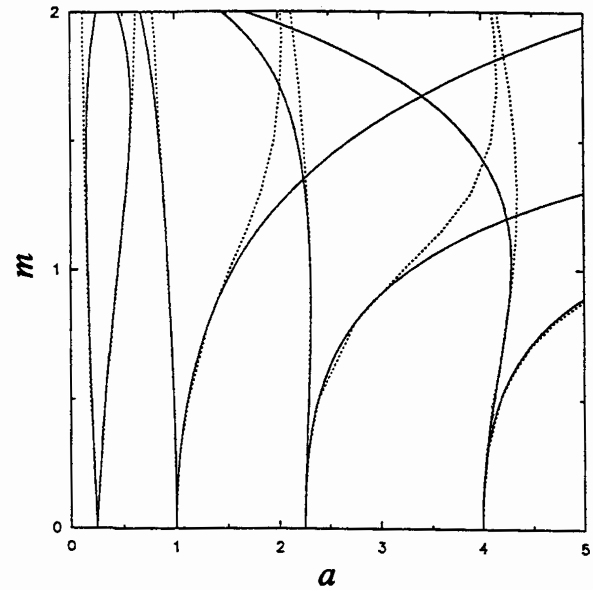

Fig. 2 Stability chart of Perturbation method (8-term) vs. Runge-Kutta scheme for Mathieu's equation

$n=3$:

$$a=(9/4)+(81/256)m^2-(729/2048)m^3+(201933/1310720)m^4-(229635/2097152)m^5+(55420767/671088640)m^6-(1452073959/53687091200)m^7$$

$$a=(9/4)+(81/256)m^2+(729/2048)m^3+(201933/1310720)m^4+(229635/2097152)m^5+(55420767/671088640)m^6+(1452073959/53687091200)m^7$$

$n=4$:

$$a=4+(8/15)m^2+(2212/3375)m^4+(1704592/5315625)m^6$$

$$a=4+(8/15)m^2-(788/3375)m^4-(59408/5315625)m^6$$

The results obtained by the proposed method were compared with those obtained through Runge-Kutta numerical scheme and perturbation solutions of the eighth order. The graphical results are shown in Fig. 1 and Fig. 2.

4.2 Stability of a Three-Bladed Helicopter Rotor (*Solution via Numerical Computation*). In order to demonstrate the capability of the technique to compute a numerical solution, the flap-lag stability of a helicopter rotor blade is studied. The model considered here was originally developed by Ormiston and Hodges (1972) and later extended for the case of forward flight by Peters (1975). The equations of motion are nonlinear; however, the stability criteria can be determined from the linearized perturbation equations about a periodic steady-state solution. Following Peters (1975), the steady-state flap (β_e), lag (ζ_e), and the pitch (θ_e) motions can be represented as

$$\beta_e(\psi)=\beta_0+\beta_c\cos\psi+\beta_s\sin\psi$$
$$\zeta_e(\psi)=\zeta_0+\zeta_c\cos\psi+\zeta_s\sin\psi$$
$$\theta_e(\psi)=\theta_o+\theta_c\cos\psi+\theta_s\sin\psi+\theta_\beta(\beta_e-\beta_{pc})+\theta_\zeta\zeta_e \quad (36)$$

where all θ's, β's, and ζ's appearing on the right-hand side of the equations are constants and ψ is the rotor azimuth angle.

For the special case of the moment trim condition, β_c and β_s are identically zero and the other parameters may be computed from the nonlinear equations using the harmonic balance method. In this study, the approximate closed-form expressions for the equilibrium values as given by Wei and Peters (1978) have been used. Denoting $\Delta\beta$ and $\Delta\zeta$ as small perturbations about β_e and ζ_e, respectively, the linearized perturbation equations take the form

$$\begin{pmatrix}\Delta\ddot{\beta}\\ \Delta\ddot{\zeta}\end{pmatrix}+\mathbf{C}(\psi)\begin{pmatrix}\Delta\dot{\beta}\\ \Delta\dot{\zeta}\end{pmatrix}+\mathbf{K}(\psi)\begin{pmatrix}\Delta\beta\\ \Delta\zeta\end{pmatrix}=0 \quad (37)$$

where matrices $[C(\psi)]$ and $[K(\psi)]$ are periodic functions of ψ. The above equations represent the linearized stability equations for a single blade only. In case of a three-bladed rotor, these equations have to be transformed from the rotating to the nonrotating frame. This can be accomplished by using the Multibladed Coordinate Transformation (MCT) matrices, as indicated in references (Hohenemser and Yin (1971), Gaonkar and Peters (1980)). The final set of equations in the nonrotating frame can be represented as

$$(\mathbf{M}+\mathbf{M}^*(\psi))\Delta\ddot{y}_{NR}+(\mathbf{C}+\mathbf{C}^*(\psi))\Delta\dot{y}_{NR}+(\mathbf{K}+\mathbf{K}^*(\psi))\Delta y_{NR}=0 \quad (38)$$

where Δy_{NR} represents the nonrotating degrees of freedom. The matrices **M**, **C**, and **K** are functions of the constant parameters θ_β, θ_ζ, β_{Pc}, γ, R, Z, σ, C_{d0}, C_T/σ, F, ω_ζ, and μ. For a three-bladed helicopter rotor, the periodic matrix $\mathbf{M}^*(\psi)$ has a Fourier expansion up to the first harmonic, and the matrices $\mathbf{C}^*(\psi)$ and $\mathbf{K}^*(\psi)$ contain Fourier expansions up to the fourth harmonic. Explicit forms of these matrices are given in reference (Hohenemser and Yin, 1971; Gaonkar and Peters, 1980) and are not repeated here for reasons of brevity.

It is observed that Eq. (38) has the same form as Eq. (17). The flap-lag stability analysis of the three-bladed helicopter rotor was studied using the following parameters where the notations are the same as in reference (Peters, 1975).

$$\theta_\beta=\theta_\zeta=\beta_{Pc}=0,\ \gamma=5,\ R=0,\ Z=0,\ \sigma=0.05,\ C_{d0}=0.01,\ C_T/\sigma=0.2$$

In order to investigate the influence of parameter change on the numerical efficiency and accuracy, the problem was studied with increasing values of advance ratio μ. Keeping $F=\omega_\zeta=1.2$, the proposed technique as well as some other standard numerical codes, such as, Runge-Kutta, Adams-Moulton, and Gear methods were used to compute the characteristic exponents associated with the FTM. The CPU time required to maintain the fifth digit accuracy in the real part of the larger exponent by each of the techniques was recorded. The results obtained are shown in Fig. 3. It is noted that in

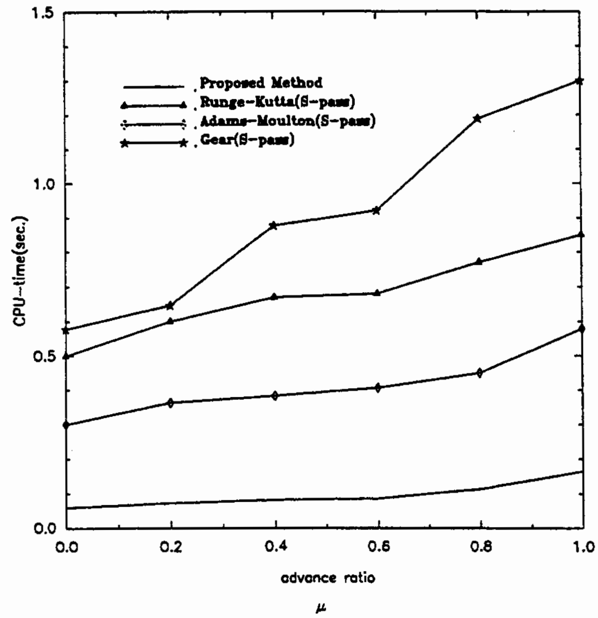

Fig. 3 A comparison of CPU time in seconds used to compute the characteristic exponents with various advance ratios for a three-bladed helicopter rotor

each case, the computation time increases with an increase in advance ratio. The numerical superiority of the proposed technique is clearly demonstrated from this figure. It should be pointed out that for the case of numerical computation the LFTRG subroutine, (a part of LINPACK algorithm) was used in solving the simultaneous set of linear equations needed to construct the proposed solution.

Finally, the CPU time taken by the proposed technique was compared with those needed by other numerical codes, to achieve the same numerical accuracy in the largest exponent. Table 1 shows the results from two case studies. In the first case the characteristic exponents converged to the fourth digit whereas in the second a seventh digit convergence was required. 18 terms were used in the former and 20 terms in the latter case for the computation of the solution vector. For both cases $F = \omega_\zeta = 1.2$ and $\mu = 0.4$.

5 Discussion and Conclusions

A novel technique has been presented for the analysis of multidimensional second-order dynamic systems with periodically varying parameters. It is shown that the original set of differential equations can be reduced to a set of linear algebraic equations by expressing the state vector and periodic coefficients matrix as finite sums of shifted Chebyshev polynomials. The accuracy of the solutions obtained through this method improves as more number of terms are taken in the expansion. The solution technique has been combined with Floquet theory to study the stability of general dynamic systems where the mass, damping, and stiffness matrices can be periodic functions of time.

To demonstrate the capability of the method to generate approximate analytical solutions via symbolic computation (MACSYMA), the Mathieu's equation is considered. It has been possible to obtain in closed-form a ninth degree polynomial in the system parameters which enables the construction of the stability chart over a wide range of parameters. To the authors' knowledge such an analytical result is obtained for the first time here. As seen from Fig. 1, the stability boundaries are in good agreement with the Runge-Kutta results even for moderately large values of both the parameters, viz., a and m. In contrast, even an eight-term perturbation solution is

Table 1 A comparison of CPU time taken in seconds to determine the characteristic exponents for a three-bladed helicopter rotor

Accuracy Level	Present Method	Runge-Kutta Method		Adams-Moulton Method		Gear Method	
		Single Pass	N-pass	Single Pass	N-pass	Single Pass	N-pass
Fourth Digit	0.063	0.342	2.947	0.205	2.145	0.264	2.546
Seventh Digit	0.082	0.641	6.114	0.372	5.274	0.876	8.094

accurate for small values of m only. For large problems, the symbolic approach faces computational problems, but so does perturbation and other methods.

The flap-lag stability analysis of a three-bladed helicopter rotor has been considered and a numerical solution of the problem is obtained. The computational superiority of the proposed technique is obvious from the analysis of the three-bladed rotor using the MCT technique. Figure 3 and Table 1 clearly indicate that this approach is much more efficient than the conventional Runge-Kutta, Adams-Moulton, and Gear algorithms applied either in an "N-pass" or "single-pass" schemes. Parametric studies with variations in the advance ratio (μ) show that for larger values of μ, more terms were needed in the expansion to maintain the same level of accuracy. Of course, this results in an increase in the CPU time. Nevertheless, as indicated in Fig. 3, all other numerical codes also behave in a similar fashion.

It is concluded that the method suggested in this paper is certainly a viable alternative method for generating approximate analytical and numerical solutions for linear systems with periodic coefficients. The technique has been found to be extremely efficient as well as accurate. The main advantage of this technique is that much of the information required to set up the problem, such as "the integration matrix," "the product matrix," and other operational matrices, can be stored in the computer in advance. In general, the periodic coefficients can be written in the forms of $\sin(k\psi)$ and/or $\cos(k\psi)$. The expansions of these quantities need to be computed only once and can be stored for future use. The entire computation process can be automated rather easily.

It is anticipated that in the near future, the proposed technique would serve as a viable computational tool in the analysis of general dynamic systems with periodically varying parameters.

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APPENDIX

$\mathbf{C}_{P1} = nq \times nq$ assembly of the product matrices $\mathbf{Q}_{ij}$ obtained for each element of $\mathbf{C}^*(t)$ where $\mathbf{Q}_{ij}$ are obtained similar to Eq. (14) for the product $c^*_{ij}(t)y_j(t)(i,j=1,2,\ldots,n)$ when $c^*_{ij}(t)$ and $y_j(t)$ are expanded in Chebyshev polynomials.

$\mathbf{C}_{P2} = nq \times nq$ assembly of the product matrices due to $\dot{\mathbf{C}}^*(t)\mathbf{y}(t)$

$\mathbf{I}_n = n \times n$ identity matrix

$\mathbf{I}_q = q \times q$ identity matrix

$\mathbf{M}_{P1} = nq \times nq$ assembly of the product matrices due to $\mathbf{M}^*(t)\mathbf{y}(t)$

$\mathbf{M}_{P2} = nq \times nq$ assembly of the product matrices due to $\dot{\mathbf{M}}^*(t)\mathbf{y}(t)$

$\mathbf{M}_{P3} = nq \times nq$ assembly of the product matrices due to $\ddot{\mathbf{M}}^*(t)\mathbf{y}(t)$

$\mathbf{K}_{P1} = nq \times nq$ assembly of the product matrices due to $\mathbf{K}^*(t)\mathbf{y}(t)$

$\mathbf{y}_0 = \{\mathbf{y}_1(0)\mathbf{y}_2(0) \ldots \mathbf{y}_n(0)\}$; $\mathbf{y}_i(0) = \{y_1(0)\ 0\ 0 \ldots 0\}$ $i=1,2,\ldots,n$

$\mathbf{y}_1 = \{\dot{\mathbf{y}}_1(0)\dot{\mathbf{y}}_2(0) \ldots \dot{\mathbf{y}}_n(0)\}$; $\dot{\mathbf{y}}_i(0) = \{\dot{y}_1(0)\ 0\ 0 \ldots 0\}$ $i=1,2,\ldots,n$