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AN EFFICIENT COMPUTATIONAL SCHEME FOR THE ANALYSIS OF PERIODIC SYSTEMS

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In this paper a new efficient numerical scheme for the stability analysis of linear systems with periodic parameters is suggested. The approach is based on the idea that the state vector and the periodic matrix of the system can be expanded in terms of Chebyshev polynomials over the principal period. Such an expansion reduces the original problem to a set of linear algebraic equations from which the solution in the interval of one period can be obtained. Furthermore, the technique is combined with Floquet theory to yield the transition matrix at the end of one period and provide the stability conditions via an eigen-analysis procedure. Two formulations are presented. The first formulation is suitable for a set of equations written in state space form, while the second can be applied directly to a set of second order equations. The application is demonstrated through practical illustrative examples including the parametric excitation problem of a fixed–fixed column. The numerical results thus obtained are compared with those obtained from DVERK, a Runge–Kutta code available in the IMSL library. An error-bound analysis is also included. It is concluded that the suggested schemes not only provide accurate results but are also computationally very efficient. In particular, the second formulation is found to be several times faster than the standard Runge–Kutta type codes.

1. INTRODUCTION

The study of systems governed by a set of ordinary differential equations with periodic coefficients is of great importance in diverse branches of science and engineering. The stability and response under various excitations are the key issues discussed in the vast amount of classical literature available on this subject. Numerous practical applications can be found in the areas of quantum mechanics, dynamic stability of structures, circuit theory, systems and control, and dynamics of rotating systems, among others.

In the past, several methods have been used to study the stability of systems with periodic coefficients. These include Hill's method [1–3, 6], the perturbation method [1, 3, 7, 8], the averaging approach [1, 3], and Floquet theory with numerical integration [9–15]. It is well known that Hill's method is not very convenient for digital computation, especially if one has to deal with systems that have a large number of degrees of freedom. This has also been pointed out by Friedmann in a recent review article [17]. The perturbation methods have their own limitations due to the fact that they can only be applied to systems where the periodic coefficients can be expressed in terms of a small parameter. The averaging techniques have similar drawbacks.

A number of authors [4, 5, 11, 12] have tried to determine the stability and response from an approximate system of equations, which are usually obtained by replacing the elements of the periodic coefficients matrix by piecewise constants or linear functions. In practical application, one can approximate the periodic matrix at most by a series of step functions and compute the transition matrix during one period, which then yields the

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stability condition etc. Such a technique was employed by Friedmann *et al.* [13] for a numerical evaluation of the transition matrix. Although the approach is straightforward, it is only a second order algorithm at the most. For more accurate solutions, it is necessary to apply higher order numerical schemes such as the Runge–Kutta–Gill method or similar algorithms, and utilize Floquet theory to establish stability conditions. This approach has been adopted by several authors [13–15] in a variety of stability and response problems. However, since n integration passes are required in the computation of the transition matrix for an $n \times n$ system, this approach tends to be expensive in terms of computer time as n becomes large. Therefore a modified fourth order numerical scheme was suggested in reference [13], which requires only a single integration pass and thereby reduces the computation time by a reasonable amount. A recent study by Gaonkar *et al.* [15] shows that Hamming's fourth order predictor–corrector method in a single-pass scheme is very likely the most economical approach. Recent survey articles on this topic have been presented by Friedmann [17] and Dugundji and Wendell [16].

In this study, it is proposed that the solution of periodic systems be represented in terms of Chebyshev polynomials of the first and second kind. The attractive feature of this technique is that it reduces the original differential system to a system of linear algebraic equations. The idea of solving differential and integral equations in terms of orthogonal polynomials is not new. The book by Fox and Parker [18] is an excellent reference on this subject, among numerous other articles available in the literature. However, the earlier investigations were restricted to scalar equations with constant coefficients only. Recently, many authors have used orthogonal polynomials to solve a number of problems arising in the field of systems and control. Chang *et al.* [20] have studied the response of linear dynamic systems, whereas Chou and Horng [21–23] have exploited the technique to study identification and optimal control problems as well as solution of integral equations. There is also some evidence that the method works well for time-varying systems, especially if one uses Chebyshev polynomials [24]. The first applications of orthogonal polynomials to differential equations with periodic coefficients were reported by Sinha and Chou [27] and Sinha *et al.* [12]. These applications were limited to second order scalar equations only. With the recent developments in the theory of various operational matrices [20, 25] associated with orthogonal polynomials, it is now possible to apply this technique to large periodic systems. First, it is necessary to review certain properties associated with these polynomials.

2. PROPERTIES OF CHEBYSHEV POLYNOMIALS

Chebyshev polynomials [18, 19, 32] of the first kind are defined by the relation

$$T_r(t) = ((-1)^r 2^r r!/(2r)!)(1-t^2)^{1/2}(\mathrm{d}/\mathrm{d}t)^r(1-t^2)^{r-1/2}, \qquad r=0, 1, 2, 3, \ldots, \tag{1}$$

and are orthogonal over the interval $[-1, 1]$ with respect to the weight function

$$w(t) = (1-t^2)^{-1/2}. \tag{2}$$

These polynomials can also be obtained from

$$T_r(t) = \cos(r\theta), \quad t = \cos\theta, \qquad 0 \leqslant \theta \leqslant \pi, \quad r=0, 1, 2, 3, \ldots. \tag{3}$$

Similarly, the Chebyshev polynomials of the second kind are defined by the relation

$$U_r(t) = csc\,\theta \sin(r+1)\theta, \quad t = \cos\theta, \qquad r=0, 1, 2, 3, \ldots. \tag{4}$$

Using the change of variable,

$$t^* = (t+1)/2, \tag{5}$$

we have the shifted Chebyshev polynomials of the first kind as

$$T_r^*(t) = T_r(2t-1), \qquad 0 \leqslant t \leqslant 1. \tag{6}$$

Similarly, the shifted Chebyshev polynomials of the second kind are defined as

$$U_r^*(t) = U_r(2t-1), \qquad 0 \leqslant t \leqslant 1. \tag{7}$$

2.1. PROPERTIES OF SHIFTED CHEBYSHEV POLYNOMIALS

We can generate the recurrence relations from equations (6) and (7), and obtain

$$T_{r+1}^*(t) = 2(2t-1)T_r^*(t) - T_{r-1}^*(t), \tag{8}$$

$$U_{r+1}^*(t) = 2(2t-1)U_r^*(t) - U_{r-1}^*(t). \tag{9}$$

The orthogonality relationships are given by

$$\int_0^1 T_r^*(t)T_k^*(t)w(t)\,\mathrm{d}t = \begin{cases} 0, & r \neq k, \\ \pi/2, & r = k \neq 0, \\ \pi, & r = k = 0, \end{cases} \tag{10}$$

where $w(t)$ is the weight function. For the shifted Chebyshev polynomials of the first kind,

$$w(t) = (t - t^2)^{-1/2}. \tag{11}$$

In the case of the shifted Chebyshev polynomials of the second kind, we have

$$\int_0^1 U_r^*(t)U_k^*(t)w(t)\,\mathrm{d}t = \begin{cases} 0, & r \neq k, \\ \pi/8, & r = k, \end{cases} \tag{12}$$

where the weight function is

$$w(t) = (t - t^2)^{1/2}. \tag{13}$$

Generally, a continuous time function $f(t)$ can be expanded into a Chebyshev series as [19, 28]:

$$f(t) = \sum_{r=0}^{\infty} a_r S_r^*(t), \qquad 0 \leqslant t \leqslant 1, \tag{14}$$

where $S_r^*(t)$ has been used to represent the shifted Chebyshev polynomials of either the first or the second kind. The Chebyshev coefficients a_r can be obtained from

$$a_r = \frac{1}{\delta} \int_0^1 w(\tau) f(\tau) S_r^*(\tau)\,\mathrm{d}\tau, \qquad r = 0, 1, 2, 3, \ldots, \tag{15a}$$

where $w(\tau)$ is the appropriate weight function and

$$\delta = \begin{cases} \pi/2, & r \neq 0, \\ \pi, & r = 0, \end{cases} \tag{15b}$$

for the shifted Chebyshev polynomials of the first kind.

For the shifted Chebyshev polynomials of the second kind,

$$\delta = \pi/8, \qquad r = 0, 1, 2, \ldots. \tag{15c}$$

TABLE 1

Chebyshev coefficients in the expansion of $\cos 2\pi\tau$

Shifted Chebyshev polynomial of the first kind		Shifted Chebyshev polynomial of the second kind	
$a_0=0{\cdot}30424219$	$a_1=0$	$a_0=-0{\cdot}18119174$	$a_1=0$
$a_2=0{\cdot}97086799$	$a_3=0$	$a_2=0{\cdot}63685850$	$a_3=0$
$a_4=-0{\cdot}30284708$	$a_5=0$	$a_4=-0{\cdot}16597055$	$a_5=0$
$a_6=0{\cdot}02909024$	$a_7=0$	$a_6=0{\cdot}01524210$	$a_7=0$
$a_8=-0{\cdot}00139122$	$a_9=0$	$a_8=-0{\cdot}00071622$	$a_9=0$
$a_{10}=0{\cdot}00004022$	$a_{11}=0$	$a_{10}=-0{\cdot}00002048$	$a_{11}=0$
$a_{12}=-0{\cdot}00000075$	$a_{13}=0$	$a_{12}=-0{\cdot}00000043$	$a_{13}=0$

For example, if we expand $\cos 2\pi\tau$ in the interval [0, 1], as a finite number of terms, i.e.,

$$\cos 2\pi\tau=\sum_{r=0}^{m-1} a_r S_r^*(\tau),$$

the a_rs can be computed easily from equation (15). They are summarized in Table 1.

We can also expand any continuous function in terms of shifted Chebyshev polynomials in the arbitrary interval $[t_1, t_2]$, if so desired. This is shown in Appendix A.1.

2.2. THE OPERATIONAL MATRICES OF INTEGRATION [24–26]

The general recursive formula for integration of the shifted Chebyshev polynomials of the first kind can be written as

$$\int_0^t T_r^*(\tau)\,\mathrm{d}\tau=\frac{1}{4}\left[\frac{T_{r+1}^*(t)}{r+1}-\frac{T_{r-1}^*(t)}{r-1}\right]-\frac{(-1)^r}{2(r^2-1)},\qquad r=0, 2, 3, 4, \ldots . \tag{16}$$

For the special case of $r=1$, we have

$$\int_0^t T_1^*(\tau)\,\mathrm{d}\tau=(T_2^*(t)-T_0^*(t))/8. \tag{17}$$

Equation (16) may be written in vector form as

$$\int_0^t \{\mathbf{T}^*(\tau))\,\mathrm{d}\tau=[\mathbf{G}]\{\mathbf{T}^*(t)\}=\{\mathbf{T}^*(t)\}^{\mathrm{T}}[\mathbf{G}]^{\mathrm{T}}, \tag{18a}$$

where [G] is the integration operational matrix, given by

$$[\mathbf{G}]=\begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & & & & & 0 \\ -1/8 & 0 & 1/8 & 0 & & & & & & 0 \\ -1/6 & -1/4 & 0 & 1/12 & 0 & & & & & 0 \\ 1/16 & 0 & -1/8 & 0 & & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & 1/4(m-1) \\ \dfrac{-(-1)^{m-1}}{2m(m-2)} & & & & & & & & \dfrac{-1}{4(m-2)} & 0 \end{bmatrix}, \tag{18b}$$

$\{\mathbf{T}^*(t)\}$ is a column vector of the polynomials, defined as

$$\{\mathbf{T}^*(t)\}=\{T_0^*(t)\quad T_1^*(t)\quad \cdots \quad T_{m-1}^*(t)\}^{\mathrm{T}}$$

and $\{\ \}^{\mathrm{T}}$ denotes the transpose of the quantity $\{\ \}$.

Similarly, for the shifted Chebyshev polynomials of the second kind, we have

$$\int_0^t U_r^*(\tau)\,d\tau = \frac{1}{4}\left[\frac{U_{r+1}^*(t)}{r+1} - \frac{U_{r-1}^*(t)}{r+1}\right] + \frac{(-1)^r}{2(r+1)}, \qquad r=0,2,3,\ldots. \tag{19}$$

For $r=1$,

$$\int_0^t U_1^*(\tau)\,d\tau = (U_2^*(t) - 3U_0^*(t))/8.$$

Equation (19) can also be written as

$$\int_0^t \{\mathbf{U}^*(\tau)\}\,d\tau = [\mathbf{G}']\{\mathbf{U}^*(t)\} = \{\mathbf{U}^*(t)\}^{\mathrm{T}}[\mathbf{G}']^{\mathrm{T}}, \tag{20a}$$

where $[\mathbf{G}']$ is the integration operational matrix associated with the shifted Chebyshev polynomials of the second kind. It can be expressed as

$$[\mathbf{G}'] = \begin{bmatrix} 1/2 & 1/4 & 0 & & & & & 0 \\ -3/8 & 0 & 1/8 & 0 & & & & 0 \\ 1/6 & -1/12 & 0 & 1/12 & & & & 0 \\ -1/8 & 0 & -1/16 & 0 & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1/4(m-1) \\ \dfrac{(-1)^{m-1}}{2m} & & & & & & \dfrac{-1}{4m} & 0 \end{bmatrix}, \tag{20b}$$

and

$$\{\mathbf{U}^*(t)\} = \{U_0^*(t) \quad U_1^*(t) \quad \cdots \quad U_{m-1}^*(t)\}^{\mathrm{T}}.$$

Note that both of the $T_m^*(t)$ and $U_m^*(t)$ terms have been truncated in the integration of $T_{m-1}^*(t)$ and $U_{m-1}^*(t)$.

From the integral property of the Chebyshev vector, it can be shown that

$$\underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{k\ \text{times}} \{\mathbf{S}^*(t')\}\,dt' = \{\mathbf{S}^*(t)\}[\bar{G}^{\mathrm{T}}]^k, \tag{20c}$$

where $\{\mathbf{S}^*(t)\}$ could be the shifted Chebyshev polynomials of either the first or the second kind, and $[\bar{\mathbf{G}}]$ represents the associated integration operational matrix.

We can also obtain the integration operational matrix of shifted Chebyshev polynomials in an arbitrary interval $[t_1, t_2]$, as shown in Appendix A.2.

2.3. THE OPERATIONAL MATRICES OF PRODUCTS

Next, we consider the relationships associated with the products of these polynomials. The scalar product of two shifted Chebyshev polynomials of the first kind is given as

$$T_r^*(t)T_k^*(t) = \tfrac{1}{2}(T_{r+k}^*(t) + T_{|r-k|}^*(t)), \qquad r,k=0,1,2,\ldots, \tag{21}$$

whereas the cross-product of two Chebyshev vectors can be expressed as

$$\{\mathbf{T}^*(t)\}\{\mathbf{T}^*(t)\}^{\mathrm{T}} = \begin{bmatrix} T_0^* & T_1^* & T_2^* & \cdots & T_{m-1}^* \\ T_1^* & \frac{1}{2}(T_0^*+T_2^*) & \frac{1}{2}(T_1^*+T_3^*) & \cdots & \\ T_2^* & \frac{1}{2}(T_1^*+T_3^*) & \frac{1}{2}(T_0^*+T_2^*) & \cdots & \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ T_{m-1}^* & \frac{1}{2}(T_m^*+T_{m-2}^*) & & \cdots & \frac{1}{2}(T_0^*+T_{2(m-1)}^*) \end{bmatrix}. \tag{22}$$

Similarly, for the shifted Chebyshev polynomials of the second kind,

$$U_r^*(t)U_k^*(t) = \sum_{h=0}^{k} U_{r+k-2h}^*(t), \qquad r, k = 0, 1, 2, \ldots, \tag{23}$$

and the cross-product of these Chebyshev vectors can be expressed as

$$\{\mathbf{U}^*(t)\}\{\mathbf{U}^*(t)\}^{\mathrm{T}} = \begin{bmatrix} U_0^* & U_1^* & U_2^* & \cdots & U_{m-1}^* \\ U_1^* & U_0^*+U_2^* & U_1^*+U_3^* & \cdots & U_m^*+U_{m-2}^* \\ U_2^* & U_1^*+U_3^* & U_0^*+U_2^*+U_4^* & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ U_{m-1}^* & U_{m-1}^*+U_{m-3}^* & & \cdots & \sum_{r=0}^{m-1} U_{2(m-r-1)}^* \end{bmatrix}. \tag{24}$$

Hence, for instance, if

$$f(t) = \sum_{r=0}^{m-1} a_r T_r^*(t), \qquad g(t) = \sum_{r=0}^{m-1} b_r T_r^*(t), \tag{25}$$

then

$$f(t)g(t) = \{a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_{m-1}\}\{\mathbf{T}^*\}^{\mathrm{T}}\{\mathbf{T}^*\}\{b_0 \quad b_1 \quad b_2 \quad \cdots \quad b_{m-1}\}^{\mathrm{T}}, \tag{26a}$$

where a_r and b_r are Chebyshev coefficients of functions $f(t)$ and $g(t)$, respectively. Using equation (22), we can rewrite equation (26a) as

$$f(t)g(t) = \{\mathbf{T}^*\}^{\mathrm{T}}[\mathbf{Q}]\{\mathbf{b}\}, \tag{26b}$$

where $[\mathbf{Q}]$ is the product operation matrix of shifted Chebyshev polynomials of the first kind, given by

$$[\mathbf{Q}] = \begin{bmatrix} a_0 & a_1/2 & a_2/2 & \cdots & a_{m-1}/2 \\ a_1 & a_0+a_2/2 & \frac{1}{2}(a_1+a_3) & \cdots & \frac{1}{2}(a_{m-1}+a_m) \\ a_2 & \frac{1}{2}(a_1+a_3) & a_0+a_4/2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m-1} & \frac{1}{2}(a_{m-2}+a_m) & & \cdots & a_0+a_{2m-2}/2 \end{bmatrix} \tag{26c}$$

and

$$\{\mathbf{b}\}=\{b_0 \quad b_1 \quad b_2 \quad \cdots \quad b_{m-1}\}^{\mathrm{T}}.$$

Similarly, if $f(t)$ and $g(t)$ are expanded in terms of the shifted Chebyshev polynomials of the second kind, we have

$$f(t)=\sum_{r=0}^{m-1} a'_r U^*_r(t), \qquad g(t)=\sum_{r=0}^{m-1} b'_r U^*_r(t), \tag{27a}$$

and utilizing the result from equation (24),

$$f(t)g(t)=\{\mathbf{U}^*\}^{\mathrm{T}}[\mathbf{Q}']\{\mathbf{b}'\}. \tag{27b}$$

The product operational matrix $[\mathbf{Q}']$ of shifted Chebyshev polynomials of the second kind is

$$[\mathbf{Q}']=\begin{bmatrix} a'_0 & a'_1 & a'_2 & \cdots & a'_{m-1} \\ a'_1 & a'_0+a'_2 & a'_1+a'_3 & \cdots & a'_{m-1}+a'_{m+1} \\ a'_2 & a'_1+a'_3 & a'_0+a'_4 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a'_{m-1} & a'_{m-1}+a'_{m+1} & & \cdots & a'_0+a'_2+a'_4+a'_6+\cdots \end{bmatrix} \tag{27c}$$

and

$$\{\mathbf{b}'\}=\{b'_0 \quad b'_1 \quad b'_2 \quad \cdots \quad b'_{m-1}\}^{\mathrm{T}}.$$

3. A BRIEF REVIEW OF FLOQUET THEORY

We briefly present some main results of Floquet theory. Consider the linear periodic system

$$\dot{\mathbf{x}}=[\mathbf{A}(t)]\mathbf{x}, \qquad \dot{\mathbf{x}}=\mathrm{d}\mathbf{x}/\mathrm{d}t, \tag{28}$$

where $\mathbf{x}$ is an $n\times 1$ vector and $[\mathbf{A}(t)]$ is an $n\times n$ matrix of principal period T, such that

$$[\mathbf{A}(t+T)]=[\mathbf{A}(t)]. \tag{29}$$

Let $[\boldsymbol{\varphi}(t)]$ be a fundamental matrix solution of equation (28); i.e., a non-singular matrix each of the columns of which is a solution of equation (28) such that $[\boldsymbol{\varphi}(0)]=[\mathbf{I}]$, the identify matrix. Then it is known that $[\boldsymbol{\varphi}(t)]$ satisfies equation (28), i.e.,

$$[\dot{\boldsymbol{\varphi}}(t)]=[\mathbf{A}(t)][\boldsymbol{\varphi}(t)], \tag{30}$$

as well as

$$[\boldsymbol{\varphi}(t+T)]=[\boldsymbol{\varphi}(t)][\mathbf{F}], \tag{31}$$

where $[\mathbf{F}]$ is a constant matrix.

We can also define a real number α_i, which is the real part of the characteristic exponent, as

$$\alpha_i=\frac{1}{T}\ln|\mu_i|, \tag{32}$$

where μ_i, the eigenvalues of [F], are called characteristic numbers, since the μ_i, in general, are complex. It is observed that matrix [F] may be obtained directly from equation (31) as

$$[\mathbf{F}]=[\boldsymbol{\varphi}(t)]^{-1}[\boldsymbol{\varphi}(t+T)]=[\boldsymbol{\varphi}(0)]^{-1}[\boldsymbol{\varphi}(T)]=[\boldsymbol{\varphi}(T)]. \tag{33}$$

These results may be summarized as follows: (i) the knowledge of $[\boldsymbol{\varphi}(T)]$ is sufficient to predict the stability of the system and the condition can be expressed as $\alpha_i<0$ or $\mu_i<1$, $i=1, 2, \ldots, n$—if $\mu_i=1$, then the systems have periodic solutions; (ii) once the transition matrix $[\boldsymbol{\varphi}(t)]$ is known in the interval $[0, T]$, the solution for all $t>T$ can simply be obtained from the semigroup property—the details can be found in Coddington and Levinson [29].

3.1. COMPUTATION OF FLOQUET TRANSITION MATRIX (FTM) USING STANDARD NUMERICAL CODES [13–15]

The FTM is the state transition matrix, of dimension $n\times n$, evaluated at the end of one period. Consequently, the state vector at the end of one period is the product of the FTM and the initial state. As discussed earlier, the FTM matrix $[\boldsymbol{\varphi}(T)]$ can be determined from the $n\times n$ matrix solution of the state equations over one period which corresponds to the initial state of the identity matrix. Several standard numerical codes have been used in the past for the computation of FTM. These numerical codes can be implemented in two different ways, as discussed in the following.

3.1.1. *N-pass scheme*

The state equation is solved for discrete time values over one period, the solution being repeated n times for n initial states which comprise the columns of the $n\times n$ identity matrix. This approach is referred to as the $N-$pass scheme.

3.1.2. *Single-pass scheme* [15]

The $n\times n$ FTM is computed in a single integration scheme as an $n^2\times 1$ state vector, the $n^2\times n^2$ modified state matrix being identified with the initial condition [1 0 0 $\cdots$; 0 1 0 0 $\cdots$; 0 0 1 0 0 $\cdots$; $\cdots$; $\cdots$; 0 0 $\cdots$ 1]. It is to be noted that the n original state equations must be reformulated before starting the integration. There are several advantages of the single-pass scheme over the N-pass scheme, although the modified state matrix is of dimension n^2. The details are given in reference [15].

4. METHOD OF ANALYSIS

4.1. STATE SPACE FORMULATION

Consider the linear system

$$[\mathbf{M}]\ddot{\mathbf{y}}(t)+[\mathbf{C}+\mathbf{C}^*(t)]\dot{\mathbf{y}}(t)+[\mathbf{K}+\mathbf{K}^*(t)]\mathbf{y}(t)=0, \tag{34}$$

where $\mathbf{y}(t)$ is an $n\times 1$ state vector, $\{y_1\ y_2\ \cdots\ y_n\}^{\mathrm{T}}$, [M] is an $n\times n$ inertia matrix, [C] is the constant damping matrix, [K] is the constant stiffness matrix, and $[\mathbf{C}^*(t)]$ and $[\mathbf{K}^*(t)]$ are periodic time-variant matrices. Letting

$$\begin{gathered}\{\mathbf{y}_1(t)\}=\{y_1(t)\quad y_2(t)\quad\cdots\quad y_n(t)\}^{\mathrm{T}},\\ \{\mathbf{y}_2(t)\}=\{\dot{y}_1(t)\quad \dot{y}_2(t)\quad\cdots\quad \dot{y}_n(t)\}^{\mathrm{T}}=\{y_{n+1}(t)\quad y_{n+2}(t)\quad\cdots\quad y_{2n}(t)\}^{\mathrm{T}},\end{gathered} \tag{35}$$

equation (34) can be rewritten as

$$\dot{\mathbf{Y}}(t)=\left[\begin{array}{c|c}\mathbf{0} & \mathbf{I}\\ \hline -\mathbf{M}^{-1}[\mathbf{K}+\mathbf{K}^*(t)] & -\mathbf{M}^{-1}[\mathbf{C}+\mathbf{C}^*(t)]\end{array}\right]\mathbf{Y}(t), \qquad \mathbf{Y}(t)=\{y_1(t) \quad y_2(t)\}^{\mathrm{T}}. \tag{36}$$

We can also express equation (36) as

$$\{\dot{\mathbf{Y}}(t)\}=[\mathbf{W}+\mathbf{A}(t)]\{\mathbf{Y}(t)\}, \tag{37}$$

where [**W**] is the constant matrix, and [$\mathbf{A}(t)$] is the periodic time-variant matrix. These matrices have the forms

$$\mathbf{W}=\left[\begin{array}{c|c}\mathbf{0} & \mathbf{I}\\ \hline -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{array}\right], \qquad \mathbf{A}(t)=\left[\begin{array}{c|c}\mathbf{0} & \mathbf{0}\\ \hline -\mathbf{M}^{-1}\mathbf{K}^*(t) & -\mathbf{M}^{-1}\mathbf{C}^*(t)\end{array}\right]. \tag{38}$$

The elements $A_{ij}(t)$ of matrix $\mathbf{A}(t)$ are periodic functions of time, with period T.

It is proposed that the solution of equation (37) be expressed in terms of Chebyshev polynomials defined over the interval [0, T]. For this purpose we let

$$y_i(t)=\sum_{r=0}^{m-1} b_r^i S_r^*(t)=\{S^*(t))^{\mathrm{T}}\{\mathbf{b}^i\}, \qquad i=1,2,3,\ldots,2n, \tag{39}$$

$$A_{ij}(t)=\sum_{r=0}^{m-1} d_r^{ij} S_r^*(t)=\{S^*(t)\}^{\mathrm{T}}\{\mathbf{d}^{ij}\}, \qquad i,j=1,2,3,\ldots,2n, \tag{40}$$

where

$$\{\mathbf{b}^i\}=\{b_0^i \quad b_1^i \quad \cdots \quad b_{m-1}^i\}^{\mathrm{T}}, \qquad \{\mathbf{d}^{ij}\}=\{d_0^{ij} \quad d_1^{ij} \quad \cdots \quad d_{m-1}^{ij}\}^{\mathrm{T}},$$

and

$$\{\mathbf{S}^*(t)\}^{\mathrm{T}}=\{S_0^*(t) \quad S_1^*(t) \quad \cdots \quad S_{m-1}^*(t)\}.$$

b_r^i and d_r^{ij} are expansion coefficients of $y_i(t)$ and $A_{ij}(t)$, respectively, and $S_r^*(t)$ represent either $T_r^*(t)$ or $U_r^*(t)$ as defined earlier.

It is to be noted that b_r^i are unknown, while d_r^{ij} are determined from the procedure described in section 2.1 (cf. equation (15a)).

For further considerations, we define the $2n\times 2nm$ Chebyshev coefficients matrix $[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}$ as

$$[\hat{\mathbf{S}}(t)]^{\mathrm{T}}=[\mathbf{I}]\otimes\{\mathbf{S}^*(t)\}^{\mathrm{T}}, \tag{41}$$

where $\otimes$ represents the Kronecker product, and [**I**] is a $2n\times 2n$ identity matrix. Thus

$$[\hat{\mathbf{S}}(t)]^{\mathrm{T}}=\begin{bmatrix}\{\mathbf{S}^*(t)\}^{\mathrm{T}} & 0 & 0 & & \cdots & & 0\\ 0 & \{\mathbf{S}^*(t)\}^{\mathrm{T}} & 0 & 0 & \cdots & & \\ 0 & 0 & \{\mathbf{S}^*(t)\}^{\mathrm{T}} & & \cdots & & 0\\ 0 & 0 & & & \cdots & & 0\\ 0 & 0 & & & \cdots & 0 & \{\mathbf{S}^*(t)\}^{\mathrm{T}}\end{bmatrix}.$$

Therefore $\mathbf{Y}(t)$ and $\mathbf{A}(t)$ can be rewritten as

$$\mathbf{Y}(t)=[\hat{\mathbf{S}}(t)]^{\mathrm{T}}\{\mathbf{B}\} \qquad \text{and} \qquad \mathbf{A}(t)=[\hat{\mathbf{A}}(t)]^{\mathrm{T}}[\mathbf{D}], \tag{42, 43}$$

where

$$\{\mathbf{B}\}=\{\mathbf{b}^1 \quad \mathbf{b}^2 \quad \cdots \quad \mathbf{b}^{2n}\}^{\mathrm{T}} \qquad \text{and} \qquad [\mathbf{D}]=[\mathbf{d}^{i1} \quad \mathbf{d}^{i2} \quad \cdots \quad \mathbf{d}^{ij}], \quad i,j=1,2,3,\ldots,2n.$$

From equation (26b) and/or (27b) and equations (42, 43), we have

$$\mathbf{A}(t)\mathbf{y}(t)=[\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\bar{\mathbf{Q}}]\{\mathbf{B}\}, \tag{44}$$

where the $2nm\times 2nm$ product matrix is given by

$$[\bar{\mathbf{Q}}]=\left[\begin{array}{c|c}\mathbf{0} & \mathbf{0}\\ \hline \mathbf{Q}_{21} & \mathbf{Q}_{22}\end{array}\right].$$

The submatrices $\mathbf{Q}_{21}$ and $\mathbf{Q}_{22}$ are constant product matrices, and $[\mathbf{0}]$ is an $nm\times nm$ null matrix.

The integral form of equation (37) is

$$\mathbf{Y}(t)-\mathbf{Y}(0)=\int_0^t \mathbf{W}\mathbf{Y}(t')\,\mathrm{d}t'+\int_0^t \mathbf{A}(t')\mathbf{Y}(t')\,\mathrm{d}t', \tag{45}$$

where t' represents a dummy variable. Substituting equations (18a), or (20a), and (44) in equation (45), yields

$$[\hat{\mathbf{S}}(t)]^{\mathrm{T}}\{\mathbf{B}\}-[\hat{\mathbf{S}}(t)]^{\mathrm{T}}\{\mathbf{Y}(0)\}=[\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\mathbf{P}]\{\mathbf{B}\}+[\hat{\mathbf{S}}(t)\}^{\mathrm{T}}[\mathbf{R}]\{\mathbf{B}\}, \tag{46}$$

where

$$[\mathbf{P}]=\left[\begin{array}{c|c}\mathbf{0} & \hat{\mathbf{G}}\\ \hline \hat{\mathbf{K}} & \hat{\mathbf{C}}\end{array}\right], \qquad [\mathbf{R}]=\left[\begin{array}{c|c}\mathbf{0} & \mathbf{0}\\ \hline \hat{\mathbf{K}}^* & \hat{\mathbf{C}}^*\end{array}\right],$$

and where $\{\mathbf{Y}(0)\}=\{\mathbf{y}_1(0) \quad \mathbf{y}_2(0) \quad \cdots \quad \mathbf{y}_{2n}(0)\}^{\mathrm{T}}$ and $\{\mathbf{y}_i(0)\}=\{y_i(0) \quad 0 \quad \cdots \quad 0\}^{\mathrm{T}}$, the initial state of $y_i(t)$. The submatrices are defined as

$$[\hat{\mathbf{G}}]=[\mathbf{I}]\otimes[\bar{\mathbf{G}}]^{\mathrm{T}}, \qquad [\hat{\mathbf{K}}]=[-\mathbf{M}^{-1}\mathbf{K}]\otimes[\bar{\mathbf{G}}]^{\mathrm{T}}, \qquad [\hat{\mathbf{C}}]=[-\mathbf{M}^{-1}\mathbf{C}]\otimes[\bar{\mathbf{G}}]^{\mathrm{T}},$$

$$[\hat{\mathbf{K}}^*]=[\hat{\mathbf{G}}][\mathbf{Q}_{21}] \qquad \text{and} \qquad [\hat{\mathbf{C}}^*]=[\hat{\mathbf{G}}][\mathbf{Q}_{22}],$$

where $[\bar{\mathbf{G}}]^{\mathrm{T}}$ is known from equations (18b) or (20b). Equating the coefficients of $[\hat{\mathbf{S}}(t)]^{\mathrm{T}}$ results in a set of linear algebraic equations in $\{\mathbf{B}\}$ such that

$$[\mathbf{I}-\mathbf{Z}]\{\mathbf{B}\}=\{\mathbf{y}(0)\}, \tag{47}$$

where $[\mathbf{Z}]=[\mathbf{P}+\mathbf{R}]$ is a $2nm\times 2nm$ constant matrix, and $\{\mathbf{B}\}$ is a $2nm\times 1$ vector. At this point, we can either directly solve equation (47) or use the partitional inversion technique. The second choice is particularly useful for the case in which the damping matrix $[\mathbf{C}+\mathbf{C}^*(t)]$ is identically zero, when it turns out to be much more efficient. Once vector $\{\mathbf{B}\}$ is determined, the solution is obtained from equation (42).

Note that a similar expansion of matrix $[\mathbf{A}(t)]$ has been used earlier by Sinha *et al.* [27] in the study of scalar periodic systems. Once the $\{\mathbf{B}\}$ are known from the solution of equation (47), the state vector $\mathbf{Y}(t)$ is readily obtained from equation (42).

4.2. DIRECT FORMULATION

First we rewrite equation (34) as

$$\ddot{\mathbf{y}}(t)+[\mathbf{V}+\mathbf{V}^*(t)]\dot{\mathbf{y}}(t)+[\mathbf{U}+\mathbf{U}^*(t)]\mathbf{y}(t)=0, \tag{48}$$

where $[\mathbf{V}+\mathbf{V}^*(t)]=[\mathbf{M}]^{-1}[\mathbf{C}+\mathbf{C}^*(t)]$ and $[\mathbf{U}+\mathbf{U}^*(t)]=[\mathbf{M}]^{-1}[\mathbf{K}+\mathbf{K}^*(t)]$. Integrating equation (48) once yields

$$\dot{\mathbf{y}}(t)-\dot{\mathbf{y}}(0)+\int_0^t \mathbf{V}\mathbf{y}(t')\,\mathrm{d}t+\int_0^t \mathbf{V}^*(t')\dot{\mathbf{y}}(t')\,\mathrm{d}t'+\int_0^t (\mathbf{U}+\mathbf{U}^*(t'))\mathbf{y}(t')\,\mathrm{d}t'=0, \tag{49}$$

where $\dot{\mathbf{y}}(0)=\{\dot{y}_1(0)\ \ \dot{y}_2(0)\ \ \cdots\ \ \dot{y}_n(0)\}^{\mathrm{T}}$ is the vector of initial conditions of $\dot{y}_i(t)$.

Using integration by parts, the fourth term in equation (49) can be expressed as

$$\int_0^t \mathbf{V}^*(t')\dot{\mathbf{y}}(t')\,\mathrm{d}t'=\mathbf{V}^*(t)\mathbf{y}(t)-\mathbf{V}^*(0)\mathbf{y}(0)-\int_0^t \dot{\mathbf{V}}^*(t')\mathbf{y}(t')\,\mathrm{d}t', \tag{50}$$

where the initial vector $\mathbf{y}(0)=\{y_1(0)\ \ y_2(0)\ \ \cdots\ \ y_n(0)\}^{\mathrm{T}}$, $\dot{\mathbf{V}}^*(t)=\mathrm{d}\mathbf{V}^*(t)/\mathrm{d}t$ and $\mathbf{V}^*(0)$ is the initial value of $\mathbf{V}^*(t)$.

At this point, the state vector $\mathbf{y}(t)$ is expanded in terms of Chebyshev polynomials:

$$y_i(t)=\sum_{r=0}^{m-1} b_r^{\prime i}S_r^*(t)=\{\mathbf{S}^*(t)\}^{\mathrm{T}}\{\mathbf{b}^{\prime i}\},\qquad \{\mathbf{b}^{\prime i}\}=\{b_0^{\prime i}\ \ b_1^{\prime i}\ \ \cdots\ \ b_{m-1}^{\prime i}\}^{\mathrm{T}}, \tag{51}$$

or

$$\mathbf{y}(t)=\{\hat{\mathbf{S}}(t)\}^{\mathrm{T}}\{\mathbf{B}'\}, \tag{52}$$

where $[\hat{\mathbf{S}}(t)]^{\mathrm{T}}=[\mathbf{I}]\otimes\{\mathbf{S}(t)\}^{\mathrm{T}}$ and $\{\mathbf{B}'\}=\{\mathbf{b}'^1\ \ \mathrm{b}'^2\ \ \cdots\ \ \mathrm{b}'^n\}^{\mathrm{T}}$.

Similarly, functions $\mathbf{V}^*(t)$, $\dot{\mathbf{V}}^*(t)$ and $\mathbf{U}^*(t)$ are also expanded in Chebyshev polynomials as

$$\mathbf{V}^*(t)=[\hat{\mathbf{S}}(t)][\mathbf{H}],\qquad \dot{\mathbf{V}}^*(t)=[\hat{\mathbf{S}}(t)][\mathbf{J}],\qquad \mathbf{U}^*(t)=[\hat{\mathbf{S}}(t)][\mathbf{N}], \tag{53}$$

where $[\mathbf{H}]$, $[\mathbf{J}]$ and $[\mathbf{N}]$ are known constant matrices.

Substituting equation (50) in equation (49), we obtain

$$\dot{\mathbf{y}}(t)-\dot{\mathbf{y}}(0)+\int_0^t \mathbf{V}\dot{\mathbf{y}}(t')\,\mathrm{d}t'+\mathbf{V}^*(t)\mathbf{y}(t)-\mathbf{V}^*(0)\mathbf{y}(0)-\int_0^t \dot{\mathbf{V}}^*(t')\mathbf{y}(t')\,\mathrm{d}t'$$
$$+\int_0^t (\mathbf{U}+\mathbf{U}^*(t'))\mathbf{y}(t')\,\mathrm{d}t'=0. \tag{54}$$

Integrating equation (54) term by term and utilizing the results given in section 2, we have

$$\int_0^t \{\dot{\mathbf{y}}(t')-\dot{\mathbf{y}}(0)\}\,\mathrm{d}t'=\mathbf{y}(t)-\mathbf{y}(0)-\int_0^t \dot{\mathbf{y}}(0)\,\mathrm{d}t'$$
$$=[\hat{\mathbf{S}}(t)]^{\mathrm{T}}\{\mathbf{B}'\}-[\hat{\mathbf{S}}(t)]^{\mathrm{T}}\{\mathbf{y}_0\}-[\hat{\mathbf{S}}(t)]^{\mathrm{T}}\{\hat{\mathbf{y}}_0\}, \tag{55}$$

$$\int_0^t\int_0^t \mathbf{V}\dot{\mathbf{y}}(t')\,\mathrm{d}t'=[\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\hat{\mathbf{V}}]\{\mathbf{B}'\}-[\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\hat{\mathbf{V}}]\{\mathbf{y}_0\}, \tag{56}$$

$$\int_0^t \mathbf{V}^*(t')\mathbf{y}(t')\,\mathrm{d}t'-\int_0^t \mathbf{V}^*(0)\mathbf{y}(0)\,\mathrm{d}t'-\int_0^t\int_0^t \dot{\mathbf{V}}^*(t')\mathbf{y}(t')\,\mathrm{d}t'$$
$$=[\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\hat{\mathbf{V}}^*]\{\mathbf{B}'\}-[\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\hat{\mathbf{V}}_0]\{\mathbf{y}_0\}-[\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\tilde{\mathbf{V}}^*]\{\mathbf{B}'\}, \tag{57}$$

and

$$\int_0^t \int_0^t \mathbf{U}\mathbf{y}(t')\,\mathrm{d}t' = [\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\bar{\mathbf{U}}]\{\mathbf{B}'\}, \tag{58}$$

$$\int_0^t \int_0^t \mathbf{U}^*(t')\mathbf{y}(t')\,\mathrm{d}t' = [\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\bar{\mathbf{U}}^*]\{\mathbf{B}'\}, \tag{59}$$

where

$$\{\mathbf{y}_0\} = \{y_1(0) \quad y_2(0) \quad \cdots \quad y_n(0)\}^{\mathrm{T}}, \qquad \{\mathbf{y}_i(0)\} = \{y_i(0) \quad 0 \quad \cdots \quad 0\}^{\mathrm{T}},$$

$$\{\hat{\mathbf{y}}_0\} = [\hat{\mathbf{G}}]\{\dot{y}_1(0) \quad \dot{y}_2(0) \quad \cdots \quad \dot{y}_n(0)\}^{\mathrm{T}}, \qquad \{\dot{\mathbf{y}}_i(0)\} = \{\dot{y}_i(0) \quad 0 \quad \cdots \quad 0\}^{\mathrm{T}},$$

$$[\hat{\mathbf{V}}] = [\mathbf{V}]\otimes[\bar{\mathbf{G}}]^{\mathrm{T}}, \qquad [\hat{\mathbf{V}}_0] = [\mathbf{V}^*(0)]\otimes[\bar{\mathbf{G}}]^{\mathrm{T}},$$

$[\hat{\mathbf{V}}^*] = [\hat{\mathbf{G}}][\bar{\mathbf{Q}}_1]$, $\quad [\bar{\mathbf{Q}}_1]$ is the product matrix of $\mathbf{V}^*(t)\mathbf{y}(t)$,

$[\bar{\mathbf{V}}^*] = [\bar{\mathbf{G}}][\bar{\mathbf{Q}}_2]$, $\quad [\bar{\mathbf{Q}}_2]$ is the product matrix of $\dot{\mathbf{V}}^*(t)\mathbf{y}(t)$,

$[\bar{\mathbf{U}}] = [\mathbf{U}]\otimes[\bar{\mathbf{G}}^{\mathrm{T}}]^2$,

$[\bar{\mathbf{U}}^*] = [\bar{\mathbf{G}}][\bar{\mathbf{Q}}_3]$, $\quad [\mathbf{Q}_3]$ is the product matrix of $\mathbf{U}^*(\mathrm{t})\mathbf{y}(t)$,

and

$$[\hat{\mathbf{G}}] = [\mathbf{I}]\otimes[\bar{\mathbf{G}}]^{\mathrm{T}}, \qquad [\bar{\mathbf{G}}] = [\mathbf{I}]\otimes[\bar{\mathbf{G}}^{\mathrm{T}}]^2.$$

Using the results from equations (55)–(59), we obtain

$$\begin{aligned}
&[\hat{\mathbf{S}}(t)]^{\mathrm{T}}\{\mathbf{B}'\} - [\hat{\mathbf{S}}(t)]^{\mathrm{T}}\{\mathbf{y}_0\} - [\hat{\mathbf{S}}(t)]^{\mathrm{T}}\{\hat{\mathbf{y}}_0\} + [\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\hat{\mathbf{V}}]\{B'\} - [\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\hat{\mathbf{V}}]\{\mathbf{y}_0\} \\
&\quad + [\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\hat{\mathbf{V}}^*]\{\mathbf{B}'\} - [\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\hat{\mathbf{V}}_0]\{\mathbf{y}_0\} - [\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\bar{\mathbf{V}}^*]\{\mathbf{B}'\} + [\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\bar{\mathbf{U}}]\{\mathbf{B}'\} \\
&\quad + [\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\bar{\mathbf{U}}^*]\{\mathbf{B}'\} = 0.
\end{aligned} \tag{60}$$

Comparing coefficients of $[\hat{\mathbf{S}}(t)]^{\mathrm{T}}$ on both sides of the above equation yields a set of linear equations in $\{\mathbf{B}'\}$ given by

$$[\mathbf{I} + \hat{\mathbf{V}} + \hat{\mathbf{V}}^* - \bar{\mathbf{V}}^* + \bar{\mathbf{U}} + \bar{\mathbf{U}}^*]\{\mathbf{B}'\} = \{\mathbf{y}_0\} + \{\hat{\mathbf{y}}_0\} + \{\hat{\mathbf{V}}_0\mathbf{y}_0\} + \{\hat{\mathbf{V}}\mathbf{y}_0\}. \tag{61}$$

The matrix on the left side of equation (61) has a dimension of $nm \times nm$, which is half the size of the matrix in equation(47) of section 4.1. Therefore, vector $\{\mathbf{B}'\}$ is obtained in a much more efficient fashion as compared to the "state space formulation" approach. $\mathbf{y}(t)$ is readily obtained from equation (52), once $\{\mathbf{B}'\}$ is known.

It is also observed from equation (54) that the velocity vector can be expressed as

$$\begin{aligned}
\dot{\mathbf{y}}(t) = \dot{\mathbf{y}}(0) &- [\mathbf{V}]\{\mathbf{y}(t) - \mathbf{y}(0)\} - [\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\bar{\mathbf{Q}}_1]\{\mathbf{B}'\} + [\mathbf{V}^*(0)]\mathbf{y}(0) \\
&+ [\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\hat{\mathbf{G}}][\bar{\mathbf{Q}}_2]\{\mathbf{B}'\} - [\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\hat{\mathbf{U}}]\{\mathbf{B}'\} - [\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\hat{\mathbf{U}}^*]\{\mathbf{B}'\},
\end{aligned} \tag{62}$$

where

$$[\hat{\mathbf{U}}] = [\mathbf{U}]\otimes[\bar{\mathbf{G}}]^{\mathrm{T}},$$

$[\hat{\mathbf{U}}^*] = [\hat{\mathbf{G}}][\bar{\mathbf{Q}}_3]$, $\quad [\bar{\mathbf{Q}}_3]$ is the product matrix of $\mathbf{U}^*(\mathrm{t})\mathbf{y}(t)$.

Since $\{\mathbf{B}'\}$ and $\mathbf{y}(t)$ are known, $\dot{\mathbf{y}}(t)$ is easily obtained. From $\mathbf{y}(t)$ and $\dot{\mathbf{y}}(t)$, we can construct the Floquet transition matrix (FTM) as shown in the next section.

4.3. COMPUTATION OF THE FLOQUET TRANSITION MATRIX AND STABILITY ANALYSIS

As discussed in section 3, for stability analysis of periodic systems, one needs to find the transition matrix $[\boldsymbol{\varphi}(T)]$ associated with the linear system given by equation (37). From equation (47), a set of $2n$ $\{\mathbf{B}\}_i$s are obtained for the $2n$ initial conditions: $\mathbf{y}_i(0)=(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1)$. It is to be noted that all $\{\mathbf{B}\}_i$s can be determined simultaneously by defining the right side of equation (47) in the matrix form. Then the FTM is given by

$$[\boldsymbol{\varphi}(T)]=[\hat{\mathbf{S}}(T)]^{\mathrm{T}}[\bar{\mathbf{B}}], \tag{63}$$

where $[\bar{\mathbf{B}}]=[\{\mathbf{B}\}_1\ \{\mathbf{B}\}_2\ \cdots\ \{\mathbf{B}\}_{2n}]$.

Similarly, for the "direct formulation" technique, a set of n $\{\mathbf{B}'\}_i$s are calculated from equation (61) with appropriate n initial conditions on $\mathbf{y}_i(0)$ and $\dot{\mathbf{y}}_i(0)$. Since $\mathbf{y}(t)$ is known from equation (62), the $2n\times 2n$ FTM can be constructed rather easily by evaluating $\mathbf{y}(T)$ and $\dot{\mathbf{y}}(T)$.

4.4. LINEAR RESPONSE

The solutions for the "state space formulation" and the "direct formulation" are defined by equations (39) and (51), respectively. These are valid in the interval, $t\in[0, T]$. As pointed out previously, the solution can be easily extended for $t>T$ by utilizing the formula

$$\mathbf{Y}(t)=[\boldsymbol{\varphi}(\eta)][\boldsymbol{\varphi}(T)]^k\mathbf{Y}(0), \tag{64}$$

where $t=kT+\eta$, $\eta\in[0, T]$, $k=1, 2, \ldots$.

5. SOME TYPICAL APPLICATIONS

5.1. APPLICATION TO MATHIEU'S EQUATION

As an example, we first consider the well known problem of the Mathieu equation,

$$\ddot{y}(t)+(v+u\cos(\Omega t))y(t)=0, \qquad t>0, \tag{65}$$

which has a period $T=2\pi/\Omega$: u and v are system parameters. With $\tau=\Omega t/2\pi$, equation (65) transforms to

$$y''(\tau)+p(v+u\cos 2\pi\tau)y(\tau)=0, \qquad p=(2\pi/\Omega)^2, \qquad y'(\tau)=\mathrm{d}y/\mathrm{d}\tau. \tag{66}$$

It is observed that the period is normalized to 1. Substituting $y_1=y$, $y_2=y'$, equation (66) can be rewritten as

$$\mathbf{y}'(\tau)=\left[\begin{array}{c|c} 0 & 1 \\ \hline -p(v+u\cos 2\pi\tau) & 0 \end{array}\right]\mathbf{y}(\tau), \tag{67}$$

where $\mathbf{y}(\tau)=\{y_1\ y_2\}^{\mathrm{T}}$.

Following the procedure outlined in section 4.1 and after some algebraic manipulations, it can be shown that

$$[\mathbf{I}-\mathbf{Z}]=\left[\begin{array}{c|c} [\mathbf{I}] & -[\bar{\mathbf{G}}]^{\mathrm{T}} \\ \hline p[\bar{\mathbf{G}}]^{\mathrm{T}}(v+u[\bar{\mathbf{Q}}]) & [\mathbf{I}] \end{array}\right], \tag{68}$$

where $[\bar{\mathbf{G}}]$ is the integration operational matrix and $[\bar{\mathbf{Q}}]$ is the product operational matrix associated with $(\cos 2\pi\tau)y(\tau)$.

Using the above expression for $[\mathbf{I}-\mathbf{Z}]$ in equation (47), the coefficient vector $\{\mathbf{B}\}$ is determined and consequently the FTM, $[\boldsymbol{\varphi}(T)]$ is computed.

For the "direct formulation" approach, equation (61) takes the form

$$[\mathbf{I}+p[\bar{\mathbf{G}}^{\mathrm{T}}]^2(v+u[\bar{\mathbf{Q}}])]\{\mathbf{B}'\}=[\bar{\mathbf{G}}]^{\mathrm{T}}\{\dot{\mathbf{y}}(0)\}+\{\mathbf{y}(0)\}. \quad (69)$$

After $\{\mathbf{B}'\}$ is obtained from the above equation, $y(t)$ is found from equation (52), whereas the velocity $\dot{y}(t)$ is obtained from equation (62) as

$$\dot{y}(t)=-pv\{\mathbf{S}^*(t)\}^{\mathrm{T}}[\bar{\mathbf{G}}]^{\mathrm{T}}\{\mathbf{B}'\}-pu\{\mathbf{S}^*(t)\}^{\mathrm{T}}[\bar{\mathbf{G}}]^{\mathrm{T}}[\bar{\mathbf{Q}}]\{\mathbf{B}'\}+\dot{y}(0). \quad (70)$$

The numerical results for the case of $v=1$, $u=-0{\cdot}32$ and $\Omega=3$ were computed over one period and are summarized in Table 2, which shows solutions of equation (65) by the use of 10 and 12 terms taken in the expansions of $y(t)$, $\dot{y}(t)$ and cos Ωt over the interval $[0, T]$. The results obtained from the shifted Chebyshev polynomials of the first kind and the second kind are tabulated for one period along with those obtained by the IMSL Runge–Kutta (DVERK) method. Both formulations yield the same numerical results. A tolerance figure of 10^{-6} was used in the DVERK routine.

TABLE 2

The solution of Mathieu's equation over one period ($v=1$, $u=-0{\cdot}32$, $\Omega=3$), m = *number of terms*

	Present method (either formulation)				
	Shifted Chebyshev polynomial (first kind)		Shifted Chebyshev polynomial (second kind)		
Time (s)	$m=10$	$m=12$	$m=10$	$m=12$	Runge–Kutta (IMSL–DVERK) method
0·0	1·000005	0·999999	1·000030	0·999997	1·000000
0·1	0·996589	0·996589	0·996582	0·996590	0·996589
0·2	0·986247	0·986242	0·986245	0·986242	0·986242
0·3	0·968621	0·968623	0·968624	0·968623	0·968623
0·4	0·943232	0·943237	0·943233	0·943238	0·943238
0·5	0·909514	0·909514	0·909512	0·909516	0·909516
0·6	0·866922	0·866917	0·866917	0·866917	0·866917
0·7	0·815046	0·815039	0·815042	0·815040	0·815039
0·8	0·753723	0·753721	0·753722	0·753721	0·753721
0·9	0·683112	0·683116	0·683114	0·683117	0·683116
1·0	0·603727	0·603733	0·603730	0·603734	0·603734
1·1	0·516423	0·516426	0·516424	0·516427	0·516426
1·2	0·422343	0·422340	0·422342	0·422341	0·422340
1·3	0·322833	0·322827	0·322830	0·322827	0·322826
1·4	0·219338	0·219333	0·219335	0·219334	0·219333
1·5	0·113291	0·113292	0·113291	0·113293	0·113293
1·6	0·006022	0·006028	0·006025	0·006029	0·006028
1·7	−0·101315	−0·101311	−0·101310	−0·101310	−0·101311
1·8	−0·207773	−0·207776	−0·207771	−0·207775	−0·207776
1·9	−0·312591	−0·312596	−0·312595	−0·312596	−0·312596
2·0	−0·415133	−0·415130	−0·415136	−0·415129	−0·415130
2·1	−0·514776	−0·514785	−0·514749	−0·514786	−0·514784

In Table 3, the results related to the convergence of FTM is reported. For this case $\Omega=2$, which corresponds to the resonant case and therefore is a good choice for the convergence test. The real part of the characteristic exponent was calculated from the FTM by various methods. It was found that the Runge–Kutta results converged with a tolerance of about 10^{-7}. In the "state space" and "direct formulation" methods, two approaches were adopted. In the first approach, the state vector and the coefficients were expanded in

TABLE 3

Convergence of characteristic exponent and CPU time (s) for Mathieu equation ($v=1$, $u=-0{\cdot}32$, $\Omega=2$); α = real part of the larger characteristic exponent

Number of terms	Result from the Shifted Chebyshev polynomials of the first kind						Runge–Kutta (IMSL–DVERK) method	
	State space formulation				Direct formulation, one step [0, T]			
	One step [0, T]		Two steps [0, T/2], [T/2, T]					
	α	CPU time	α	CPU time	α	CPU time	α	CPU time
8	0·0795592	0·128	0·0797723	0·252	0·0795592	0·084	0·0797617	0·125
10	0·0797818	0·222	0·0797613	0·434	0·0797818	0·145		
12	0·0797602	0·346	0·0797617	0·675	0·0797602	0·223		
14	0·0797607	0·522	0·0797617	1·043	0·0797607	0·337		
16	0·0797616	0·746			0·0797616	0·476		
20	0·0797617	1·352			0·0797617	0·626		
30	0·0797617	6·910			0·0797617	1·143		

the entire interval of $[0, T]$, while in the second, the expansions were done in the stages of $[0, T/2]$ and $[T/2, T]$. The first approach is labeled as the "one-step" and the second one is called the "two-step" method. The results from the "direct formulation" method are also included in Table 3 where the entire period $[0, T]$ was used in the expansions. The CPU time for each technique is also tabulated for comparison.

It is noticed that the results from the two formulations are exactly the same: however, the CPU times are significantly different due to the difference in the number of equations that one has to solve. For the "state space formulation", the "two-step" approach takes only 12 terms to achieve the convergence, as opposed to 20 terms taken by the "one-step" approach. Nevertheless, the computational efficiencies are about the same, as indicated by the CPU times. All computations were carried out on a Harris 8685 system in double precision.

The second example is selected to test the computational efficiency of the proposed technique for moderately large periodic systems. ·

5.2. THE CLAMPED–CLAMPED COLUMN SUBJECTED TO AN AXIAL PERIODIC LOAD

Consider the parametric response of an uniform elastic clamped–clamped column which is under a periodic axial load, as shown in Figure 1. The equation of motion is well known to be

$$m\frac{\partial^2 y}{\partial t^2}+EI\frac{\partial^4 y}{\partial x^4}+(P_1+P_2\cos\Omega)\frac{\partial^2 y}{\partial x^2}=0, \tag{71}$$

where EI is the bending stiffness of the column, P_1 is the constant part of the axial load, P_2 is the amplitude of the periodic axial load, Ω is the circular frequency and m is the mass of the column per unit length. The boundary conditions are

$$y(x, t)=\partial y(x, t)/\partial x=0 \qquad \text{at } x=0, L. \tag{72}$$

This problem has been investigated by several authors [14, 30] in the past, and a variety of techniques have been used in the analysis. Following Friedmann *et al.* [14], the spatial properties of equation (71) can be discretized through an application of the finite element

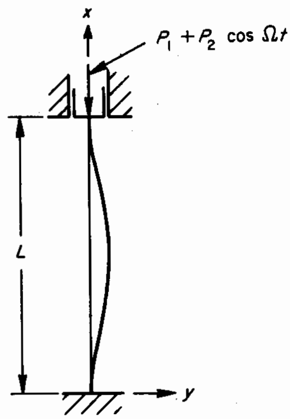

Figure 1. A clamped–clamped column subjected to a periodic load.

method, and a system of second order ordinary differential equations is obtained. Then equation (71) takes the form of a set of n coupled Mathieu equations, given by

$$\mathbf{M}\ddot{\mathbf{q}} + [\mathbf{K} + \mathbf{K}_G(t)]\mathbf{q} = 0, \tag{73}$$

where

$$[\mathbf{M}] = ml[\mathbf{M}'], \qquad [\mathbf{K}] = (EI/l^3)[\mathbf{K}'],$$
$$[\mathbf{K}_G(t)] = [(P_1 + P_2 \cos \Omega t)/l][\mathbf{K}'_G],$$

and $\{\mathbf{q}\} = \{q_1 \ q_2 \ \cdots \ q_n\}^{\mathrm{T}}$ is the state variable. $[\mathbf{M}']$, $[\mathbf{K}']$ and $[\mathbf{K}'_G]$ are the global dimensionless square matrices the elements of which are determined by the boundary conditions, and l is the element length. The axial load is introduced through the concept of the geometric stiffness matrix, $\mathbf{K}_G(t)$, associated with uniform elements of length l. Mass and geometric stiffness matrices for the element are given in Appendix B. The global system (73) is found from the element properties, and the transformation matrix is discussed in reference [14].

The global matrices $[\mathbf{M}']$, $[\mathbf{K}']$ and $[\mathbf{K}'_G]$ are obtained by dividing the column into j elements of equal length, and therefore the dimension of FTM associated with this system depends on j. The "direct formulation" technique can be applied to equation (73): however, for the "state space formulation", equation (73) is transformed to

$$\dot{\mathbf{Y}}(t) = [\mathbf{W} + \mathbf{A}(t)]\mathbf{Y}(t), \tag{74}$$

where $\mathbf{Y}(t) = \{\mathbf{q}_1 \ \mathbf{q}_2\}^{\mathrm{T}}$, $\mathbf{q}_1 = \mathbf{q}$, $\mathbf{q}_2 = \dot{\mathbf{q}}$, and

$$\mathbf{W} = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{I} \\ \hline -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{array}\right] \quad \text{and} \quad \mathbf{A}(t) = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline -\mathbf{M}^{-1}\mathbf{K}_G(\tau) & \mathbf{0} \end{array}\right].$$

It is observed from equations (73) and (74) that the period of the periodic coefficient is $T = \Omega/2\pi$. In order to normalize the expansion interval to [0, 1], once again a scaling parameter τ was introduced such that $\tau = \Omega t/2\pi$. The state vector as well as the periodic coefficients were expanded in the interval [0, 1] and the Floquet transition matrices were obtained by the procedure described in section 4 for various numbers of elements, i.e., $j =$ 2, 3, 4 and 5.

Numerical results were computed for several typical values of the system parameters. The results for the case of $P_1 = 0$, $P_2/P_{cr} = 0{\cdot}1$, $\Omega/2\omega_2 = 2{\cdot}525$ (where $P_{cr} = 4\pi^2 EI/L^2$), the first Euler buckling load and ω_2 is the second natural frequency of the unloaded fixed–fixed column) are shown in Table 4, as well as in Figure 2.

TABLE 4

CPU time (s) for computation of FTM for the column problem (double precision)

		10-term shifted Chebyshev polynomial				
		State space formulation			Runge–Kutta (IMSL–DVERK) method	
Size of FTM	No. of elements	Direct inversion	Partitional inversion	Direct formulation	Single-pass	N-pass
4	2	1·55	1·05	0·63	3·65	7·20
8	3	10·02	7·61	2·46	13·20	17·63
12	4	44·54	24·75	5·27	49·87	55·87
16	5	78·46	58·19	12·65	107·77	122·60

These results are based on a 10-term expansion of the shifted Chebyshev polynomials of the first kind. The polynomials of the second kind yield almost identical results. The eigenvalues of the FTM computed by the 10-term approximation and the Runge–Kutta (DVERK) routine matched up to five significant digits, which is more than sufficient for practical purposes. In Table 4 is shown a comparison of CPU times taken by the "state space formulation", the "direct formulation" and the Runge–Kutta "single-pass" and "N-pass" schemes in the computation of various sizes of FTM. In the "state space formulation" it was found that equation (47) may be solved in a much more efficient fashion by utilizing the results from a partitional inversion lemma (see Appendix C). Partitional inversion is efficient only in this particular problem, due to the fact that the two block-diagonal matrices appearing in $[\mathbf{I}-\mathbf{Z}]$, equation (68), happen to be identity matrices. This would be the case for all systems for which the damping matrix $[\mathbf{C}+\mathbf{C}^*(t)]$ is identically zero.

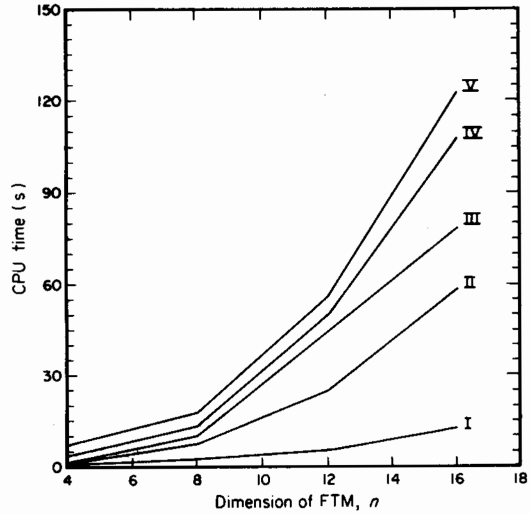

Figure 2. A comparison of CPU times taken by various methods in the computation of FTM. I, direct formulation; II, partitional inversion (state space); III, direct inversion (state space); IV, Runge–Kutta (single pass); V, Runge–Kutta (N-pass).

It is also observed from Table 4 that the "single-pass" scheme does not turn out to be significantly more efficient than the "N-pass" scheme in this case. It was reported by Gaonkar *et al.* [15] that the "single-pass" scheme could save up to 50–60% of computer time as compared to the "N-pass" scheme. This is found to be true only if all the computations are carried out with single precision arithmetic. All results reported in this paper are based on double precision arithmetic.

A plot of CPU time taken by various methods as a function of the size of the Floquet transition matrix is shown in Figure 2.

6. ERROR ANALYSIS

In this section a quantitative estimate of the error expected in such a formulation is presented. Consider the expansion of an arbitrary continuous scalar function $f(t)$ into a series of Chebyshev polynomials as

$$f_{ex}(t)=\sum_{r=0}^{\infty} a_r S_r^*(t), \tag{75}$$

where $f_{ex}(t)$ simply indicates that $f(t)$ has an exact representation if all the terms on the right side are included in the expansion. Equation (75) can be rewritten as

$$f_{ex}(t)=\bar{f}(t)+\varepsilon_m, \tag{76}$$

where

$$\bar{f}(t)=\sum_{r=0}^{m-1} a_r S_r^*(t) \qquad \text{and} \qquad \varepsilon_m=\sum_{r=m}^{\infty} a_r S_r^*(t). \tag{77}$$

It is known [32] that the remainder ε_m satisfies the inequality

$$|\varepsilon_m| \leqslant |a_m S_m^*(t)|, \tag{78}$$

and

$$a_m=\frac{f^{(m)}(t')}{m!\,Z_m}, \qquad 0 \leqslant t' \leqslant 1, \tag{79}$$

where $f^{(m)}(t')$ denotes the mth derivative and Z_m is a constant.

If we define $f^*(t)$ as

$$f^*(t)=\sum_{r=0}^{m-1} a_r S_r^*(t)+a_m S_m^*(t)=\bar{f}(t)+a_m S_m(t), \tag{80}$$

then, from equations (76)–(80), it is seen that the error $E(t)$ satisfies

$$E(t)\equiv|f_{ex}(t)-\bar{f}(t)| \leqslant |f^*(t)-\bar{f}(t)|. \tag{81}$$

This idea can be extended to obtain an error estimate for the state vector $\mathbf{Y}(t)$ appearing in the periodic system given by equation (37). The analysis is partially an extension of the developments suggested by Chen and Chen [31] for the case of systems with constant coefficients. Without loss of generality, the constant matrix [W] in equation (37) can be set to a null matrix, so that we have

$$\dot{\mathbf{Y}}(t)=\mathbf{A}(t)\mathbf{Y}(t). \tag{82}$$

Let $\bar{\mathbf{Y}}(t)$ represent the m-term Chebyshev polynomials solution of equation (82) given by

$$\bar{\mathbf{Y}}(t)=[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}\{\mathbf{B}\}. \tag{83}$$

$\mathbf{A}(t)$ has the same representation as given in equation (43). Following the procedure described in section 4.1, we obtain

$$[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}\{\mathbf{B}\}-[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}\{\mathbf{Y}(0)\}=[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}[\hat{\mathbf{G}}][\bar{\mathbf{Q}}]\{\mathbf{B}\}. \tag{84}$$

At this point, we define $\mathbf{Y}^*(t)$ as the solution of equation (82) obtained via an $(m+1)$-term Chebyshev expansion such that

$$\mathbf{Y}^*(t)=[\hat{\mathbf{S}}(t)]^{\mathrm{T}}\{\mathbf{B}^*\}+S_m^*(t)\{\mathbf{b}_m^*\},\qquad \mathbf{A}(t)=[\hat{\mathbf{S}}(t)]^{\mathrm{T}}[\mathbf{D}]+S_m^*(t)[\mathbf{d}_m], \tag{85, 86}$$

where $\{\mathbf{b}_m^*\}$ is a $2n$ vector, and the elements of matrix $[\mathbf{d}_m]$ are the $(m+1)$th terms in the expansions of the elements $A_{ij}(t)$.

From equations (85) and (86), we obtain

$$\mathbf{A}(t)\mathbf{Y}^*(t)=[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}[\bar{\mathbf{Q}}]\{\mathbf{B}^*\}+[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}[\mathbf{Q}^*]\{\mathbf{b}_m^*\}+S_m^*(t)([\mathbf{d}_m]\{\mathbf{b}_0^*\}+[\mathbf{d}_0]\{\mathbf{b}_m^*\}), \tag{87}$$

where

$$[\mathbf{Q}^*]=\begin{bmatrix}\bar{\mathbf{d}}^{11} & \bar{\mathbf{d}}^{12} & \cdots & \bar{\mathbf{d}}^{ij}\\ \bar{\mathbf{d}}^{21} & & \cdots & \\ \vdots & \vdots & \cdots & \vdots\\ \bar{\mathbf{d}}^{il} & & \cdots & \bar{\mathbf{d}}^{ij}\end{bmatrix},\qquad i,j=1,2,3,\ldots,2n,$$

$\{\bar{\mathbf{d}}^{ij}\}=\frac{1}{2}\{d_m^{ij}\ d_{m-1}^{ij}\ \cdots\ d_1^{ij}\}^{\mathrm{T}}$ which are $A_{ij}(t)$'s expansion coefficients, $\{\mathbf{b}_0^*\}=\{b_0^1\ b_0^2\ \cdots\ b_0^n\}^{\mathrm{T}}$, and

$$[\mathbf{d}_r]=\begin{bmatrix}d_r^{11} & d_r^{12} & \cdots & d_r^{ij}\\ d_r^{21} & & \cdots & \\ \vdots & \vdots & \cdots & \vdots\\ d_r^{il} & & \cdots & d_r^{ij}\end{bmatrix},\qquad i,j=1,2,3,\ldots,2n.$$

d_r^{ij} is the $(r+1)$th coefficient in the expansion of $A_{ij}(t)$.

Integrating equation (87) yields

$$\int_0^t \mathbf{A}(t')\mathbf{Y}^*(t')\,\mathrm{d}t=[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}[\hat{\mathbf{G}}][\bar{\mathbf{Q}}]\{\mathbf{B}^*\}+[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}[\hat{\mathbf{G}}]\{\mathbf{Q}^*\}\{\mathbf{b}_m^*\}$$
$$+[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}\{\mathbf{R}(m,t)\}, \tag{88}$$

where

$$[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}\{\mathbf{R}(m,t)\}=\mathbf{K}\int_0^t S_m^*(t')\,\mathrm{d}t'+S_m^*(t)([\mathbf{Q}^{**}]\{\mathbf{B}^*\}+[\mathbf{d}_1]\{\mathbf{b}_m^*\})/4m,$$

$$\mathbf{K}=([\mathbf{d}_m]\{\mathbf{b}_0^*\}+[\mathbf{d}_0]\{\mathbf{b}_m^*\}),$$

$$[\mathbf{Q}^{**}] = \begin{bmatrix} \mathbf{q}^{11} & \mathbf{q}^{12} & \cdots & \mathbf{q}^{lj} \\ \mathbf{q}^{21} & & \cdots & \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{q}^{il} & & \cdots & \mathbf{q}^{ij} \end{bmatrix}, \qquad i, j = 1, 2, 3, \ldots, 2n,$$

and $\{\mathbf{q}^{ij}\}$ is the last row vector of $[\mathbf{Q}^{ij}]$, which is the product matrix of $A_{ij}(t)y(t)$.

Substituting equations (85), (86) and (88) in the integral equation (45) (with $\mathbf{W} \equiv 0$), we obtain

$$[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}\{\mathbf{B}^*\} + S_m^*(t)\{\mathbf{b}_m^*\} - [\hat{\mathbf{s}}^*(t)]^{\mathrm{T}}\{\mathbf{Y}(0)\}$$
$$= [\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}[\hat{\mathbf{G}}][\bar{\mathbf{Q}}]\}\mathbf{B}^*\} + [\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}[\hat{\mathbf{G}}][\mathbf{Q}^*]\{\mathbf{b}_m^*\} + [\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}\{\mathbf{R}(m, t)\}. \tag{89}$$

Subtracting equation (84) from equation (89) and equating the coefficients of $[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}$ yields

$$[\mathbf{I} - \hat{\mathbf{G}}\mathbf{Q}]\{\mathbf{B}^* - \mathbf{B}\} = [\hat{\mathbf{G}}][\mathbf{Q}^*]\{\mathbf{b}_m^*\} + \{\bar{\mathbf{R}}(m, t)\}, \tag{90}$$

where the vector $\{\bar{\mathbf{R}}(m, t)\}$ is defined by

$$[\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}\{\bar{\mathbf{R}}(m, t)\} = [\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}\{\mathbf{R}(m, t)\} - S_m^*(t)\{\mathbf{b}_m^*\}. \tag{91}$$

At this point, an expression for the solution difference $\mathbf{Y}^*(t) - \bar{\mathbf{Y}}(t)$ can be obtained from equations (83) and (85), as

$$\mathbf{Y}^*(t) - \bar{\mathbf{Y}}(t) = [\hat{\mathbf{S}}^*(t)]^{\mathrm{T}}\{\mathbf{B}^* - \mathbf{B}\} + S_m^*(t)\{\mathbf{b}_m^*\}. \tag{92}$$

The right side can be evaluated because $\{\mathbf{B}^* - \mathbf{B}\}$ is known from equation (90). As anticipated, the solution difference is directly proportional to the magnitude of the last coefficient vector $\{b_m^*\}$ in the representation of $\mathbf{Y}^*(t)$. As $m \to \infty$, $\{\mathbf{b}_m^*\} \to 0$ (cf. equation (79)) and also $\{\mathbf{B}^* - \mathbf{B}\} \to 0$. Thus convergence is guaranteed. Equation (92) provides a quantitative estimate of the difference between each component $y_i^*(t)$ and $\bar{y}_i(t)$. Therefore, from inequality (81), an error bound for each $y_i(t)$ is given by

$$|E_i(t)| \leqslant |y_i^*(t) - \bar{y}_i(t)|, \qquad i = 1, 2, 3, \ldots, 2n. \tag{93}$$

These results imply that an error bound for the $(m-1)$-term approximation can always be determined once the solution with an m-term expansion has been computed. It should be noted that $E_i(t)$ can take positive as well as negative values.

As an example, consider the scalar periodic system given by

$$\dot{y}(t) = (\cos 3t)y(t), \qquad y(0) = 1. \tag{94}$$

The exact solution, of course, is

$$y(t) = \exp\left(\tfrac{1}{3}\sin 3t\right). \tag{95}$$

An approximate solution with a 12-term expansion was computed, and the error between

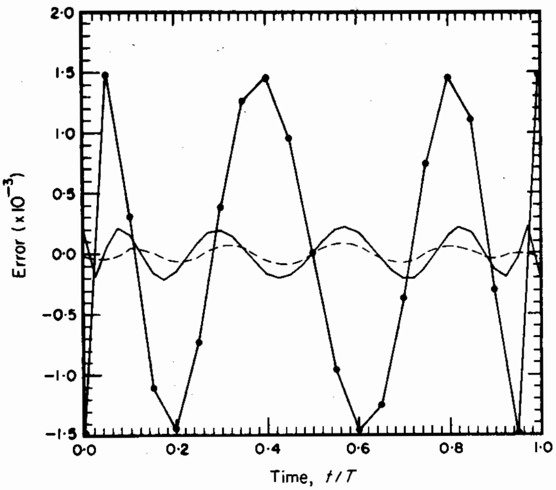

Figure 3. Error bounds for the scalar equation (94): ——, error bound with 12 terms; – – –, actual error with 12 terms; —●—●—, error bound with eight terms.

exact and approximate solutions is plotted in Figure 3, as indicated by the dotted line. In order to compute the error bound for the 12-term approximation, a 13-term solution was obtained, because the knowledge of b^*_{13} is explicitly required in the computation of $E(t)$. The error bound is shown by the solid line in Figure 3. An error bound for the eight-term expansion is also shown in the figure for comparison. It was observed that the major contribution in $E(t)$ came from the $\{\mathbf{b}^*_m\}$ term rather than the $\{\mathbf{B}^* - \mathbf{B}\}$ term.

7. DISCUSSION AND CONCLUSIONS

In this study, an efficient numerical technique suitable for the analysis of linear systems with periodic parameters has been presented. It has been shown that by expressing the state vector and the periodic coefficient matrix as finite sums of shifted Chebyshev polynomials, the original dynamic equations can be reduced to a set of linear algebraic equations. Two algorithms, the "state space formulation" and the "direct formulation", have been suggested. The former is suitable for a set of equations written in the state space form, whereas the latter is directly applicable to a system of second order equations. The main advantage of the "direct formulation" is that it yields only half the number of equations, as compared to the "state space formulation", and therefore it is much more efficient. The solution technique has been successfully combined with Floquet theory to study the stability characteristics of general periodic systems.

The results from the analysis of Mathieu's equation show that the shifted Chebyshev polynomials of both kinds provide very accurate solutions with 10 or 12 terms. This is indicated by Table 2. Since the characteristic exponent of an FTM determines the stability of the system, the new algorithm was applied to study the convergence of the larger exponent resulting in the Mathieu's equation. It is observed from Table 3 that convergence is achieved only if more than 16 terms are used in the expansion. It is also noted that the "direct formulation" is much more efficient than the "state space formulation"; however, neither of these are better than the IMSL–DVERK subroutine.

The computational superiority of the suggested technique is obvious from the analysis of the column problem. It is indicated in Figure 2 that both formulations are much more efficient than the conventional Runge–Kutta algorithm applied either in an "N-pass" or "single-pass';' scheme. In particular, the "direct formulation" approach is clearly several times faster than all other techniques. The computational efficiency is found to increase as the dimension of the FTM becomes larger and larger. Calculations with several other values of P_1, P_2 and Ω were also performed. It was observed that for larger values of P_1, and P_2 and small values of Ω, more terms were needed to maintain the same level of accuracy and the algorithm became relatively less efficient. On the other hand if P_1 and P_2 were chosen smaller and Ω larger, then the numerical efficiency increased. Such variations were also noted for the Runge–Kutta scheme.

The error analysis, presented in section 6, not only provides a mathematical foundation for the technique but also a simple approach for determining the error bound.

It is concluded that the numerical technique suggested in this paper is certainly a viable alternate method for the analysis of large-scale linear periodic systems. The primary reason for computational advantage lies in the fact that much of the information needed to set up the problem can be stored in the computer in advance. the "product" and "integration" matrices associated with the shifted Chebyshev polynomials can be readily constructed from the general expressions and do not require any computation. In general, the terms periodic in $\mathbf{A}(t)$ have the forms sin $(n\pi t/T)$ and/or cos $(n\pi t/T)$. The expansions of these quantities can be made a part of the subroutine, and one does not have to compute the expansion coefficients each time. The entire computation process can be automated rather easily. From the error bound analysis, it appears that it would be possible to incorporate a tolerance parameter in the computer program to ensure a desired accuracy. However, no such attempt was made in this study. Since the application of finite elements or other discretization procedures generally leads to a large set of equations, it is anticipated that this technique will be very useful in the analysis of elastic and aeroelastic rotating systems where the coefficients are always periodic.

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APPENDIX A

A.1

Any funciton which is analytic in the interval [0, 1] may be expanded in a series of shifted Chebyshev polynomials. The function can also be expanded in an arbitrary interval $[t_1, t_2]$ by means of a linear transformation

$$t^* = (t - t_1)/\beta, \qquad t = \beta t^* + t_1, \tag{A1}$$

where $\beta = t_2 - t_1$. Thus a new set of shifted Chebyshev polynomials is obtained as

$$S_r^*(t^*) = S_r^*((t - t_1)/\beta). \tag{A2}$$

A function analytic in $[t_1, t_2]$ then has the representation

$$f(t) = \sum_{r=0}^{\infty} a_r S_r^*(t^*), \tag{A3}$$

where

$$a_r = \frac{1}{\delta} \int_0^1 w(t^*) f(\beta t^* + t_1) S_r^*(t^*) \, \mathrm{d}t^*, \qquad r = 0, 1, 2, 3, \ldots, \tag{A4}$$

and $w(t^*)$ is the appropriate weight function.

For shifted Chebyshev polynomials of the first kind,

$$w(t^*) = (t^* - t^{*2})^{-1/2}$$

and

$$\delta = \begin{cases} \pi/2, & r \neq 0, \\ \pi, & r = 0, \end{cases}$$

whereas for the second kind, $w(t^*) = (t^* - t^{*2})^{1/2}$ and

$$\delta = \pi/8, \qquad r = 0, 1, 2, \ldots .$$

Note that $S_r^*(t)$ has been used to represent the shifted Chebyshev polynomials of either the first or second kind.

A.2

An integration matrix in an arbitrary interval $[t_1, t_2]$ can also be obtained by means of a linear transformation as shown in section 1 of this appendix. From equation (A1), since $\mathrm{d}t^* = \mathrm{d}t/\beta$, we have

$$\int_{t_1}^{t_2} S_r^*(t) \, \mathrm{d}t = \beta \int_0^1 S_r^*(t^*) \, \mathrm{d}t^*. \tag{A5}$$

From section 2.2, we can obtain the operational matrix of integration of shifted Chebyshev polynomials as

$$\int_0^t \mathbf{S}^*(\tau) \, \mathrm{d}\tau = \beta [\bar{\mathrm{G}}] \{\mathbf{S}^*(t)\}, \qquad t \in [0, 1],$$

where $\{\mathbf{S}^*(t)\}$ is $\{S_0^*(t) \;\; S_1^*(t) \;\; S_2^*(t) \;\; \cdots \;\; S_{m-1}^*(t)\}^{\mathrm{T}}$.

APPENDIX B

The element mass, stiffness and geometric stiffness properties are given as follows:

$$[\mathbf{k}]=EI/l^3\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}, \tag{B1}$$

$$[\mathbf{m}]=ml/420\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}, \tag{B2}$$

$$[\mathbf{k}_G]=(P_1+P_2\cos\Omega t)/l\begin{bmatrix} 6/5 & 1/10 & -6/5 & 1/10 \\ 1/10 & 2/15 & -1/10 & -1/30 \\ -6/5 & -1/10 & 6/5 & -1/10 \\ 1/10 & -1/30 & -1/10 & 2/15 \end{bmatrix}. \tag{B3}$$

APPENDIX C

Partitional inversion is as follows:

$$[\mathbf{A}]=\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}, \tag{C1}$$

$$[\mathbf{A}^{-1}]=\begin{bmatrix} (\mathbf{A}_1-\mathbf{A}_2\mathbf{A}_4^{-1}\mathbf{A}_3)^{-1} & -\mathbf{A}_1^{-1}\mathbf{A}_2(\mathbf{A}_4-\mathbf{A}_3\mathbf{A}_1^{-1}\mathbf{A}_2)^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3(\mathbf{A}_1-\mathbf{A}_2\mathbf{A}_4^{-1}\mathbf{A}_3)^{-1} & (\mathbf{A}_4-\mathbf{A}_3\mathbf{A}_1^{-1}\mathbf{A}_2)^{-1} \end{bmatrix}. \tag{C2}$$

For $\mathbf{A}_1=\mathbf{A}_4=[\mathbf{I}]$,

$$[\mathbf{A}^{-1}]=\begin{bmatrix} (\mathbf{I}-\mathbf{A}_2\mathbf{A}_3)^{-1} & -\mathbf{A}_2(\mathbf{I}-\mathbf{A}_3\mathbf{A}_2)^{-1} \\ -\mathbf{A}_3(\mathbf{I}-\mathbf{A}_2\mathbf{A}_3)^{-1} & (\mathbf{I}-\mathbf{A}_3\mathbf{A}_2)^{-1} \end{bmatrix}. \tag{C3}$$

APPENDIX D: NOMENCLATURE

a_r, a'_r	the $(r+1)$th Chebyshev polynomials expansion coefficients
$[\mathbf{A}(t)]$	periodic matrix
A_{ij}	element of matrix $[\mathbf{A}(t)]$
b_r, b'_r	the $(r+1)$th Chebyshev polynomials expansion coefficients
b^i_r	the $(r+1)$th coefficients in the expansion of $y_i(t)$
$\{\mathbf{b}\}$, $\{\mathbf{b}'\}$	vectors of expansion coefficients defined by equations (26b) and (27b), respectively
$\{\mathbf{b}^i\}$	$\{b^i_0\ b^i_1\ \cdots\ b^i_{m-1}\}^{\mathrm{T}}$, vector of expansion coefficient of $y_i(t)$
$\{\mathbf{b}^*_m\}$	defined by equation (85)
$\{\mathbf{B}\}$, $\{\mathbf{B}'\}$	defined by equations (42) and (52), respectively
$\{\underline{\mathbf{B}}^*\}$	defined by equation (85)
$[\bar{\mathbf{B}}]$	defined by equation (63)
$[\mathbf{C}^*(t)]$	periodic damping matrix
$[\mathbf{C}]$	constant damping matrix
d^{ij}_r	the $(r+1)$th coefficient in the expansion of A_{ij}
$[\mathbf{d}_r]$	defined by equation (87)
$[\mathbf{D}]$	constant matrix
$[\mathbf{E}(t)]$	defined by equation (81)

$E_i(t)$	defined by equation (93)
EI	bending rigidity
$f_{ex}(t)$	defined by equation (75)
$f^*(t)$	defined by equation (80)
$\bar{f}(t)$	defined by equation (77)
[F]	matrix defined by equation (33)
[G]	general integration matrix
[G]	defined by equation (18a)
[G′]	defined by equation (20a)
[H]	the m-term Chebyshev polynomials expansion coefficients matrix of $[\mathbf{V}^*(t)]$
[I]	identity matrix
[J]	the m-term Chebyshev polynomials expansion coefficients matrix of $[\dot{\mathbf{V}}^*(t)]$
[k]	element stiffness matrix (Appendix B)
$[\mathbf{k}_G]$	element geometric stiffness matrix (Appendix B)
$[\mathbf{K}_G]$, $[\mathbf{K}'_G]$	system geometric stiffness matrix
$[\mathbf{K}^*(t)]$	periodic stiffness matrix
[K], [K′]	constant stiffness matrices
l	element length
L	column length
[M]	system mass matrix
[m]	element mass matrix (Appendix B)
m	mass per unit length of beam
n	order of FTM
[N]	the m-term Chebyshev polynomials expansion coefficients matrix of $[\mathbf{U}^*(t)]$
P_{cr}	critical buckling load for fixed end column
P_1	constant part of axial load (Figure 1)
P_2	amplitude of the periodic axial load (Figure 1)
[P]	constant matrix
[R]	constant matrix
$[\mathbf{Q}_{21}]$	the product matrix due to $-\mathbf{M}^{-1}\mathbf{K}^*(t)\mathbf{y}_1(t)$
$[\mathbf{Q}_{22}]$	the product matrix due to $-\mathbf{M}^{-1}\mathbf{C}^*(t)\mathbf{y}_2(t)$
[Q]	general product matrix
[Q]	defined by equation (26b)
[Q′]	defined by equation (27b)
[Q*]	defined by equation (87)
[Q**]	defined by equation (88)
$[\mathbf{Q}_1]$	the product matrix of $\mathbf{V}^*(t)\mathbf{y}(t)$
$[\mathbf{Q}_2]$	the product matrix of $\dot{\mathbf{V}}^*(t)\mathbf{y}(t)$
$[\mathbf{Q}_3]$	the product matrix of $\mathbf{U}^*(t)\mathbf{y}(t)$
$[\mathbf{Q}^{ij}]$	the product matrix of $A_{ij}(t)y(t)$
{q}	system generalized co-ordinates vector
$\{\mathbf{q}^{ij}\}$	the last row vector of $[\mathbf{Q}^{ij}]$
$\{\mathbf{q}_1\}$, $\{\mathbf{q}_2\}$	vectors used in state variable representation
$\{\mathbf{R}(m, t)\}$	defined by equation (89)
$\{\bar{\mathbf{R}}(m, t)\}$	defined by equation (91)
$[\hat{\mathbf{S}}(t)]$	defined by equation (41)
$S_r^*(t)$	represents either T_r^* or U_r^*
t	time variable
t_1, t_2	fixed time lengths
t'	dummy variable
T	period
$T_r(t)$	Chebyshev polynomials of the first kind
$T_r^*(t)$	shifted Chebyshev polynomials of the first kind
$[\mathbf{T}^*(t)]$	defined by equation(18b)
[V]	constant matrix, defined by equation (48)
$[\mathbf{V}^*(t)]$	periodic matrix, defined by equation(48)
$U_r(t)$	Chebyshev polynomials of the second kind
$U_r^*(t)$	shifted Chebyshev polynomials of the second kind
$[\mathbf{U}^*(t)]$	defined by equation (20b)
[U]	constant matrix, defined by equation (48)

$[\mathbf{U}^*(t)]$	periodic matrix, defined by equation (48)
$w(t)$	weight function
w_2	the second natural frequency of fixed end column
$[\mathbf{W}]$	constant matrix defined by equation (38)
$y_i(t)$	state variable
$y_i(0)$	initial value of state variable $y_i(t)$
$\dot{y}_i(0)$	initial value of state variable $\dot{y}_i(t)$
$\{\mathbf{y}\}$	system generalized co-ordinates vector
$\{\mathbf{y}_1\}$, $\{\mathbf{y}_2\}$	state vectors defined by equation (35)
$\{\mathbf{Y}(t)\}$	defined by equation (36)
$\{\bar{\mathbf{Y}}(t)\}$	defined by equation (83)
$\{\mathbf{Y}^*(t)\}$	defined by equation (85)
$[\boldsymbol{\varphi}(t)]$	Floquet transition matrix (FTM)
α_i	defined by equation (32)
μ_i	eigenvalues of FTM
ε_m	the remainder defined by equation (77)
δ	defined by equation (15)
Ω	the circular forcing frequency
β	defined by equation (A1)
τ	non-dimensional time variable
$\otimes$	Kronecker product
$\{\dot{\ }\}$	time derivative
$[\ ]^{\mathrm{T}}$	transpose of the quantity [ ]
$[\ ]^{-1}$	inverse matrix