[*Reprinted from* THE AERONAUTICAL JOURNAL OF THE ROYAL AERONAUTICAL SOCIETY, *JANUARY 1994*]

A new approach in the analysis of linear systems with periodic coefficients for applications in rotorcraft dynamics

D.-H. Wu
Department of Mechanical Engineering,
National Pingtung Polytechnic Institute,
Pingtung, Taiwan, ROC
and
S.C. Sinha
Nonlinear Systems Research Laboratory,
Department of Mechanical Engineering,
Auburn University, USA

PRINTED BY MANOR PARK PRESS LIMITED, EDISON ROAD, HAMPDEN PARK, EASTBOURNE, SUSSEX, ENGLAND
AND PUBLISHED BY THE ROYAL AERONAUTICAL SOCIETY, 4 HAMILTON PLACE, LONDON W1V 0BQ, ENGLAND

A new approach in the analysis of linear systems with periodic coefficients for applications in rotorcraft dynamics

D.-H. Wu
Department of Mechanical Engineering,
National Pingtung Polytechnic Institute,
Pingtung, Taiwan, ROC
and
S.C. Sinha
Nonlinear Systems Research Laboratory,
Department of Mechanical Engineering,
Auburn University, USA

ABSTRACT

A numerical technique for the stability analysis of linear mechanical dynamic systems with periodically varying parameters is proposed. The technique is based on representation of the solution vector in terms of Chebyshev polynomials defined over the principal period. Two formulations have been presented. The first formulation is suitable for systems described by state space equations, while the second can be applied directly to a set of second order equations with periodically varying mass, damping and stiffness matrices. As an illustrative example, the flap-lag stability of a multi-bladed rotor is examined. The numerical accuracy and efficiency of the proposed technique is compared with standard numerical codes based on Runge-Kutta, Adams-Moulton and Gear algorithms. The results indicate that the suggested approach is by far the most efficient one, particularly for systems with larger dimensions.

NOTATION

$\mathbf{A}$, $\mathbf{A}^*(t)$	matrices defined in Equation (7)
$\mathbf{B}$, $\mathbf{B}^*(t)$	matrices defined in Equation (8)
$\mathbf{C}^*(t)$, $\dot{\mathbf{C}}^*(t)$	damping matrix and its derivative matrix
C_{d0}	blade profile drag coefficient
C_T	thrust coefficient
$d_l^{ij}(t)$	known Chebyshev coefficients of the mass, damping and stiffness matrices, ($l = 1, \dots, 5$)
$\mathbf{f}$, $\hat{\mathbf{f}}$	vectors as defined in Equation (5) and Equation (12), respectively
F_β, F_ζ	nondimensional aerodynamic loads
$\mathbf{G}$	integration operational matrix
$\mathbf{K}$	stiffness matrix
$\mathbf{M}^*(t)$, $\dot{\mathbf{M}}^*(t)$	mass matrix and their derivatives
$m_{ij}^*(t)$ $\dot{m}_{ij}^*(t)$	elements of mass matrix and their derivatives
n	number of degrees of freedom of the system
p	dimensionless flapping frequency
q	number of Chebyshev polynomials
R	the blade radius
$\hat{R}$	elastic coupling parameter
$\mathbf{s}^T(t)$	vector of shifted Chebyshev polynomials
t	time in seconds
T	period in seconds
$T_k(t)$	shifted Chebyshev polynomial of order k
$\mathbf{W}$, $\hat{\mathbf{W}}$	matrices defined in Equation (5) and Equation (12), respectively
W'	lead-lag frequency squared
Z	stiffness parameter
α, β	unknown Chebyshev vectors
β	flap angle
$\Phi(t)$	state transition matrix
γ	lock number
λ_j	eigenvalues of the FTM
μ	advance ratio
θ	pitch angle
σ	rotor solidity
υ_j	real part of the characteristic exponent
ψ	rotor azimuth angle
ζ	lead-lag angle
ω_β	flap frequency
ω_ζ	lead-lag frequency

Subscripts

C	corresponding to first harmonic cosine function in the Fourier form
NR	corresponding to non-rotating frame
S	corresponding to first harmonic sine function in the Fourier form
0	denotes the constant part in the Fourier form

$$\mathbf{X}_0 = [\dot{\mathbf{y}}_1(0), \dot{\mathbf{y}}_2(0), \dots, \dot{\mathbf{y}}_n(0), \mathbf{y}_1(0), \mathbf{y}_2(0), \mathbf{y}_n(0)]^T$$

where

$$\dot{\mathbf{y}}_i(0) = [\dot{y}_i(0), 0, 0, \dots, 0]^T_{q\times 1}$$

Manuscript received 19 August 1992, revised version received 26 May 1993, accepted 20 October 1993.
Paper No. 1916.

and

$$\mathbf{y}_i(0) = \left[y_i(0), 0, 0, \ldots, 0\right]^{\mathrm{T}}_{q\times 1}$$

$$\mathbf{y}_0 = \left[\mathbf{y}_1(0), \mathbf{y}_2(0), \ldots, \mathbf{y}_n(0)\right]^{\mathrm{T}}$$

where

$$\mathbf{y}_i(0) = \left[y_1(0), 0, 0, \ldots, 0\right]^{\mathrm{T}}_{q\times 1}$$

$$\mathbf{y}_1 = \left[\dot{\mathbf{y}}_1(0), \dot{\mathbf{y}}_2(0), \ldots, \dot{\mathbf{y}}_n(0)\right]^{\mathrm{T}}$$

where

$$\dot{\mathbf{y}}_i(0) = \left[\dot{y}_1(0), 0, 0, \ldots, 0\right]^{\mathrm{T}}_{q\times 1}$$

1. INTRODUCTION

The study of systems governed by a set of ordinary linear or non-linear differential equations with periodic coefficients is of great importance in many diverse branches of science and engineering. The investigation of stability and response prediction are the two most significant dynamic problems associated with such systems. One of the well known problems which yield such mathematical systems is the study of helicopter dynamics. In forward flight, the rotor blade motions are governed by a set of nonlinear differential equations with periodic coefficients. For stability analyses, one may linearise the equations of motion for small perturbations about a periodic equilibrium position. Linearisation of perturbed equations results in a set of ordinary linear differential equations with periodic coefficients.

In the past, several methods have been used to investigate the stability of rotorcraft systems. In 1973, Friedmann and Tong(1) studied the non-linear flap-lag dynamics of a hingeless helicopter blade in hover and in forward flight by using the perturbation method in time domain. Johnson(2, 3) presented the perturbation solutions of helicopter rotor flapping stability in forward flight. Crespo da Silva and Hodges(4) have also applied the perturbation method to rotorcraft dynamics in forward flight through the symbolic manipulation program Macsyma.

A number of authors have also tried to determine the stability and response from an approximate system of equations which are obtained by replacing the elements of a periodic coefficient matrix by piecewise constant or linear functions. Friedmann and Silverthorn(5) presented a method for approximating the Floquet transition matrix based on the idea suggested by Hsu(6) called the generalisation of the "rectangular ripple" method. They also studied the coupled flap-lag aeroelastic motion of a hingeless rotor blade in forward flight. Crimi(7) used Hill's infinite determinant method to analyse the stability of a helicopter rotor in forward flight.

Although the above mentioned techniques are quite useful, for more accurate results most authors have relied upon the application of Floquet theory to such systems. In 1971, Peters and Hohenemser(8) first used numerical integration of the periodic coefficient equations over one period by generating the Floquet transition matrix and found the stability of the blade flapping problem in forward flight. Later, Friedmann and Silverthorn(9) and Friedmann and Shamie(10) extended the applications to flap-lag stability of various rotor blades in forward flight.

During the course of these studies, it was realised that the computation of the Floquet transition matrix (FTM) can be very time consuming for large-dimensional systems. Subsequently, several studies(11–14) were undertaken toward developing techniques for efficient computation of the FTM. Hammond(11) suggested a single-pass scheme for numerical integration as opposed to the N-pass approach previously used by several authors. This scheme is also described in the papers by Friedmann *et al* (12) and Gaonkar *et al* (13). In Ref. 13, the flap-lag stability of a multi-bladed rotor was studied via Floquet theory. The FTMs were computed through the applications of several numerical codes such as Hamming predictor-corrector, Runge-Kutta, Gear methods, etc. It was reported that the single-pass scheme could save as much as 50%–60% of the computation time depending on the number of blades. The computational efficiency increased as the number of blades was increased from two to five.

Sinha and Wu(14) recently presented a new numerical scheme for the computation of Floquet transition matrix associated with a class of periodic systems. In this approach, the solution vector is expanded in terms of the shifted Chebyshev polynomials. The attractive feature of this technique is that it reduces the original differential system to a system of linear algebraic equations from which the solutions in the interval of one period can be obtained easily. Applications included in Ref. 14 indicate that the technique can be extremely efficient as well as accurate. However, the analysis is not applicable to systems where the mass matrices are periodic functions of time, which is certainly the case when one attempts to analyse the stability problem associated with a multi-bladed rotor. Moreover, the examples included in Ref. 14 are too simple to provide any insurance whether, or not, this is indeed a viable approach for large-scale systems.

The technique is generalised here for systems whose mass, damping and stiffness matrices can have periodic variations in time. Further, it is shown that the technique can be applied to investigate the flap-lag stability of a multi-bladed rotor. The FTMs associated with one, three, four and five bladed rotor problems are calculated and the computation times were compared with those obtained from Adams-Moulton, Runge-Kutta and Gear methods. It is found that the suggested technique is several times faster than any other algorithm.

2. DEVELOPMENT OF THE ANALYSIS TECHNIQUE

In the following, two formulations are presented. The first is suitable for the direct integration of second order equations while the second can be applied to a system of equations expressed in state-space form.

2.1 Direct formulation

Consider a set of second-order nonlinear ordinary differential equations with periodic coefficients given by

$$\left[\mathbf{M}+\mathbf{M}^*(t)\right]\ddot{\mathbf{y}}(t)+\left[\mathbf{C}+\mathbf{C}^*(t)\right]\dot{\mathbf{y}}(t)+\left[\mathbf{K}+\mathbf{K}^*(t)\right]\mathbf{y}(t)=0 \qquad \ldots (1)$$

with appropriate initial conditions $\mathbf{y}(0)$ and $\dot{\mathbf{y}}(0)$. $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the $n \times n$ time-invariant mass, damping and the stiffness matrices, respectively, n being the number of degrees of freedom of the system. The starred quantities are the respective periodic matrices and $\mathbf{y}(t)$ is an $n \times 1$ state vector.

Integrating equation (1) once yields

$$\begin{aligned}&\mathbf{M}\dot{\mathbf{y}}(t)-\mathbf{M}\dot{\mathbf{y}}(0)+\mathbf{M}^*(t)\dot{\mathbf{y}}(t)-\mathbf{M}^*(0)\dot{\mathbf{y}}(0)\\&-\dot{\mathbf{M}}^*(t)\mathbf{y}(t)+\dot{\mathbf{M}}^*(0)\mathbf{y}(0)+\int_0^t\ddot{\mathbf{M}}^*(\eta)\mathbf{y}(\eta)d\eta\\&+\mathbf{C}\mathbf{y}(t)-\mathbf{C}\mathbf{y}(0)+\mathbf{C}^*(t)\mathbf{y}(t)-\mathbf{C}^*(0)\mathbf{y}(0)\\&-\int_0^t\dot{\mathbf{C}}^*(\eta)\mathbf{y}(\eta)d\eta+\int_0^t\left(\mathbf{K}+\mathbf{K}^*(\eta)\right)\mathbf{y}(\eta)d\eta=0\end{aligned} \qquad \ldots (2)$$

where integration by parts has been used where necessary. At this stage, the elements of the periodic matrices $\dot{\mathbf{M}}^*(t)$, $\ddot{\mathbf{M}}^*(t)$, $\mathbf{C}^*(t)$ and $\mathbf{K}^*(t)$ are expanded in shifted Chebyshev polynomials of the first kind(15) with known coefficients. The state vector $\mathbf{y}$ is expanded in Chebyshev polynomials with unknown coefficients. These are given by

$$
\begin{aligned}
&\dot{m}_{ij}(t) = \mathbf{s}^{\mathbf{T}}(t)d_1^{ij}\\
&\dot{m}_{ij}^{*}(t) = \mathbf{s}^{\mathbf{T}}(t)d_2^{ij}\\
&c_{ij}^{*}(t) = \mathbf{s}^{\mathbf{T}}(t)d_3^{ij}\\
&\dot{c}_{ij}^{*}(t) = \mathbf{s}^{\mathbf{T}}(t)d_4^{ij}\\
&k_{ij}^{*}(t) = \mathbf{s}^{\mathbf{T}}(t)d_5^{ij}\\
&y_j(t) = \mathbf{s}^{\mathbf{T}}(t)\alpha^{\mathbf{j}} \qquad (i,j=1,2,\dots,n)\\
&\mathbf{s}^{\mathbf{T}}(t) = \left[T_0^*(t) \quad T_1^*(t) \quad T_2^*(t) \quad \dots \quad T_{(q-1)}^*(t)\right]
\end{aligned}
\qquad \dots(3)
$$

where $T_r^*(t)$, $0 \leq t \leq 1$ are the shifted Chebyshev polynomials of the first kind. $\alpha^{\mathbf{j}}$ and d_l^{ij} are the unknown and known Chebyshev vector coefficients, respectively. $\dot{m}_{ij}^*(t)$, $\ddot{m}_{ij}^*(t)$, $c_{ij}^*(t)$, $\dot{c}_{ij}^*(t)$ and $k_{ij}^*(t)$ are the elements of the matrices $\dot{\mathbf{M}}^*(t)$, $\ddot{\mathbf{M}}_{\mathbf{ij}}^*(t)$, $\mathbf{C}_{\mathbf{ij}}^*(t)$, $\dot{\mathbf{C}}_{\mathbf{ij}}^*(t)$ and $\mathbf{K}_{\mathbf{ij}}^*(t)$, respectively.

Applying Equation (3) to Equation (2) yields

$$
\begin{aligned}
\left(\mathbf{M}+\mathbf{M}^*(t)\right)\dot{y}(t) &= \left(\mathbf{I_n}\otimes\mathbf{s}^{\mathbf{T}}(t)\right)\begin{bmatrix}\mathbf{M_{p2}}-(\mathbf{I_n}\otimes\mathbf{G})\mathbf{M_{p3}}-\mathbf{C}\otimes\mathbf{I_q}\\ -\mathbf{C_{PI}}+(\mathbf{I_n}\otimes\mathbf{G})\mathbf{C_{p2}}-\mathbf{K}\otimes\mathbf{G}\\ -(\mathbf{I_n}\otimes\mathbf{G})\mathbf{K_{p1}}\end{bmatrix}\alpha\\
&+\left(\mathbf{I_n}\otimes\mathbf{s}^{\mathbf{T}}(t)\right)\left[\mathbf{C}^*(0)\otimes\mathbf{I_q}+\mathbf{C}\otimes\mathbf{I_q}-\dot{\mathbf{M}}^*(0)\otimes\mathbf{I_q}\right]\mathbf{y_0}\\
&+\left(\mathbf{I_n}\otimes\mathbf{s}^{\mathbf{T}}(t)\right)\left[\mathbf{M}\otimes\mathbf{I_q}+\mathbf{M}^*(0)\otimes\mathbf{I_q}\right]\mathbf{y_1}
\end{aligned}
\qquad \dots(4)
$$

where the symbol $\otimes$ refers to the Kronecker product as defined by Bellman[16] and the matrices and vectors in Equation (4) are defined in the Appendix. It should be noted that the "operational matrices" defined earlier in Sinha and Wu[14] are used in Equation (4).

Integrating Equation (4) and again using the Chebyshev expansions for the state vector, the elements of the matrix $\mathbf{M}^*(t)$ ($m_{ij}^* = \mathbf{s}^{\mathbf{T}}(t)d_6^{ij}$) and the "operational matrices" finally results in the following set of algebraic equations for the unknown vector α,

$$
\mathbf{W}\alpha = \mathbf{f} \qquad \dots(5)
$$

where

$$
\begin{aligned}
\mathbf{W} &= \mathbf{M}\otimes\mathbf{I_q}+\mathbf{M_{p1}}-2(\mathbf{I_n}\otimes\mathbf{G})\mathbf{M_{p2}}+\left(\mathbf{I_n}\otimes\mathbf{G}^2\right)\mathbf{M_{p3}}\\
&+\mathbf{C}\otimes\mathbf{G}+(\mathbf{I_n}\otimes\mathbf{G})\mathbf{C_{p1}}-\left(\mathbf{I_n}\otimes\mathbf{G}^2\right)\mathbf{C_{p2}}+\mathbf{K}\otimes\mathbf{G}^2\\
&+\left(\mathbf{I_n}\otimes\mathbf{G}^2\right)\mathbf{K_{p1}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{f} &= \begin{pmatrix}\mathbf{M}\otimes\mathbf{I_q}+\mathbf{M}^*(0)\otimes\mathbf{I_q}-\dot{\mathbf{M}}^*(0)\otimes\mathbf{G}\\ +\mathbf{C}^*(0)\otimes\mathbf{G}+\mathbf{C}\otimes\mathbf{G}\end{pmatrix}\mathbf{y_0}+\\
&\left(\mathbf{M}\otimes\mathbf{G}+\mathbf{M}^*(0)\otimes\mathbf{G}\right)\mathbf{y_1}
\end{aligned}
$$

2.2 State-space formulation

The dynamical system given by the Equation (1) can be rewritten in state-space form given by

$$
\left[\mathbf{A}+\mathbf{A}^*(t)\right]\dot{\mathbf{x}}(t)+\left[\mathbf{B}+\mathbf{B}^*(t)\right]\mathbf{x}(t)=0 \qquad \dots(6)
$$

where

$$
\mathbf{x}(t) = \begin{bmatrix}\dot{y}_1(t)\\ \dot{y}_2(t)\\ \vdots\\ \dot{y}_n(t)\\ y_1(t)\\ y_2(t)\\ \vdots\\ y_n(t)\end{bmatrix}
$$

and $\mathbf{A}$, $\mathbf{A}^*(t)$, $\mathbf{B}$ and $\mathbf{B}^*(t)$ are $2n \times 2n$ square matrices given by

$$
\mathbf{A} = \begin{bmatrix}0 & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix} \qquad
\mathbf{A}^*(t) = \begin{bmatrix}0 & \mathbf{M}^*(t)\\ \mathbf{M}^*(t) & \mathbf{C}^*(t)\end{bmatrix} \qquad \dots(7)
$$

$$
\mathbf{B} = \begin{bmatrix}-\mathbf{M} & 0\\ 0 & \mathbf{K}\end{bmatrix} \qquad
\mathbf{B}^*(t) = \begin{bmatrix}-\mathbf{M}^*(t) & 0\\ 0 & \mathbf{K}^*(t)\end{bmatrix} \qquad \dots(8)
$$

Integrating Equation (6) once yields

$$
\begin{aligned}
&\mathbf{Ax}(t)-\mathbf{Ax}(0)+\mathbf{A}^*(0)\mathbf{x}(0)-\mathbf{A}^*(t)\mathbf{x}(t)\\
&-\int_0^t\dot{\mathbf{A}}^*(\eta)\mathbf{x}(\eta)d\eta+\int_0^t\left(\mathbf{B}+\mathbf{B}^*(\eta)\right)\mathbf{x}(\eta)d\eta=0
\end{aligned}
\qquad \dots(9)
$$

The elements of the matrices $\mathbf{A}^*(t)$, $\dot{\mathbf{A}}^*(t)$, $\mathbf{B}^*(t)$ and the state vector are expanded in shifted Chebyshev polynomials of the first kind and are given by

$$
\begin{aligned}
&a_{ml}^*(t) = \mathbf{s}^{\mathbf{T}}d_7^{ml}\\
&\dot{a}_{ml}^*(t) = \mathbf{s}^{\mathbf{T}}d_8^{ml}\\
&b_{ml}^*(t) = \mathbf{s}^{\mathbf{T}}d_9^{ml}\\
&x_l(t) = \mathbf{s}^{\mathbf{T}}(t)\beta^l \qquad (m,l=1,2,\dots,2n)\\
&\mathbf{s}^{\mathbf{T}}(t) = \left[T_0^*(t) \quad T_1^*(t) \quad T_2^*(t) \quad \dots \quad T_{(q-1)}^*(t)\right]
\end{aligned}
\qquad \dots(10)
$$

Use of these expansions in Equation (9) in conjunction with the operational matrices (given in the Appendix) results in the following equation

$$
\begin{aligned}
&\left(\mathbf{I_{2n}}\otimes\mathbf{s}^{\mathbf{T}}(t)\right)\left[\left(\mathbf{A}\otimes\mathbf{I_q}\right)\beta-\left(\mathbf{A}\otimes\mathbf{I_q}\right)\mathbf{x_0}+\mathbf{A_{p1}}\beta\right]\\
&-\left(\mathbf{I_{2n}}\otimes\mathbf{s}^{\mathbf{T}}(t)\right)\left[\left(\mathbf{A}^*(0)\otimes\mathbf{I_q}\right)\mathbf{x_0}+\left(\mathbf{I_{2n}}\otimes\mathbf{G}\right)\mathbf{A_{p2}}\beta\right]\\
&+\left(\mathbf{I_{2n}}\otimes\mathbf{s}^{\mathbf{T}}(t)\right)\left[(\mathbf{B}\otimes\mathbf{G})\beta+\left(\mathbf{I_{2n}}\otimes\mathbf{G}\right)\mathbf{B_p}\beta\right]=0
\end{aligned}
\qquad \dots(11)
$$

Cancelling the term $(\mathbf{I_{2n}} \otimes \mathbf{s}^{\mathbf{T}}(t))$ on either side of Equation (11) gives a set of algebraic equations for the coefficients of the state vector given by

$$
\hat{\mathbf{W}}\beta = \hat{\mathbf{f}} \qquad \dots(12)
$$

where the matrices $\hat{\mathbf{W}}$ and $\hat{\mathbf{f}}$ are defined by

$$
\hat{\mathbf{W}} = \left(\mathbf{A}\otimes\mathbf{I_q}\right)+\mathbf{A_{p1}}-\left(\mathbf{I_{2n}}\otimes\mathbf{G}\right)\mathbf{A_{p2}}+(\mathbf{B}\otimes\mathbf{G})+\left(\mathbf{I_{2n}}\otimes\mathbf{G}\right)\mathbf{B_p}
$$

$$
\hat{\mathbf{f}} = \left(\mathbf{A}^*(0)\otimes\mathbf{I_q}+\mathbf{A}\otimes\mathbf{I_q}\right)\mathbf{x_0}
$$

Here β is the unknown Chebyshev vector and the subscripted matrices are defined in the Appendix.

It is interesting to note that the present method does not require the mass matrix to be inverted even when the equations are rewritten in state-space form.

3. COMPUTATION OF FTM AND STABILITY ANALYSIS

For the stability analysis of periodic systems, one needs to find the state transition matrix evaluated at $t = T$, $[\Phi(T)]$, associated with the linear system given by Equation (6). From Equation (12), a set of $2n$ β_is are obtained for the $2n$ initial conditions:

$\mathbf{x}_i(0) = (1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1)$.

It is to be noted that all β_is can be determined simultaneously by defining the right hand side of Equation (12) in the matrix form. Then the FTM is given by

$$[\Phi(T)] = \mathbf{s}^{\mathbf{T}}(T)\beta \qquad \ldots (13)$$

Similarly, for the direct formulation technique, a set of n α_is are calculated from Equation (5) with appropriate n initial conditions on $\mathbf{y}_i(0)$ and $\dot{\mathbf{y}}_i(0)$. Since $\dot{\mathbf{y}}_i(t)$ can be calculated from Equation (2), the $(2n \times 2n)$ FTM can be constructed rather easily by evaluating $\mathbf{y}(T)$ and $\dot{\mathbf{y}}(T)$. It is well known[17] that the stability can be guaranteed if the magnitude of the eigenvalues λ_i of the FTM $\Phi(T)$ are less than unity. One can also define υ_i, the real part of the characteristic exponent as

$$\upsilon_i = \frac{1}{T}\ln|\lambda_i| \qquad \ldots (14)$$

Then for stability, $\upsilon_i < 0$.

4. FLAP-LAG STABILITY OF A HELICOPTER ROTOR BLADE

To demonstrate the application of the technique presented in the previous section, the flap-lag stability of a helicopter rotor blade in forward flight is considered. The task of deriving a helicopter rotor blade dynamic equations and performing a stability analysis on the system is very challenging and involves many simplifications. The model considered here was originally developed by Ormiston and Hodges[18] and later discussed by Peters[19], who extended the problem for the case of forward flight. The ideal model of a hingeless rotor blade consists of a slender rigid blade, centrally hinged for rotation, with spring restrained at the hinge. Aerodynamic force is based on the linear, quasi-steady strip theory and induced inflow is assumed to be uniform and is obtained from the simple momentum theory.

The nonlinear equations of motion[19] for the flap-lag dynamics of a rigid blade with contributions from the aerodynamic, inertial, and elastic forces are obtained as

$$\ddot{\beta} + \mathrm{Sin}\beta\mathrm{Cos}\beta\left(1+\dot{\zeta}\right)^2 + (P-1)\left(\beta-\beta_{pc}\right) + Z\zeta = \frac{1}{\Omega^2 I}\int_0^1 F_\beta \bar{r}\,d\bar{r}$$

$$\mathrm{Cos}^2\beta\ddot{\zeta} - 2\mathrm{Sin}\beta\mathrm{Cos}\beta\left(1+\dot{\zeta}\right)\dot{\beta} + W'\zeta + Z\left(\beta-\beta_{pc}\right) = \frac{\mathrm{Cos}\beta}{\Omega^2 I}\int_0^1 F_\zeta \bar{r}\,d\bar{r} \qquad \ldots (15)$$

where F_β and F_ζ are the nondimensional aerodynamic loads which are perpendicular to the flap rotational direction and parallel to the lead rotational direction, respectively, R = the blade radius and $\bar{r} = r/R$ is a nondimensional coordinate. The various other parameters of Equation (15) are defined as

$$P = 1 + \left(\frac{1}{\Delta}\right)\left[\omega_\beta^2 + \hat{R}\left(\omega_\zeta^2 - \omega_\beta^2\right)\mathrm{Sin}^2\theta\right]$$

$$W' = \left(\frac{1}{\Delta}\right)\left[\omega_\zeta^2 + \hat{R}\left(\omega_\zeta^2 - \omega_\beta^2\right)\mathrm{Sin}^2\theta\right]$$

$$Z = \left(\frac{\hat{R}}{2\Delta}\right)\left(\omega_\zeta^2 - \omega_\beta^2\right)\mathrm{Sin}2\theta$$

$$\Delta = 1 + \hat{R}\left(1-\hat{R}\right)\mathrm{Sin}^2\theta\frac{\left(\omega_\zeta^2 - \omega_\beta^2\right)}{\left(\omega_\zeta^2\omega_\beta^2\right)}$$

Here $\hat{R}$ is the parameter which indicates the measure of the elastic coupling between the hub flap, blade flap and lag spring. In the following, $\hat{R}$ is assumed to be zero and therefore there is no elastic coupling and hence the flap frequency squared, $P = 1 + \omega_\beta^2 = p^2$; the lead-lag frequency squared, $W' = \omega_\zeta^2$ and the stiffness parameter Z , are all zero.

The stability of any periodic equilibrium solution of Equation (15) is determined by the dynamics of the linearised variational equation constructed about that solution for a given trim condition. In general, in the study of rotor blade dynamics there are three different trim conditions[19] which are known as the untrimmed, the propulsive trim, and the moment trim conditions. In the untrimmed condition, the resulting cyclic flapping produces hub moments in roll and pitch. Propulsive trim simulates the real forward flight condition. This is achieved by tilting the rotor shaft and maintaining zero rolling and pitching moments on the blade. Moment trim or windtunnel trim, eliminates the rolling and pitching moments, which is achieved with cyclic pitch sufficient to suppress the first harmonic cyclic flapping, i.e., $\beta_s = \beta_c = 0$. Therefore in moment trim, the steady-state cyclic flapping angle is zero, the shaft angle-of-attack is zero, and the rotor force is approximately vertical. In this study, we focus on moment trim only. Stability analysis of the trim condition first requires a determination of the equilibrium solutions, θ_e, ζ_e and β_e for a given trim condition. Following Wei and Peters[20], the steady-state flap (β_e), lag (ζ_e) and the pitch (θ_e) motions can be described by the truncated Fourier series as follows.

$$\beta_e(\psi) = \beta_0 + \beta_c\mathrm{Cos}\psi + \beta_s\mathrm{Sin}\psi$$

$$\zeta_e(\psi) = \zeta_0 + \zeta_c\mathrm{Cos}\psi + \zeta_s\mathrm{Sin}\psi \qquad \ldots (16)$$

$$\theta_e(\psi) = \theta_0 + \theta_c\mathrm{Cos}\psi + \theta_s\mathrm{Sin}\psi$$

where β_0 is the coning angle independent of ψ, and β_c and β_s generate once-per-revolution variations of the flap angle. ζ_0 is the lag angle independent of ψ, ζ_c and ζ_s generating once-per-revolution variations of the lag angle. θ is the collective pitch and θ_c and θ_s are cyclic pitch angles. However, for the moment trim condition β_c and β_s are identically zero. Substituting Equation (16) into Equation (15) and applying the harmonic balance method, the values of β_0, ζ_0, ζ_c, ζ_s, θ_0, θ_c and θ_s can be obtained in a closed form[20] for a moment trim flight with constant thrust coefficient C_T as

$$\phi_e = \frac{3}{4}\left\{\frac{1}{2}\left(\sqrt{C_T^2+\mu^4} - \mu^2\right)\right\}^{\frac{1}{2}}$$

$$\theta_0 = \left(\frac{6C_T}{\sigma a} + \frac{9}{8}\phi_c\right) + \left(15\frac{C_T}{\sigma a} + \frac{9}{16}\phi_e\right)\mu^2$$

$$\theta_s = -\left(16\frac{C_T}{\sigma a} + \frac{3}{2}\phi_e\right)\mu - \left(16\frac{C_T}{\sigma a} - \frac{3}{4}\phi_e\right)\mu^3$$

$$\theta_c = \frac{\gamma}{p^2}\left(\frac{C_T}{\sigma a} + \frac{1}{48}\phi_e\right)\mu - \frac{\gamma}{p^2}\left(\frac{5}{4}\frac{C_T}{\sigma a} + \frac{1}{16}\phi_e\right)\mu^3$$

$$\beta_0 = \frac{\gamma}{8p^2}\left(6\frac{C_T}{\sigma a} + \frac{1}{8}\phi_e\right) - \frac{\gamma}{p^2}\left(\frac{1}{24}\frac{C_T}{\sigma a} + \frac{5}{128}\phi_e\right)\mu^2$$

where $a = 2\pi$ is the lift curve slope.

Denoting $\Delta\beta$ and $\Delta\zeta$ as small perturbations about β_e and ζ_e, respectively, Equation (15) is linearised to the following form

$$\begin{bmatrix}\Delta\ddot{\beta}\\ \Delta\ddot{\zeta}\end{bmatrix}+\left[\mathbf{C}(\psi)\right]\begin{bmatrix}\Delta\dot{\beta}\\ \Delta\dot{\zeta}\end{bmatrix}+\left[\mathbf{K}(\psi)\right]\begin{bmatrix}\Delta\beta\\ \Delta\zeta\end{bmatrix}=0 \quad \ldots (18)$$

where matrices $[\mathbf{C}(\psi)]$ and $[\mathbf{K}(\psi)]$ are periodic functions of ψ.

The above equations represent the linearised stability equations for a single blade only with two degrees-of-freedom. In case of a b-bladed rotor, the corresponding $2b$ equations have to be transformed from the rotating to the nonrotating frame. This can be accomplished by using the multi-bladed coordinate transformation (MCT) matrices as indicated by Hohenemser and Yin[21]. The final set of equations in the nonrotating frame can be represented by

$$\left[\mathbf{M}+\mathbf{M}^*(\psi)\right]\Delta\ddot{\mathbf{y}}_{\mathbf{NR}}+\left[\mathbf{C}+\mathbf{C}^*(\psi)\right]\Delta\dot{\mathbf{y}}_{\mathbf{NR}}+\left[\mathbf{K}+\mathbf{K}^*(\psi)\right]\Delta\mathbf{y}_{\mathbf{NR}}=0 \quad \ldots (19)$$

where $\Delta\mathbf{y}_{\mathbf{NR}}$ represents the non-rotating degrees of freedom,

$$\mathbf{M}^*(\psi)=\mathbf{M}_{1c}\mathrm{Cos}(\psi)+\mathbf{M}_{2c}\mathrm{Cos}(2\psi)+\ldots+\mathbf{M}_{ic}\mathrm{Cos}(i\psi)$$
$$+\mathbf{M}_{1s}\mathrm{Sin}(\psi)+\mathbf{M}_{2s}\mathrm{Sin}(2\psi)+\ldots+\mathbf{M}_{is}\mathrm{Sin}(i\psi)$$
$$\mathbf{C}^*(\psi)=\mathbf{C}_{1c}\mathrm{Cos}(\psi)+\mathbf{C}_{2c}\mathrm{Cos}(2\psi)+\ldots+\mathbf{C}_{jc}\mathrm{Cos}(j\psi)$$
$$+\mathbf{C}_{1s}\mathrm{Sin}(\psi)+\mathbf{C}_{2s}\mathrm{Sin}(2\psi)+\ldots+\mathbf{C}_{js}\mathrm{Sin}(j\psi)$$
$$\mathbf{K}^*(\psi)=\mathbf{K}_{1c}\mathrm{Cos}(\psi)+\mathbf{K}_{2c}\mathrm{Cos}(2\psi)+\ldots+\mathbf{K}_{kc}\mathrm{Cos}(k\psi)$$
$$+\mathbf{K}_{1s}\mathrm{Sin}(\psi)+\mathbf{K}_{2s}\mathrm{Sin}(2\psi)+\ldots+\mathbf{K}_{ks}\mathrm{Sin}(k\psi)$$

and the integers i, j and k are dependent on the number of blades b. For $b = 3$, $i = 1$, $j = 3$, $k = 4$; for $b = 4$, $i = 1$, $j = 3$, $k = 4$; and for $b = 5$, $i = 2$, $j = 4$, $k = 5$. It is observed that Equation (19) has the same form as Equation (1).

4.1 Numerical computation

Numerical results associated with the flap-lag stability analyses of a single blade as well as multi-bladed rotors are presented via the proposed Chebyshev method and the standard numerical algorithms. All calculations are based on the assumptions that (i) stall, compressibility and reverse flow are neglected, (ii) inflow is uniform, and (iii) elastic coupling parameter $\hat{R} = 0$. Unless otherwise specified, all calculations are carried out for $\gamma = 5$, $\hat{R} = 0$, $C_T/\sigma = 0{\cdot}2$, $C_{d0} = 0{\cdot}01$, $Z = 0$, $\sigma = 0{\cdot}05$. All calculations are performed on a Cray XMP/24 Supercomputer.

To establish that the Chebyshev method is accurate in comparison to other existing methods, the flap-lag stability boundaries for an isolated blade in the p-ω_ζ plane were computed under powered conditions. The 4 × 4 FTM was calculated using 20 terms in the Chebyshev polynomial approximation. The results are shown in Fig. 1 along with those obtained by the perturbation method in Wei and Peters[20]. It is observed that for small values of advance ratio μ, the two results are identical. For $\mu = 0{\cdot}4$, the boundaries are observed to be somewhat different and this is expected since the perturbation approach is not expected to yield good results in this case. However, it was observed that the boundary obtained by the Chebyshev method matched exactly with the boundary obtained by a sixth-order Runge-Kutta scheme with adaptive step size. All computations were performed in double precision.

To investigate the influence of parameter change on the numerical efficiency and accuracy, the same problem was studied by increasing values of advance ratio μ gradually. Keeping $p = \omega_\zeta = 1{\cdot}2$, the proposed technique as well as some other standard numerical codes, such as Runge-Kutta, Adams-Moulton, and Gear methods were used to compute the characteristic exponents associated with the FTM. These codes constitute a part of the IMSL subroutines and are available on the Cray supercomputer.

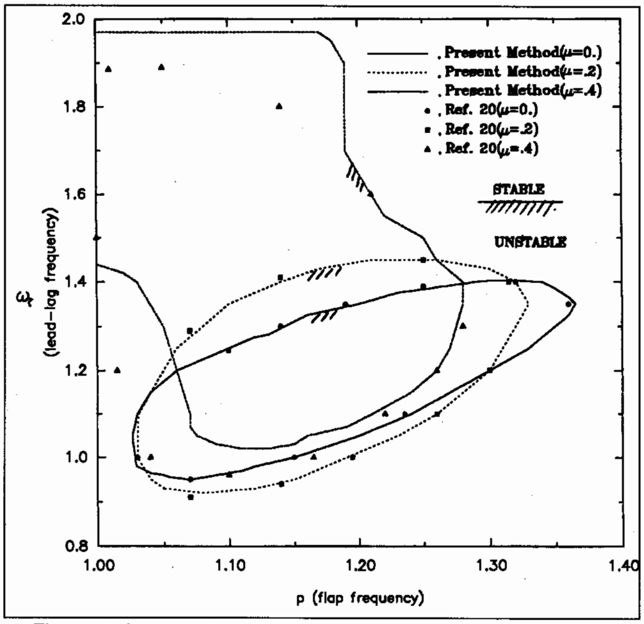

Figure 1. A comparison of stability boundaries in forward flight with $R = 0$, $\gamma = 5$, $C_T/\sigma = 0{\cdot}2$, $C_{d0} = 0{\cdot}01$, $\sigma = 0{\cdot}01$.

The CPU time required to maintain the fifth digit accuracy in the real part of the largest exponent in each of the techniques was recorded. An 18-22 Chebyshev terms expansion was found adequate in meeting the fifth digit accuracy in the largest characteristic exponents. The results are shown in Fig. 2. It is noted that the computation time increases with an increase in advance ratio. Also, it is observed that both the state space and direct formulation due to Chebyshev method is faster than that of any other numerical procedures. In particular, the direct formulation is found to be much more efficient than the state space formulation itself. The result from a similar study for the three-bladed rotor is also provided in Fig. 3 and a similar trend is observed. From these figures, the superiority of the proposed technique is clearly demonstrated in terms of the computational efficiency.

Finally, the characteristic exponents of the FTM for three, four, and five bladed rotors were computed. For a multi-bladed rotor,

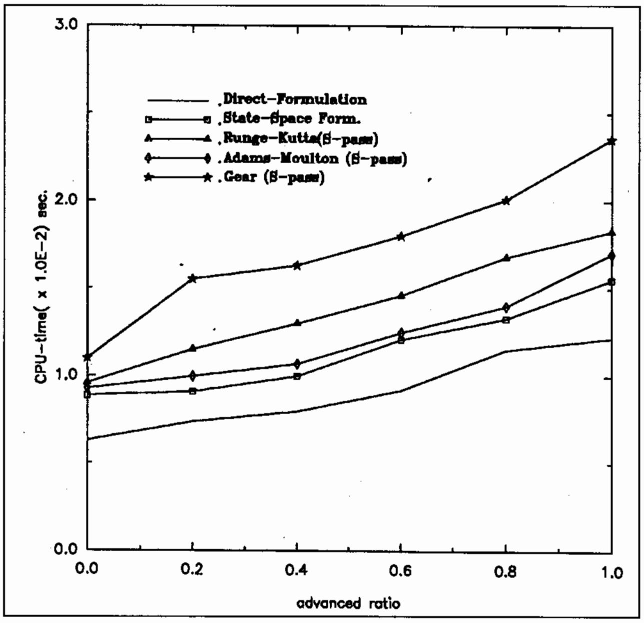

Figure 2. A comparison of CPU time used to compute the characteristic exponents with various advance ratios for a single bladed rotor.

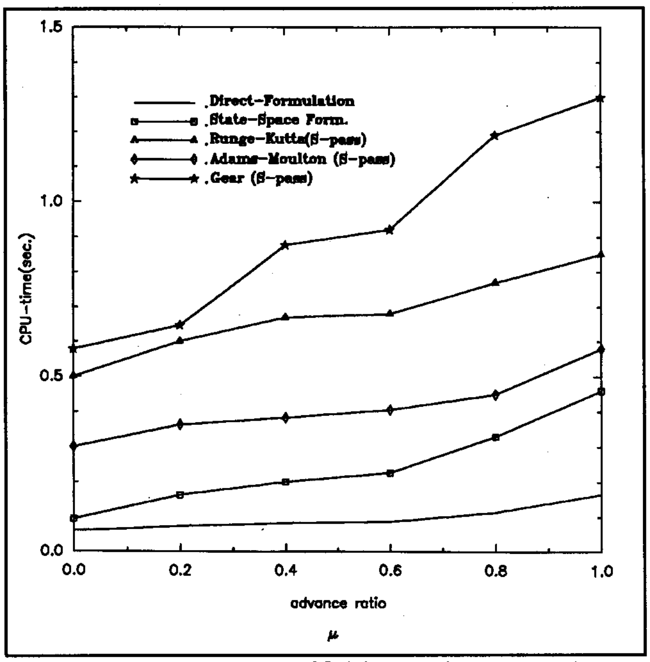

Figure 3. A comparison of CPU time used to compute the characteristic exponents with various advance ratios for a three bladed rotor.

$N/4$ repeated sets of characteristic exponents are obtained where N is the number of FTMs. The CPU times taken by the direct formulation and the state-space approaches were compared with those needed by other numerical codes in order to provide the same numerical accuracy in the largest exponent. Several accuracy conditions were studied. Here, the computational details of only two cases are presented. In both cases, $p = \omega_\zeta = 1{\cdot}2$ and $\mu = 0{\cdot}4$. The results shown in Table 1 correspond to the case where the characteristic exponents converge to the fourth digit accuracy. To achieve this accuracy, 18-22 terms were used in the expansion of the solution vector. Table 2 contains entries for the case when a seventh digit accuracy was required in the convergence. In this case a 20-25 terms expansion was needed, the largest number being used for the case of the five-bladed rotor. The results for this case are also provided in Fig. 4.

5. DISCUSSION AND CONCLUSIONS

A new numerical technique suitable for the stability analysis of linear dynamic systems with periodic coefficients has been presented. It is shown that the original set of differential equations can be reduced to a set of linear algebraic equations by expressing the state vector and periodic coefficients matrix as finite sums of shifted Chebyshev polynomials. Two algorithms, *viz*, direct formulation and state-space approach are suggested. The former is directly applicable to a dynamic system described by a set of second order equations, whereas the latter is suitable for a set of equations in the state-space form. The solution technique has been combined with the Floquet theory to study the stability of general dynamic systems where the mass, damping, and stiffness matrices can be periodic functions of time.

The computational superiority of the proposed technique is obvious from the analysis of multi-bladed rotor using the MCT technique. Figure 4 and Tables 1 and 2 clearly indicate that both formulations are much more efficient than the conventional Runge-Kutta, Adams-Moulton, and Gear algorithms applied either in N-pass or single-pass schemes. In particular, the direct formulation method is several times faster than all other techniques. The computational efficiency is found to increase as the dimension of FTM becomes larger and larger (c.f. Fig. 4). Parametric studies with variations in the advance ratio (μ) show that for larger values of μ, more terms were needed in the expansion to maintain the same level of accuracy. Of course, this results in an increase in the CPU time. Nevertheless, as indicated in Figs 2 and 3, all other numerical codes behave in a similar fashion. It is also noted that the relative computational efficiencies of the proposed technique improve considerably from a single blade rotor to a three-bladed one (c.f. Figs 2 and 3).

Table 1
(fourth digit accuracy)

		Present method		Runge-Kutta method (IVPCK)		Adams-Moulton method (IVPAG)		Gear method (IVPAG)	
No of blades	Dimension of FTM	I Direct formulation	II State space	III Single pass	IV N-pass	V Single pass	VI N-pass	VII Single pass	VIII N-pass
1	4	0·008	0·013	0·015	0·035	0·010	0·031	0·012	0·039
3	12	0·063	0·148	0·342	2·947	0·205	2·145	0·264	2·546
4	16	0·148	0·368	0·482	5·064	0·249	3·527	0·365	5·568
5	20	0·212	0·536	1·169	11·170	0·541	7·579	0·767	10·436

Table 2
(seventh digit accuracy)

		Present method		Runge-Kutta method (IVPCK)		Adams-Moulton method (IVPAG)		Gear method (IVPAG)	
No of blades	Dimension of FTM	I Direct formulation	II State space	III Single pass	IV N-pass	V Single pass	VI N-pass	VII Single pass	VIII N-pass
1	4	0·010	0·016	0·016	0·038	0·015	0·042	0·021	0·071
3	12	0·082	0·175	0·641	6·114	0·372	5·274	0·876	8·094
4	16	0·181	0·403	0·966	9·981	0·601	7·964	1·328	13·211
5	20	0·368	0·809	2·562	15·843	1·367	14·756	3·258	18·978

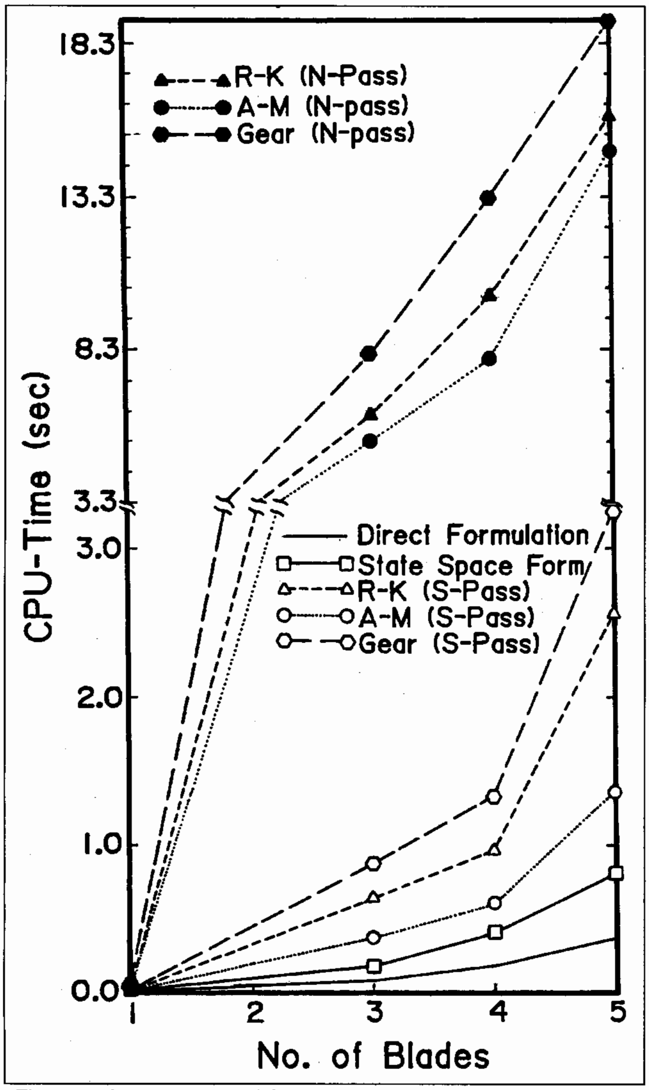

Figure 4. A comparison of CPU time for various methods (stability analysis of a multi-bladed rotor).

It is concluded that the numerical method suggested is certainly a viable alternate method for stability analysis of large-scale linear systems with periodic coefficients. The technique has been found to be extremely efficient as well as accurate when used to compute the Floquet transition matrices associated with a multi-bladed rotor. The main advantage of this technique is that much of the information required to set up the problem, such as the integration matrix, the product matrix, and other operational matrices, can be stored in the computer in advance. In general, the periodic coefficients can be written in the forms of $\text{Sin}(k\psi)$ and/or $\text{Cos}(k\psi)$. The expansions of these quantities can be computed one time only and stored for future use. The entire computation process can be automated rather easily. It should be pointed out that LFTRG subroutine, (a part of a Linpack algorithm) was used in solving the simultaneous set of linear equations needed to construct the proposed solution.

The coefficient matrix associated with these equations is sparse due to the sparse nature of the operational matrices associated with Chebyshev polynomials. The accuracy of computation, of course, depends on the number of terms used in the expansion of the solution vector $\mathbf{y}(t)$. Proof of convergence and an error bound analysis have been carried out in the paper by Sinha and Wu[14].

It is anticipated that, in near future, the proposed technique would serve as a viable computational tool in the analysis of general dynamic systems with periodic coefficients.

ACKNOWLEDGEMENT

Financial support for this work was provided by the Army Research Office, monitored by Dr Gary Anderson under contract DAAL03-89-k-0172. The Cray computer time provided by the Alabama Supercomputer Network and Auburn University is also acknowledged.

REFERENCES

1. Friedmann, P. and Tong, P. Nonlinear flap lag dynamics of hingeless helicopter blades in hover and in forward flight, *J Sound Vib*, 1973, **30**, (1), pp 9-31.
2. Johnson, W. A perturbation solution of helicopter rotor flapping stability, NASA TM X-62, 1972, pp 165.
3. Johnson, W. Perturbation solution for the influence of forward flight on helicopter rotor flapping stability, NASA TM X-62, 1974, pp 361.
4. Crespo da Silva, M. and Hodges, D. The role of computerized symbolic manipulation in rotorcraft dynamic analysis, *Computers Maths Appl*, 1986, **12A**, (1), pp 161-172.
5. Friedmann, P. and Silverthorn, L.J. Aeroelastic stability of periodic systems with application to rotor blade flutter, *AIAA J*, 1974, **12**, (11), pp 1559-1565.
6. Hsu, C.S. On approximating a general linear periodic system, *J Math Anal Appl*, 1974, **45**, pp 234-251.
7. Crimi, P.A. Method for analyzing the aeroelastic stability of a helicopter rotor in the forward flight, NASA CR-1332, 1969.
8. Peters, D.A. and Hohenemser, K.H. Application of the Floquet transition matrix to problems of lifting rotor stability, *J Amer Hel Soc*, 1971, **16**, (2), pp. 25-33.
9. Friedmann, P. and Silverthorn, L.J. Aeroelastic stability of coupled flap-lag motion of hingeless helicopter blades at arbitrary advance ratio, *J Sound Vib*, 1975, **39**, (4), pp 409-428.
10. Friedmann, P. and Shamie, J. Aeroelastic stability of trimmed helicopter blades in forward flight, *Vertica*, 1979, **3**, (2), pp 189-211.
11. Hammond, C.E. An application of Floquet theory to the prediction of mechanical instability, *J Amer Hel Soc*, 1974, **19**, (4), pp 14-23.
12. Friedmann, P., Hammond, C.E. and Woo, T.H. Efficient numerical treatment of periodic systems with application to stability problems, *Int J Numer Meth Eng*, 1977, **11**, pp 1117-1136.
13. Gaonkar, G.H., Simha Prasad, D.S. and Sastry, D. On computing Floquet transition matrices of rotorcraft, *J Amer Hel Soc*, 1981, **26**, (3), pp 56-61.
14. Sinha, S.C., and Wu, D.-H., An efficient computational scheme for the analysis of periodic systems, *J Sound Vib*, 1991, **151**, pp 91-117.
15. Fox, L. and Parker, I.B. *Chebyshev Polynomials in Numerical Analysis*, Oxford University Press, 1968.
16. Bellman, R. *Introduction to Matrix Analysis*, McGraw-Hill, 1970.
17. Coddington, E.A. and Levinson, N. *Theory of Ordinary Differential Equations*, McGraw Hill, 1955.
18. Ormiston, R.A. and Hodges, D.H. Linear flap-lag dynamics of hingeless helicopter rotor blades in hover, *J Amer Hel Soc*, 1972, **17**, (2), pp 2-14.
19. Peters, D.A. Flap-lag stability of helicopter rotor blades in forward flight, *J Amer Hel Soc*, 1975, **20**, (4), pp 2-13.
20. Wei, F-S, and Peters, D.A. Lag damping in autorotation by a perturbation method, 34th Annual National Forum of the American Helicopter Society, Washington DC, May 1978.
21. Hohenemser, K.H. and Yin, S.K. Some applications of the method of multi-blade coordinates, *J Amer Hel Soc*, 1972, **17**, pp 3-12.

APPENDIX

$\mathbf{A}_{p1}$ = $2nq \times 2nq$ assembly of the "product operational matrices"[14] $\mathbf{Q}(d_1^{kl})$ for the multiplication of $a^*_{kl}(t)x_1(t)$.

$\mathbf{A}_{p2}$ = $2nq \times 2nq$ assembly of the product operational matrices $\mathbf{Q}(d_8^{kl})$ for the multiplication of $\dot{a}^*_{kl}(t)x_1(t)$.

$\mathbf{B}_p$ = $2nq \times 2nq$ assembly of the product operational matrices $\mathbf{Q}(d_9^{kl})$ for the multiplication of $b^*_{kl}(t)x_1(t)$.

$\mathbf{C_{p1}}$ = $nq \times nq$ assembly of the product operational matrices $\mathbf{Q}(d_3^{kl})$ for the multiplication of $c_{ij}^*(t)y_j(t)$.

$\mathbf{C_{p2}}$ = $nq \times nq$ assembly of the product operational matrices $\mathbf{Q}(d_4^{kl})$ for the multiplication of $\dot{c}_{ij}^*(t)y_j(t)$.

$\mathbf{G^T}$ = $q \times q$ integration operational matrix[14] given by

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ -\frac{1}{8} & 0 & \frac{1}{8} & 0 & \cdot & \cdot & \cdot & 0 \\ -\frac{1}{6} & -\frac{1}{4} & 0 & \frac{1}{12} & \cdot & \cdot & \cdot & 0 \\ \frac{1}{16} & 0 & -\frac{1}{8} & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{(4q-1)} \\ \frac{(-1)^q}{2q(q-2)} & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{-1}{4(q-2)} & 0 \end{bmatrix}$$

$\mathbf{I_n}$ = $n \times n$ identity matrix.

$\mathbf{I_{2n}}$ = $2n \times 2n$ identity matrix.

$\mathbf{I_q}$ = $q \times q$ identity matrix.

$\mathbf{I_{2nq}}$ = $2nq \times 2nq$ identity matrix.

$\mathbf{M_{p1}}$ = $nq \times nq$ assembly of the product operational matrices $\mathbf{Q}(d_6^{ij})$ for the multiplication of $m_{ij}^*(t)y_j(t)$.

$\mathbf{M_{p2}}$ = $nq \times nq$ assembly of the product operational matrices $\mathbf{Q}(d_1^{ij})$ for the multiplication of $\dot{m}_{ij}^*(t)y_j(t)$.

$\mathbf{M_{p3}}$ = $nq \times nq$ assembly of the product operational matrices $\mathbf{Q}(d_2^{ij})$ for the multiplication of $\ddot{m}_{ij}^*(t)y_j(t)$.

$\mathbf{Q}(d_j)$ = $q \times q$ product operational matrix given by

$$\begin{bmatrix} d_0 & \frac{d_1}{2} & \frac{d_2}{2} & \cdot & \cdot & d_{q-1} \\ d_1 & d_0 + \frac{d_2}{2} & \frac{(d_1+d_3)}{2} & \cdot & \cdot & \frac{(d_{q-2}+d_q)}{2} \\ d_2 & \frac{(d_1+d_3)}{2} & d_0 + \frac{d_4}{2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{q-1} & \frac{(d_{q-2}+d_q)}{2} & \cdot & \cdot & \cdot & d_0 + \frac{d_{2q-2}}{2} \end{bmatrix}$$