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A GENERAL APPROACH IN THE DESIGN OF ACTIVE CONTROLLERS FOR NONLINEAR SYSTEMS EXHIBITING CHAOS

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A general framework for local control of nonlinearity in nonautonomous systems using feedback strategies is considered in this work. In particular, it is shown that a system exhibiting chaos can be driven to a desired periodic motion by designing a combination of feedforward controller and a time-varying controller. The design of the time-varying controller is achieved through an application of Lyapunov–Floquet transformation which guarantees the local stability of the desired periodic orbit. If it is desired that the chaotic motion be driven to a fixed point, then the time-varying controller can be replaced by a constant gain controller which can be designed using classical techniques, viz. pole placement, etc. A sinusoidally driven Duffing's oscillator and the well-known Rossler system are chosen as illustrative examples to demonstrate the application.

1. Introduction

Bifurcations and chaotic phenomena in nonlinear dynamical systems have been investigated extensively in the last two decades. Chaotic motions have also been shown to occur in nonlinear control systems with feedback [Baillieul *et al.*, 1980; Holmes, 1985; Mareels & Bitmead, 1986]. Recently, control strategies to suppress bifurcations and chaos in nonlinear systems have been proposed in the literature. Some of these employ feedback and the most popular among such schemes is the so-called OGY (Ott–Grebogi–Yorke) method [Ott *et al.*, 1990; Shinbrot *et al.*, 1993]. This method relies on the facts that chaotic systems are extremely sensitive to initial conditions and that there are typically an infinite number of unstable periodic orbits that coexist with chaotic motion. These properties are exploited to stabilize unstable periodic orbits embedded in the chaotic attractor with control perturbations. However, in real applications, one requires a continuous analysis of the state of the system. The changes in parameters can only be discrete since the OGY method uses the Poincaré map of the system. As pointed out by Kapitaniak [1996] this leads to some serious limitations. The method can be used to stabilize only those orbits whose largest Lyapunov exponent is small compared to the reciprocal of the time interval between parameter changes. This can be avoided by designing a continuous time control system. One such approach has been suggested by Pyragas [1992]. In this approach, the stabilization is achieved either by periodic external perturbation or feedback control with time delay. Engineering control literature makes extensive use of Lyapunov

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first and second methods in the analysis and design of nonlinear control systems. These techniques have also been used to suppress chaos in several nonlinear dynamical systems. Vincent and Yu [1991] have developed a linear state feedback controller for a system described by Lorenz equations. A detailed literature review on "control of chaos" has been provided by Chen and Dong [1993a], Jackson [1991] and Abed *et al.* [1995].

Most of the researchers have dealt with autonomous systems (both discrete and continuous cases). In physics literature, a nonautonomous system is converted to an autonomous, discrete system using a Poincaré map where the fixed points of the Poincaré map correspond to periodic orbits of the original system. In this case the problem of stabilization of periodic orbits of the nonautonomous system is reduced to the stabilization of fixed points of the resulting discrete autonomous system. However, it may be pointed out that, except in very special cases (such as the kicked rotator), it is not easy to obtain an analytical expression for the Poincaré map and one needs to obtain an approximate expression based on either numerical or experimental data. Local stabilization of periodic orbits in a chaotic Duffing's oscillator (among others) using a linear state feedback controller has been considered by Chen and Dong [1993a, 1993b] and Chen [1996]. However, in all these papers, the nonautonomous nature of the problem is ignored and the stability analysis is carried out using the Routh–Hurwitz criteria as if the systems were autonomous. Since the problem of stabilization of a periodic orbit leads to a linear system with periodic coefficients, Floquet theory must be used to guarantee the necessary and sufficient conditions. This has also been pointed out by Leung *et al.* [1995]. In a recent paper Chen [1997] has presented some conventional linear and nonlinear feedback controller designs and controllability conditions were derived via the second method of Lyapunov. Lyapunov functions were constructed to guarantee the global asymptotic stability of the desired trajectories in Chua's circuit and Duffing's oscillator undergoing chaotic motion. In both examples the unstable orbits of the original (uncontrolled) systems were stabilized.

In the present work, a general technique for local control of chaos in nonautonomous systems is presented. Standard results in nonlinear and time-varying control systems theory are used to put the problem of control of chaos in the proper perspective. Similar to the studies reported by several authors mentioned above (e.g. [Chen, 1997]), the controller design is set up as a tracking problem. The control law considered in this work consists of a combination of a feedforward and linear time-varying feedback. It is shown that this control law can be used to direct the chaotic motion to any desired periodic orbit or to a fixed point. The fact that the *desired orbit need not be a solution of the uncontrolled system* is a novelty of this approach. The problem is formulated as an asymptotic stabilization problem of the origin of a nonlinear, nonautonomous system. Under certain assumptions (stated later), the first method of Lyapunov guarantees the local stability of the original system on the basis of the linearized system. In the present case, this linearized system is time-varying and periodic. Hence one can make use of the method proposed by Sinha and Joseph [1994] to design a time-periodic controller via Lyapunov–Floquet transformation. This approach has been successfully employed to study other problems [Boghiu *et al.*, 1998; Sinha *et al.*, 1998]. If it is desired that the chaotic motion be driven to a fixed point, then the local stabilization can be achieved via Routh–Hurwitz criteria and pole placement because the linearized system matrix is time-invariant. A sinusoidally driven Duffing's oscillator and the Rossler system are considered to exemplify the design of the control strategy presented in this work.

2. Local Stabilization

Consider a nonautonomous system with the control law u, given by

$$\dot{x} = f[t, x(t)] + u(t) \tag{1}$$

When the control term $u(t)$ is absent, the above system has a chaotic attractor for a given set of parameter values. In many problems of engineering (such as vibration isolators, buckled beam dynamics and ship dynamics) and physics (such as Josephson junction, plasma oscillations and optical bistability) chaotic response is not desirable and the objective is to choose a control law to drive the chaotic response to a desired periodic orbit or bring the system to rest. Let $y(t)$ be the desired orbit which may be an unstable periodic orbit of Eq. (1) without the control term u. In this paper we assume $y(t)$ to be any arbitrary desired smooth periodic orbit. Consider a control law consisting of two parts, viz. feedforward

(u_f) and a linear time-varying feedback (u_t) as

$$u(t) = u_f + u_t \tag{2}$$

where

$$u_f = \dot{y} - f[t, y(t)], \quad u_t = F(t)(x - y), \tag{3}$$

and $F(t)$ is the time-varying linear state feedback matrix.

With this control law, Eq. (1) can be written as

$$\dot{x} = f[t, x(t)] + u_f + u_t. \tag{4}$$

Defining

$$e = x - y \tag{5}$$

as the error between the actual and desired trajectories, the objective is to choose $u(t)$ such that

$$e \to 0 \tag{6}$$

as $t \to \infty$.

Using Eq. (5) and u_f defined in Eq. (3), Eq. (4) can be written as

$$\dot{e} = g[t, e(t)] + u_t \tag{7}$$

where the nonlinear function $g[\cdot]$ is appropriately defined in terms of $f[\cdot]$. It may be noted that when the feedforward part is incorporated into Eq. (4), it ensures that the origin $e = 0$ is an equilibrium point of Eq. (7), even if the desired orbit $y(t)$ is not a solution of the uncontrolled version of Eq. (1). Of course, when $y(t)$ is a solution of Eq. (1) without the control term, the feedforward part (u_f) is identically zero. In this case one recovers the formulation for stabilization of the unstable periodic orbit embedded in the basin of a chaotic attractor as considered in [Shinbrot *et al.*, 1993; Chen & Dong, 1993b]. It is observed that a combination of feedback and feedforward controller has also been considered by Jackson and Grosu [1995] and Chen [1996]. Now in the new set up, the objective stated in Eq. (6) is equivalent to the asymptotic stabilization of the origin of Eq. (7). A brief discussion of a well-known result regarding the linearization of nonlinear, nonautonomous systems [Vidyasagar, 1993] is given below.

Assume that

$$g(t, 0) = 0, \quad \forall t \geq 0 \tag{8}$$

where g is a C^1 function. Define

$$A(t) = A = \left[\frac{\partial g(t, e)}{\partial e}\right]_{e=0} \tag{9}$$

and

$$h(t, e) = g(t, e) - A(t)e + u_t. \tag{10}$$

Then, if the condition

$$\lim_{\|e\| \to 0} \sup_{t \geq 0} \frac{\|h(t, e)\|}{\|e\|} = 0 \tag{11}$$

holds, then the system

$$\dot{e} = A(t)e(t) + u_t \tag{12}$$

is called the linearization of Eq. (7) around the origin. The stability behavior of the origin of the linearized system can be completely characterized in terms of the state transition matrix $\Phi(t, t_0)$ of the system. It can be proved that the origin is globally uniformly asymptotically stable if the state transition matrix satisfies the inequality

$$\|\Phi(t, t_0)\| < ke^{-\gamma(t - t_0)}, \quad \forall t \geq t_0 \geq 0, \tag{13}$$

for some positive constants k and γ. The local stability of Eq. (7) is guaranteed by the global asymptotic stability of the linearized system (12). One can use Floquet theory and in some cases, classical perturbation techniques [Nayfeh & Mook, 1979] to arrive at the stability regions in the particular case when $A(t)$ is periodic. Alternately, modified Nyquist criteria such as circle criterion and Popov criterion may also be used to derive the stability criteria in certain cases [Narendra & Taylor, 1973].

It is important to note that for time-varying systems, uniform asymptotic stability cannot be inferred by studying only the eigenvalues of the matrix for each fixed time instant. For a simple example to illustrate this point, one can refer to [Vidyasagar, 1993, p. 207]. Ignoring these subtleties, stability criteria were derived by Chen and Dong [1993a, 1993b] and Chen [1996] using the Routh–Hurwitz criteria on the time-periodic Jacobian matrix resulting from the linearization process.

3. Full State Feedback Controller Design Via Lyapunov–Floquet Transformation

Consider the general form of system (12)

$$\dot{z}(t) = A(t)z(t) + B(t)u_t \tag{14}$$

where matrices $A(t)$ and $B(t)$ has periodic coefficients with period T, and pair $[A, B]$ is controllable. In this section the idea of Lyapunov–Floquet transformation is employed to design a full state feedback control law of the type

$$u_t = F(t)z(t) \tag{15}$$

where $F(t)$ is the time-varying feedback matrix.

On the basis of Floquet theory, the state transition matrix (STM) $\Phi(t)$ associated with Eq. (14) can be factored as (see [Yakubovich & Starzhinskii, 1975; Grimshaw, 1993; Sinha *et al.*, 1996])

$$\Phi(t) = L(t)e^{Ct}\,, \tag{16}$$

where $L(t)$ and C, in general, are complex matrices. $L(t)$ is periodic with period T while C is a constant matrix.

$\Phi(t)$ may also be written as

$$\Phi(t) = Q(t)e^{Rt}\,, \tag{17}$$

where $Q(t)$ is a real periodic matrix with period $2T$, and R is a real constant matrix.

Matrices $L(t)$ and $Q(t)$ are called the Lyapunov–Floquet transformation matrices. In this study, however, we shall deal with $Q(t)$ and R matrices, only. Applying the transformation

$$z(t) = Q(t)q(t)\,, \tag{18}$$

Eq. (14) can be transformed to

$$\dot{q}(t) = Rq(t) + Q(t)^{-1}B(t)u_t(t)\,. \tag{19}$$

Matrix R can be computed from the expression

$$R = \frac{1}{2T}\ln\,\Phi(2T) = \frac{1}{2T}\ln(\Phi^2(T))\,. \tag{20}$$

$Q^{-1}(t)$ may be computed by considering the adjoint system

$$\overline{z} = -A(t)^T\overline{z}(t)\,, \tag{21}$$

and using the relationship

$$\Phi^{-1}(t) = \Psi(t)^T\,, \tag{22}$$

where $\Psi(t)$ is the STM of Eq. (21).

The computational details of obtaining the Lyapunov–Floquet (L–F) transformation matrix $Q(t)$ are given by Sinha and Joseph [1994], Pandiyan and Sinha [1995] and Sinha *et al.* [1996]. These authors have shown that the STM of a linear time-periodic system can be efficiently computed in terms of the shifted Chebyshev polynomials and using Eqs. (16) or (17) the L–F transformation can be easily computed. Since $L(t)$ and $Q(t)$ are periodic these can be conveniently expressed in Fourier series.

In order to design a controller for the system given by Eq. (19), an auxiliary time-invariant system of the type

$$\overline{q}(t) = R\overline{q} + B_0v(t)\,, \tag{23}$$

is constructed. In Eq. (23), B_0 is a full rank constant matrix, such that the pair (R, B_0) is controllable. The control vector $v(t)$ is determined by designing a full state feedback controller using either the pole placement technique or the optimal control theory. Thus one can write

$$v(t) = F_0\overline{q}(t)\,. \tag{24}$$

where F_0 is a constant feedback gain matrix. Let $\varepsilon(t) \equiv \overline{q}(t) - q(t)$ be the dynamic error between the state vectors $\overline{q}$ and q. Subtracting Eq. (23) from Eq. (19), one obtains

$$\dot{\varepsilon}(t) = (R + B_0F_0)\varepsilon(t) + Q^{-1}(t)B(t)u_t(t) - B_0F_0q(t)\,. \tag{25}$$

Since $(R + B_0F_0)$ is the stability matrix, the systems defined by Eqs. (19) and (23) can be made equivalent if

$$Q^{-1}(t)B(t)u_t(t) = B_0F_0q(t) \tag{26}$$

Because condition (26) cannot be exactly satisfied, the two systems can be made equivalent in the least square sense only. The error vector is defined as

$$\eta = B(t)u_t(t) - Q(t)B_0F_0q(t)\,, \tag{27}$$

and the control vector $u(t)$ is computed such that the performance index $\eta^T\eta$ is minimized. This procedure yields [Sinha & Joseph, 1994; Boghiu *et al.*, 1997]

$$u_t(t) = B^*(t)Q(t)B_0F_0q(t)\,, \tag{28}$$

where $B^*(t)$ is the generalized inverse of the matrix $B(t)$, defined as

$$B^* = (B^TB)^{-1}B^T\,. \tag{29}$$

Applying the inverse Lyapunov–Floquet transformation to Eq. (28), we obtain

$$u_t(t) = B^*(t)Q(t)B_0F_0Q^{-1}(t)z(t)\,. \tag{30}$$

Comparing Eq. (30) with Eq. (15), the desired feedback gain matrix $F(t)$ is

$$F(t) = B^*(t)Q(t)B_0F_0Q^{-1}(t)\,. \tag{31}$$

It can be shown that one can always choose F_0 such that the asymptotic stability of the original system is guaranteed. Further, it should be observed that the feedback matrix from Eq. (30) can be computed offline and stored into the computer memory and therefore this technique is very much suitable for a real time implementation.

4. Applications

4.1. *Duffing's Oscillator*

Duffing's oscillator is one of the paradigms of nonlinear dynamics. In this section, the design of linear state feedback control law is illustrated for a sinusoidally driven Duffing's oscillator.

Consider the Duffing's oscillator with the control law $u(t)$ given in the form

$$\ddot{x} = -\alpha x - x^3 - 2\zeta\dot{x} + f\cos\omega t + u \tag{32}$$

where α is the stiffness parameter, ζ ($\zeta > 0$) is the viscous damping coefficient, f and ω are the amplitude and frequency of the external input, respectively. It is well-known that in the absence of the control term, for certain values of system parameters, Eq. (32) possesses a chaotic attractor. One of the fundamental aspects of strange attractors is that there are typically an infinite number of unstable periodic orbits that coexist with the chaotic motion. The objective is to choose the feedback control law such that the response of the controlled Duffing's equation results in an asymptotically stable desired periodic orbit or a fixed point.

4.1.1. *Control to an arbitrary periodic orbit*

Let $y(t)$ be a desired periodic orbit with period $T = 2\pi/\omega$ given by

$$\begin{aligned} y(t) &= a_0 + a_1\cos\omega t + b_1\sin\omega t \\ &\quad + a_2\cos 2\omega t + b_2\sin 2\omega t + \cdots \end{aligned} \tag{33}$$

The feedforward part of the control law is given by

$$u_f(t) = \ddot{y} + \alpha y + y^3 - 2\zeta\dot{y} - f\cos\omega t\,. \tag{34}$$

It is obvious from Eq. (34) that $u_f = 0$ when y is a solution of Eq. (32) without the control term. Defining $z = x - y$, Eq. (12) takes the following form in this particular case

$$\ddot{z} = -(\alpha + 3y^2(t))z - 2\zeta\dot{z} + u_t\,. \tag{35}$$

In state space form Eq. (35) can be written as

$$\dot{z}(t) = A(t)z(t) + Bu_t(t) \tag{36}$$

where

$$z = \begin{Bmatrix} z \\ \dot{z} \end{Bmatrix},$$

$$A(t) = \begin{bmatrix} 0 & 1 \\ -(\alpha + 3y^2(t)) & -2\zeta \end{bmatrix}, \tag{37}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It should be observed that matrix $A(t)$ is periodic due to the appearance of $y^2(t)$ and time-invariant techniques such as the Routh–Hurwitz criteria are not applicable. This is exactly the same matrix as reported by Chen and Dong [1993a, 1993b]. Using the design procedure described in Sec. 3, the chaotic motion of the Duffing's oscillator is tracked to the desired periodic orbit $y(t)$.

For numerical computation, we consider the specific case of Eq. (32) with $\alpha = -1$, $\zeta = 0.125$, $f = 1$ and $\omega = 1$

$$\ddot{x} = x - x^3 - 0.25\dot{x} + \cos t + u \tag{38}$$

with a desired simple periodic orbit of

$$y_1(t) = 2\cos t + \sin t\,. \tag{39}$$

For the given parameter values, the system undergoes chaotic motion, as shown in Figs. 1 and 2. Following the procedure outlined in Sec. 2, a feedforward control is designed as

$$u_f = \ddot{y}_1 - y_1 + y_1^3 + 0.25\dot{y}_1 - \cos t\,. \tag{40}$$

The linearized equation in the error state $z(t)$ is given by

$$\begin{aligned} \dot{z}(t) &= \left(\begin{bmatrix} 0 & 1 \\ -\dfrac{13}{2} & -\dfrac{1}{4} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\dfrac{9}{2} & 0 \end{bmatrix} \cos 2t \right. \\ &\quad \left. + \begin{bmatrix} 0 & 0 \\ -6 & 0 \end{bmatrix} \sin 2t \right) z(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t\,. \end{aligned} \tag{41}$$

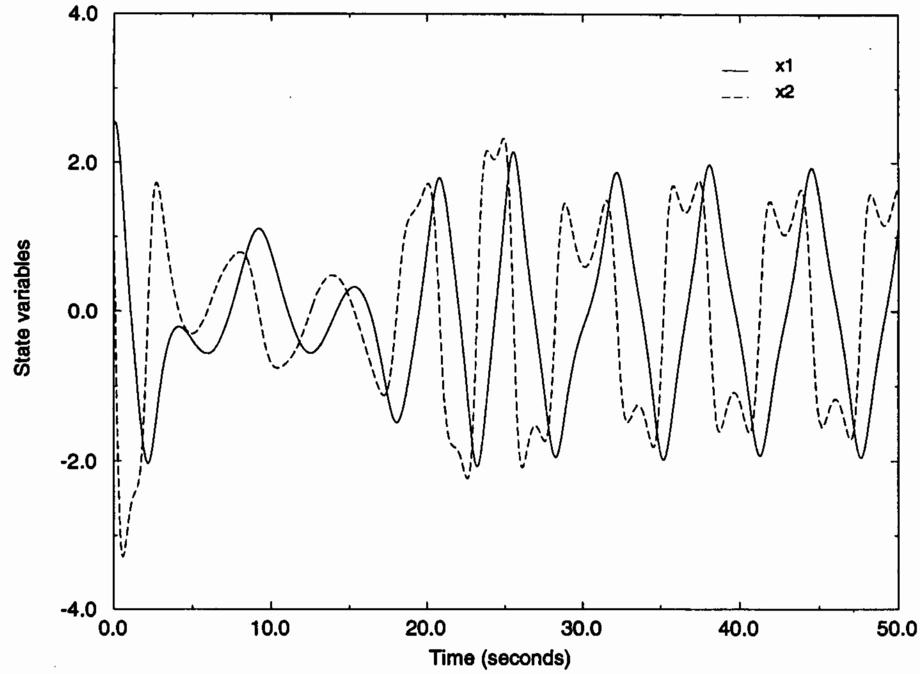

Fig. 1. Time trace of uncontrolled Duffing's oscillator.

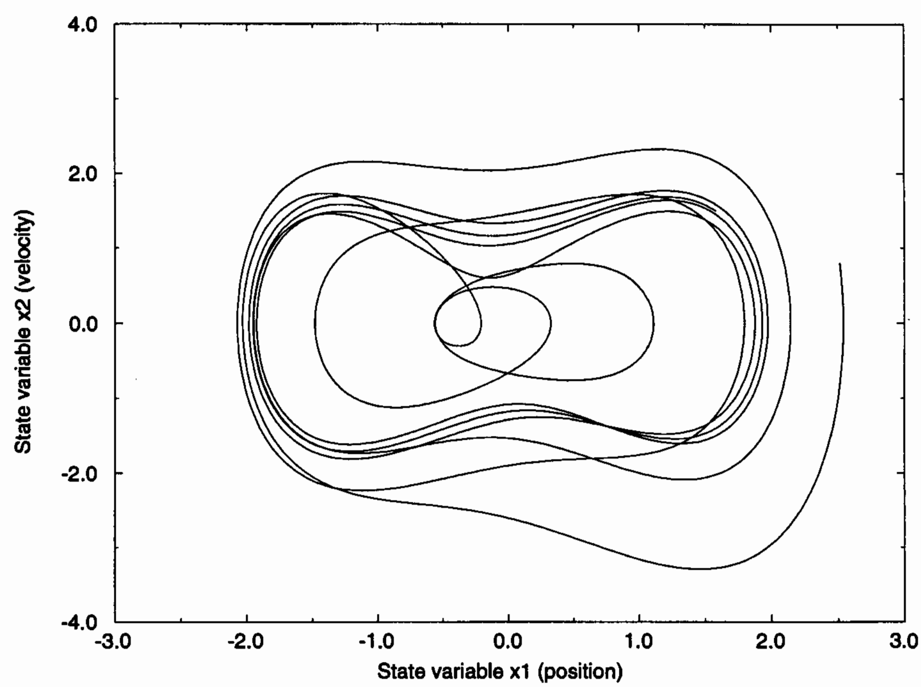

Fig. 2. Phae portrait of uncontrolled Duffing's oscillator.

One of the Floquet multipliers of this system is 1.25, which indicates instability. From Eq. (17), the state transition matrix for this system may be factored as

$$\Phi(t) = Q(t)e^{Rt} \tag{17}$$

where $Q(t)$ is a real $2T$-periodic matrix and R is a real constant matrix. Following Sinha and Joseph [1994] and Sinha *et al.* [1996], the constant matrix R, the L–F transformation matrices $Q(t)$ and $Q^{-1}(t)$ are computed and with $z(t) = Q(t)q(t)$, Eq. (41) transforms to

$$\dot{q}(t) = \begin{bmatrix} -0.052 & -0.009 \\ -27.66 & -0.734 \end{bmatrix} q(t) + Q^{-1}(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t . \tag{42}$$

The eigenvalues of the system matrix are $\lambda_{1,2} = -0.9973,\ 0.2113$, indicating that the uncontrolled system is unstable. Following the procedure outlined in Sec. 3, a feedback controller is designed using the pole placement technique. The poles are placed at -2, -3. Figures 3 and 4 show how the addition of full state feedback to the existing feedforward control drives the trajectory of the system to the desired simple periodic orbit.

If one desires that the system achieve a more complex periodic orbit such that

$$y_2(t) = 2\cos t + \sin t + 0.5\cos 2t + \sin 2t \tag{43}$$

then the feedforward control is chosen as

$$u_f = \ddot{y}_2 - y_2 + y_2^3 + 0.25\dot{y}_2 - \cos t \tag{44}$$

which results in the linearized periodic system

$$\dot{z}(t) = \left(\begin{bmatrix} 0 & 1 \\ -\frac{67}{8} - 6\cos t - \frac{9}{2}\cos 2t + \frac{9}{8}\cos 4t - \frac{9}{2}\sin t - 6\sin 2t - \frac{15}{2}\sin 3t - \frac{3}{2}\sin 4t & -\frac{1}{4} \end{bmatrix} \right) z(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t . \tag{45}$$

After the application of Lyapunov–Floquet transformation, the corresponding constant coefficient system is found to be

$$\dot{q}(t) = \begin{bmatrix} -1.686 & -0.058 \\ 1.292 & 0.115 \end{bmatrix} q(t) + Q^{-1}(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t . \tag{46}$$

Once again the eigenvalues of the transformed system are $\lambda_{1,2} = -1.64,\ 0.072$, indicating that the system is unstable. Placing the poles at -2 and -3, the feedback control u_t is designed to drive the trajectory of the nonlinear system to the specified complex periodic orbit, as shown in Figs. 5 and 6.

4.1.2. *Control to a fixed point*

In this case let the desired trajectory be of the form

$$y(t) = y_0 . \tag{47}$$

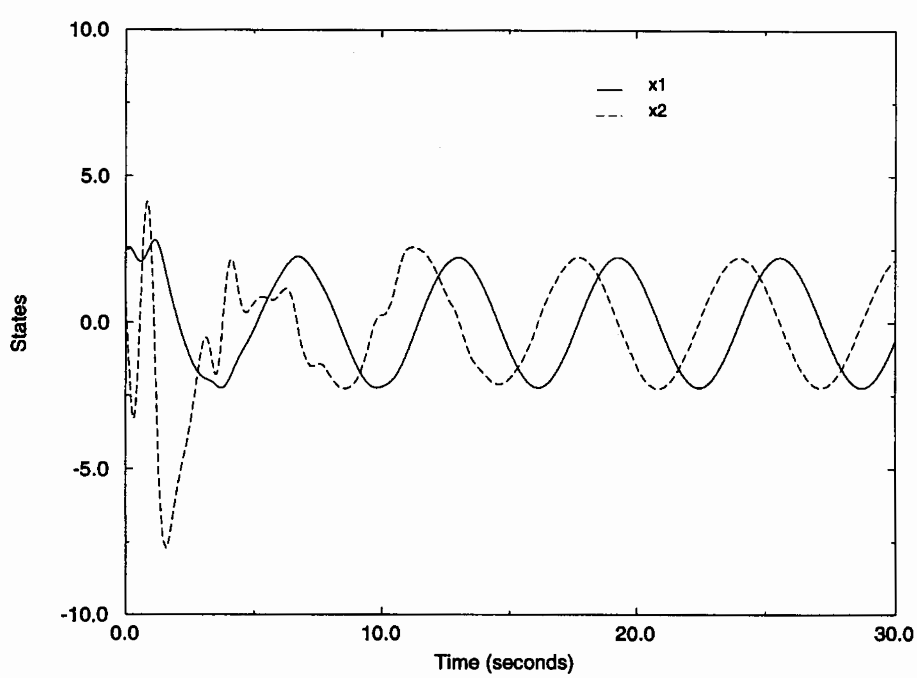

Fig. 3. Time trace of Duffing's oscillator controlled to periodic orbit $y = 2\cos t + \sin t$.

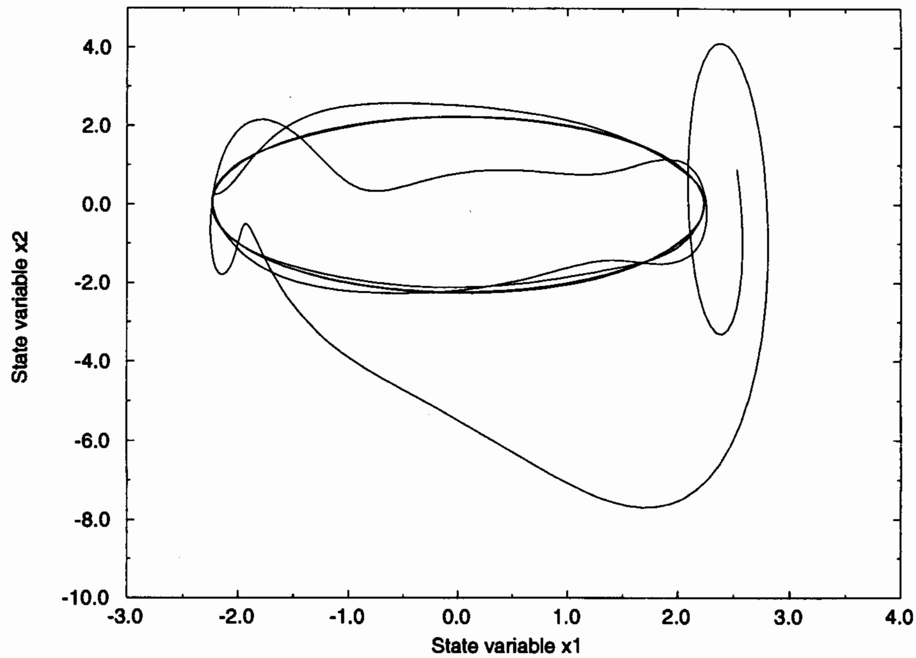

Fig. 4. Phase portrait of Duffing's oscillator controlled to periodic orbit $y = 2\cos t + \sin t$.

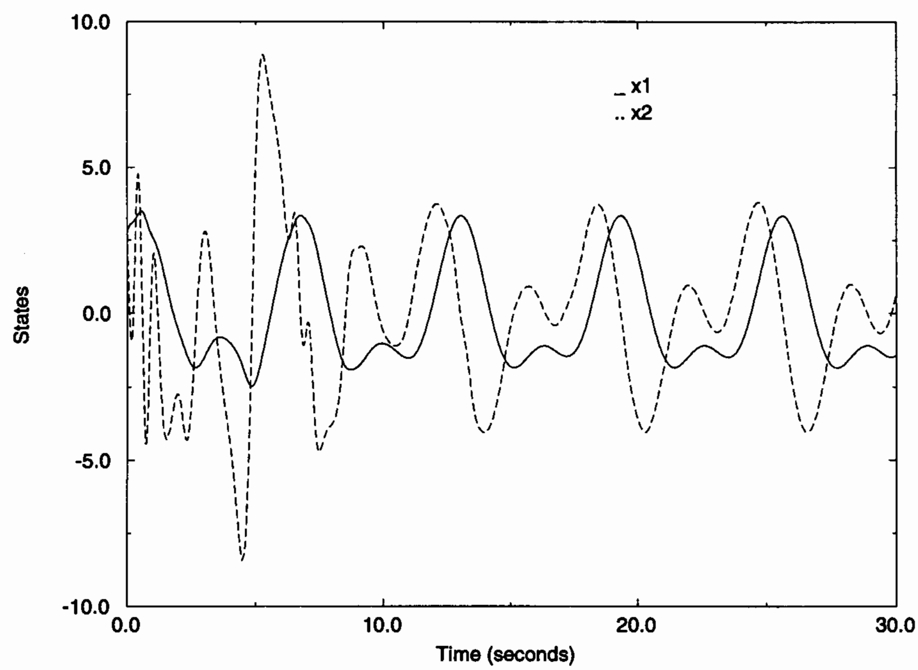

Fig. 5. Time trace of Duffing's oscillator controlled to periodic orbit $y = 2\cos t + \sin t + 0.5\cos 2t + \sin 2t$.

The feedforward part of the control law is given by

$$u_f = \alpha y_0 + y_0^3 - f\cos\omega t\,. \tag{48}$$

Matrix $A(t)$ of Eq. (36) now has constant coefficients, i.e.

$$A(t) = A = \begin{bmatrix} 0 & 1 \\ -(\alpha + 3y_0^2) & -2\zeta \end{bmatrix}. \tag{49}$$

The controller can be designed using standard techniques, viz. pole placement, etc.

As an example, consider the same system as given by Eq. (38) with a desired fixed point of

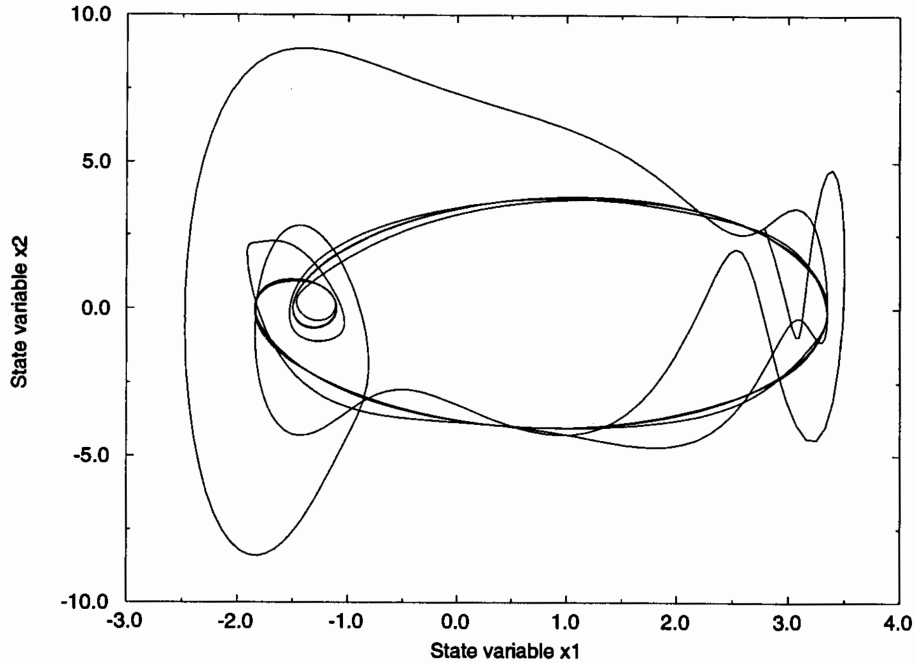

Fig. 6. Phase portrait of Duffing's oscillator controlled to periodic orbit $y = 2\cos t + \sin t + 0.5\cos 2t + \sin 2t$.

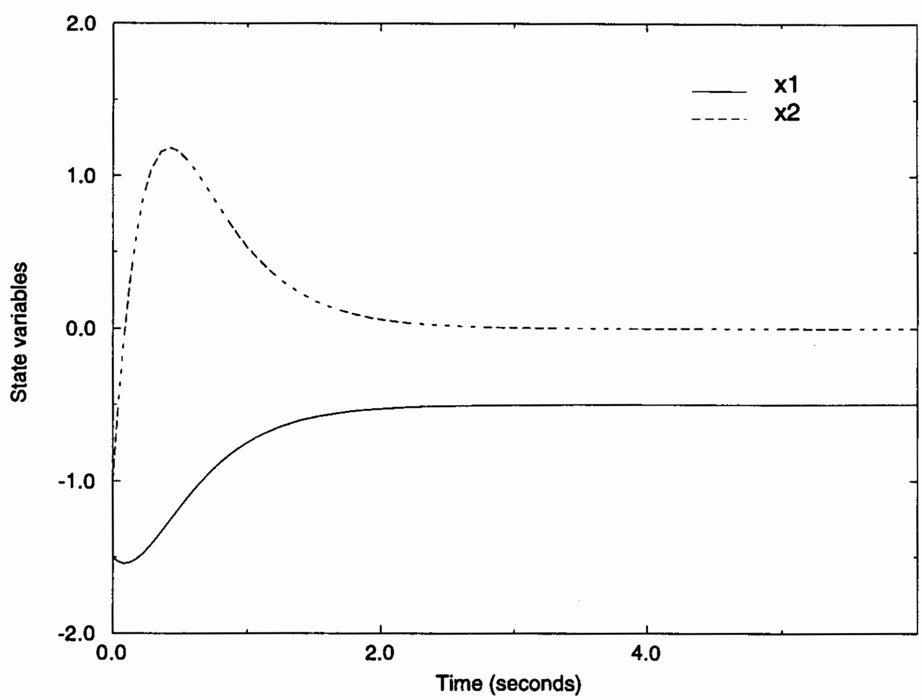

Fig. 7. Time trace of Duffing's oscillator controlled to fixed point $y = -0.5$.

$y_0 = -0.5$. Here, Eq. (41) takes the form

$$\begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.25 & -0.25 \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \,. \quad (50)$$

Note that the resulting system has constant coefficients. In this case no L–F transformation is required; we may directly employ pole placement or Routh–Hurwitz criteria to generate the feedback portion of the control law. Figures 7 and 8 demonstrate the effectiveness of this control scheme in driving the system to the desired fixed point.

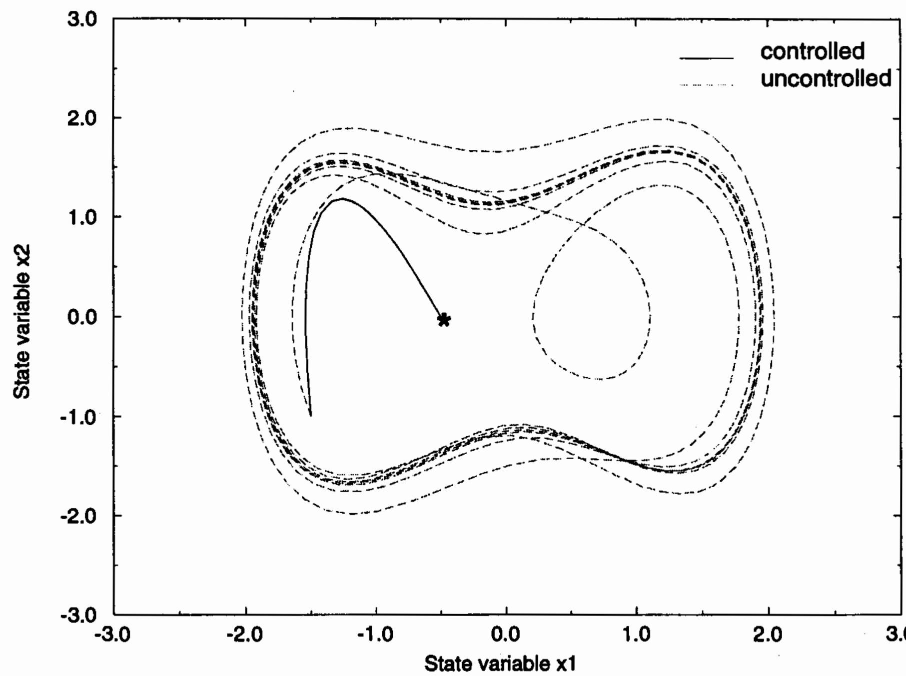

Fig. 8. Phase portrait of Duffing's oscillator controlled to fixed point $y = -0.5$.

4.2. *Rossler's System*

The Rossler equations form a nonlinear system notable for its recognizable strange attractor (the "Rossler bands") and the example it presents of the low degree of nonlinearity necessary to induce chaos. The Rossler equations are given by

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3\\ \dot{x}_2 &= x_1 + 0.2x_2\\ \dot{x}_3 &= 0.2x_3(x_1 - a)\end{aligned} \tag{51}$$

where a is the critical parameter. Note that there is only a single mild nonlinearity, located in the third state equation. Nonetheless, for $a = 5.7$, Eqs. (51) possess a familiar chaotic attractor ("Rossler bands") shown in Figs. 9 and 10.

Consider an arbitrary desired periodic trajectory (as shown in Figs. 9 and 10)

$$y = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 5 + \cos t \\ \sin t \\ \sin t \end{Bmatrix}. \tag{52}$$

With feedforward and feedback Eqs. (51) take the form

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3 + u_{f,1} + u_t\\ \dot{x}_2 &= x_1 + 0.2x_2 + u_{f,2}\\ \dot{x}_3 &= 0.2x_3(x_1 - 5.7) + u_{f,3}\end{aligned} \tag{53}$$

Following the formulation of Sec. 2, a feedforward control is chosen as

$$\begin{aligned}u_{f,1} &= \dot{y}_1 - f_1(y_1, t)\\ u_{f,2} &= \dot{y}_2 - f_2(y_2, t)\\ u_{f,3} &= \dot{y}_3 - f_3(y_3, t)\end{aligned} \tag{54}$$

where $f = \{f_1 f_2 f_3\}^T$ is the right-hand side of Eqs. (51). For the desired periodic orbit of Eq. (52), Eq. (54) becomes

$$\begin{aligned}u_{f,1} &= \sin t\\ u_{f,2} &= -5 - 0.2 \sin t\\ u_{f,3} &= -1 + \cos t + 0.1 \sin t - 1.14 \sin 2t\,.\end{aligned} \tag{55}$$

By including this feedforward control and linearizing about the goal trajectory, we can express the linearized Rossler system as

$$\begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0.2 \sin t & 0 & -0.14 + 0.2 \cos t \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_t \tag{56}$$

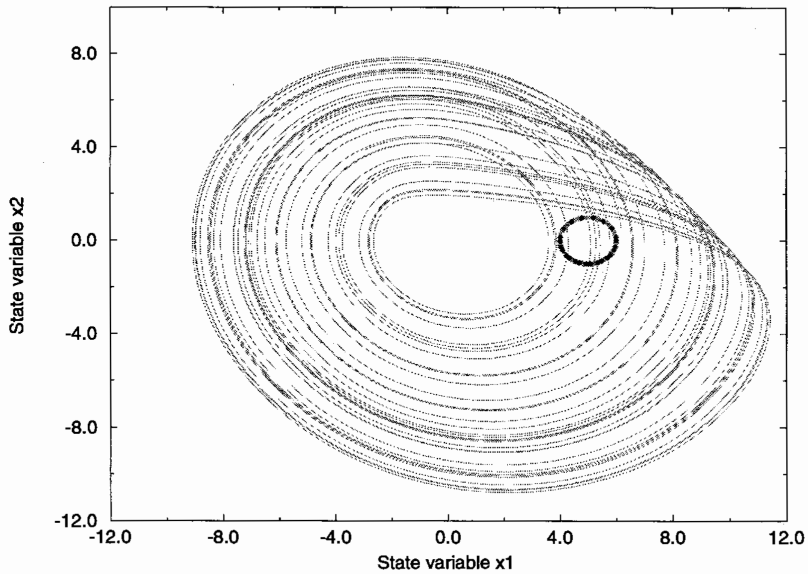

Fig. 9. Phase plane $x1$–$x2$ behavior of uncontrolled Rossler system with desired periodic orbit highlighted.

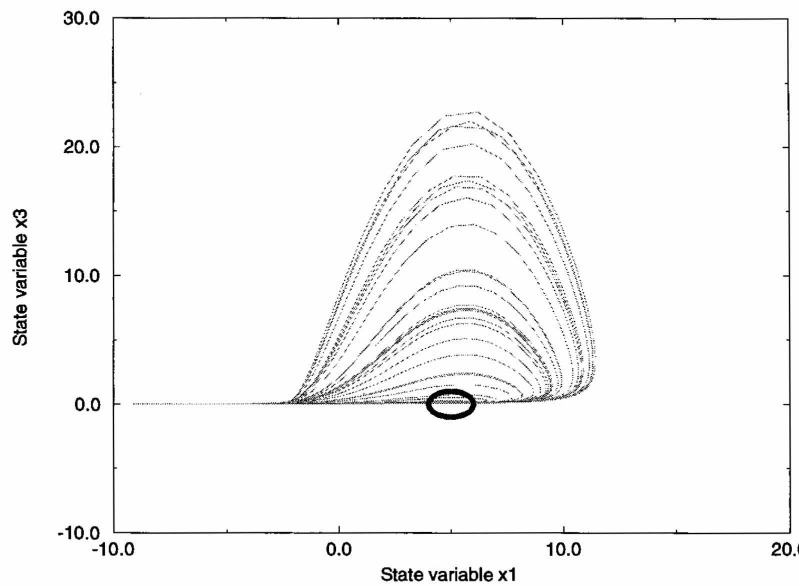

Fig. 10. Phase plane $x1$–$x3$ behavior of uncontrolled Rossler system with desired periodic orbit highlighted.

where $e = x - y$, the error between the actual trajectory and the goal trajectory. Again, as the control is to a periodic orbit, periodic terms appear in the Jacobian. Employing Chebyshev polynomial expansion and Floquet theory, we can determine the Floquet multipliers of the system. Here, one of these multipliers has a magnitude of 1.88, indicating an unstable error state.

From the calculated STM, we can find $Q(t)$ and $Q^{-1}(t)$, and apply the L–F transformation to convert the linear periodic system into its time-invariant form, yielding

$$\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} = \begin{bmatrix} 1.59 & 1.61 & 2.57 \\ -6.09 & -9.67 & -16.2 \\ 1.67 & 2.95 & 4.93 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} + Q^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_t \tag{57}$$

where the *eigenvalues* of the R matrix are $\lambda_{1,2,3} = -3.65,\ -0.1298,\ 0.6298$, the last of these indicating the instability of the time-invariant transformed system. The time invariant gain matrix $\overline{K}$ is again determined from pole placement as

$$\overline{K} = \begin{bmatrix} -3.59 & -1.61 & -2.57 \\ 6.09 & 6.67 & 16.2 \\ -1.67 & -2.95 & -8.93 \end{bmatrix} \tag{58}$$

giving poles at $s = -2, -3, -4$. The state feedback control is then calculated from

$$u_t(t) = -B^* Q(t)\overline{K}Q^{-1}(t)e(t)\,. \tag{59}$$

The compound control vector $u_t + u_f$ drives the Rossler system to the desired periodic orbit (Figs. 11 and 12).

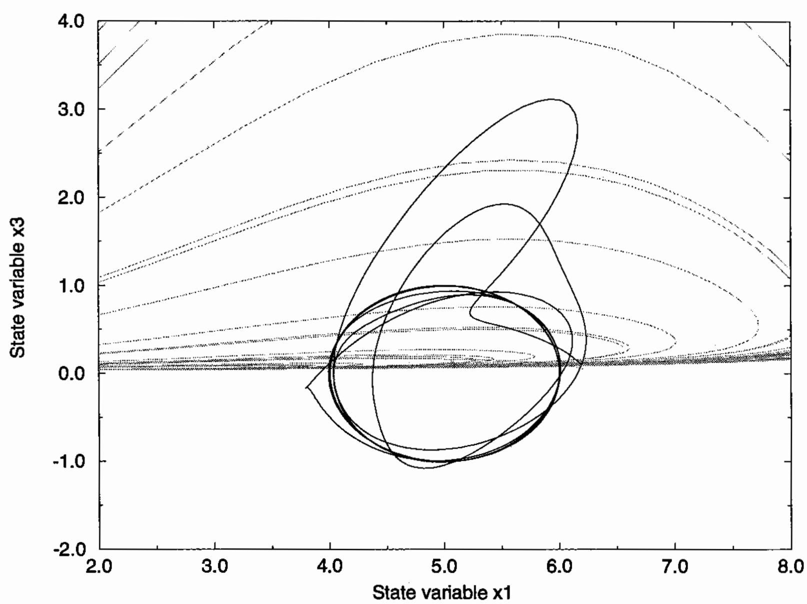

Fig. 11. Phase plane $x1$–$x3$ behavior of Rossler system subject to feedforward and full state feedback control.

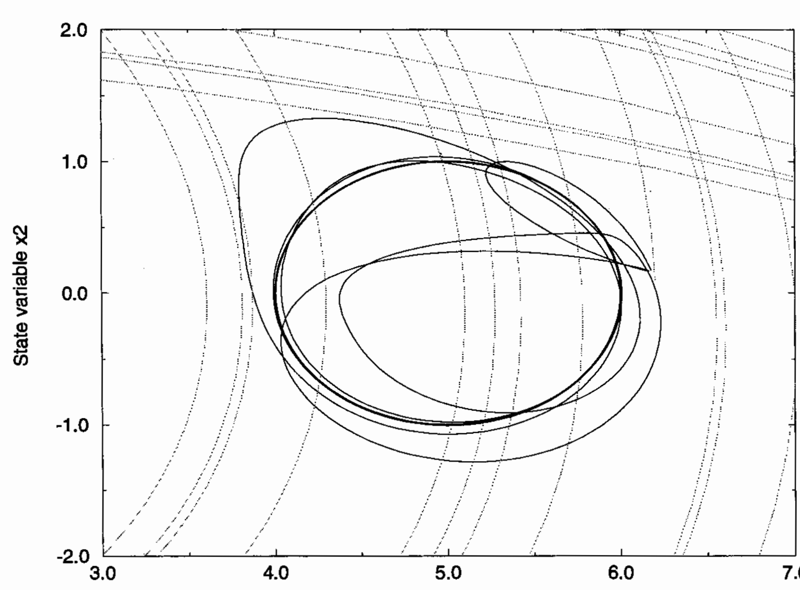

Fig. 12. Phase plane $x1$–$x2$ behavior of Rossler system subject to feedforward and full state feedback control.

5. Discussion and Conclusions

A comprehensive procedure for designing active controllers for nonlinear nonautonomous systems exhibiting chaos is outlined. The chaotic trajectory can be directed to any desired orbit or to a fixed point. The desired periodic motion does not have to be embedded in the chaotic attractor. The control vector consists of two parts, viz. a feedforward component and a feedback component with a time-varying gain. It is shown that the linearization about the desired periodic orbit leads to a set of linear equations with periodic coefficients. The control of the resulting periodic system is achieved through an application of Lyapunov–Floquet transformation which permits the use of standard time-invariant control techniques. It has been pointed out that the controller gain cannot be designed via Routh–Hurwitz criteria due to the nonautonomous nature of the problem. In the event when the chaotic motion is driven to a desired fixed point, then the linearized system matrix is time-invariant and classical control techniques may be employed. Numerical results are presented for the Duffing's oscillator and the Rossler system to show the effectiveness of the proposed technique.

It is observed that the methodology presented above makes use of full state feedback. In practice, all states are generally not measurable and an observer based controller must be designed. Using the dual system approach [Kwakernaak & Sivan, 1972], Sinha and Joseph [1994] suggested the design of an observer based controller for time-periodic systems via Lyapunov–Floquet transformation which is very similar to the approach presented here for the full state feedback systems. Their technique can be directly extended to address the problem at hand where an observer must be implemented.

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