A STRATEGY FOR CHAOS CONTROL OF NONLINEAR NONAUTONOMOUS SYSTEMS

S. C. Sinha

Nonlinear Systems Research Laboratory
Dept. of Mechanical Engineering
Auburn University, AL 36849
Phone: (334) 844 3325
Fax: (334) 844 3307
E-mail: ssinha@eng.auburn.edu

Abstract

A general framework for local linear control of nonlinear nonautonomous systems using feedback strategies is considered. In particular, it is shown that a parametrically excited nonlinear system exhibiting chaos can be driven to a desired periodic motion or to a fixed point by designing a combination of a feed forward controller and a feedback time-periodic controller. The design of feedback controller can be achieved via two different techniques. In the first approach, the open-loop system is transformed to a time-invariant form through an application of Lyapunov-Floquet transformation. Then by constructing an auxiliary system , the feedback controller is designed in time -invariant domain by employing classical techniques, such as pole placement or optimal control. In the second approach, a symbolic technique is implemented to compute the Floquet transformation matrix (monodromy matrix) of the closed -loop system containing the unknown gain parameters of the controller. Then by using Jury's criteria, the gain parameters can be selected to guarantee the orbital stability. Examples are presented to show engineering applications.

Introduction

Bifurcations and chaotic phenomena in nonlinear dynamical systems have been investigated extensively in the last two decades. Recently, control strategies to suppress bifurcations and chaos in nonlinear systems have been proposed in the literature. Some of these employ feedback and the most popular among such schemes is the so-called OGY

(Ott-Grebogi-Yorke) method [Ott *et al.*, 1990, Shinbrot *et al.*, 1993]. This method relies on the facts that chaotic systems are extremely sensitive to initial conditions and that there are typically an infinite number of unstable periodic orbits that co-exist with chaotic motion. These properties are exploited to stabilize unstable periodic orbits embedded in the chaotic attractor with control perturbations. However, in real applications, one requires a continuous analysis of the state of the system. The changes in parameters can only be discrete since the OGY method uses the Poincare map of the system. As pointed out by Kapitaniak [1996] this leads to some serious limitations. The method can be used to stabilize only those orbits whose largest Lyapunov exponent is small compared to the reciprocal of the time interval between parameter changes. This can be avoided by designing a continuous time control system. Local stabilization of periodic orbits in a chaotic Duffing's oscillator (among others) using a linear state feedback controller has been considered by Chen and Dong [1993a, b] and Chen [1996]. However, in all these papers the non-autonomous nature of the problem is ignored and the stability analysis is carried out using the Routh-Hurwitz criteria as if the systems were autonomous. Since the problem of stabilization of a periodic orbit leads to a linear system with periodic coefficients, Floquet theory must be used to guarantee the necessary and sufficient conditions. This has also been pointed out by Leung *et al.* [1995]. In a recent paper Chen (1997) has used Lyapunov functions to guarantee the global asymptotic stability of the desired trajectories in Chua's circuit and Duffing's oscillator undergoing chaotic motion.

In the present work a general technique for local control of chaos in non-autonomous systems is presented. Standard results in nonlinear and time-varying control systems theory are used to put the problem of control of chaos in the proper perspective. Similar to the studies reported by several authors mentioned above (for example, Chen, 1997), the controller design is set up as a tracking problem. The control law considered in this work consists of a combination of a feedforward and a linear time varying feedback. It is shown that this control law can be used to direct the chaotic motion to any desired periodic orbit or to a fixed point. The fact that the *desired orbit need not be a solution of the uncontrolled system* is a novelty of this approach. The problem is formulated as an asymptotic stabilization problem of the origin of a nonlinear, non-autonomous system. Under certain assumptions (stated later), the first method of Lyapunov guarantees the local stability of the original system on the basis of the linearized system. In the present case this linearized system is time-varying and periodic. Hence one can make use of the method proposed by Sinha and Joseph [1994] to design a time periodic controller via Lyapunov-Floquet transformation. This approach has been successfully employed to study other problems [Boghiu *et al.*, 1998]. If it is desired that the chaotic motion be driven to a fixed point, then the local stabilization can be achieved via Routh-Hurwitz criteria and pole placement because the linearized system matrix is time-invariant On the other hand a symbolic computation technique (proposed by Sinha and Butcher (1997)) can be used to compute the state transition matrix (STM) of a periodic system in a symbolic form as a function of the parameters. The procedure employs Chebyshev polynomial expansion and Picard iteration. In the proposed method we compute the STM as a function of the control gains. Then using Floquet's stability theorem the control gains are determined by placing the Floquet multipliers in any desired positions. This can be done by Jury's criteria. A Rossler system and a base-excited pendulum are considered to exemplify the design of the control strategy presented in this work.

2. Local stabilization

Consider a non-autonomous system with the control law $\boldsymbol{u}$, given by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}[t, \boldsymbol{x}(t)] + \boldsymbol{u}(t) \quad \textbf{(1)}$$

When the control term $\boldsymbol{u}(t)$ is absent, the above system has a chaotic attractor for a given set of parameter values. Let $\boldsymbol{y}(t)$ be the desired orbit which may be an unstable periodic orbit of equation (1) without the control term $\boldsymbol{u}$. In this paper we assume $\boldsymbol{y}(t)$ to be any arbitrary desired smooth periodic orbit. Consider a control law consisting of two parts,

$$\boldsymbol{u}(t) = \boldsymbol{u}_f + \boldsymbol{u}_t \quad \textbf{(2)}$$

viz., feedforward ($\boldsymbol{u}_f$) and a linear time-varying feedback ($\boldsymbol{u}_t$), where

$$\boldsymbol{u}_f = \dot{\boldsymbol{y}} - \boldsymbol{f}[t, \boldsymbol{y}(t)] \quad \textbf{(3)}$$

With this control law, equation (1) can be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}[t, \boldsymbol{x}(t)] + \boldsymbol{u}_f + \boldsymbol{u}_t \quad \textbf{(4)}$$

Defining

$$\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{y} \quad \textbf{(5)}$$

as the error between the actual and desired trajectories, the objective is to choose ***u(t)*** such that ***e*** 60 as *t* 6 4.
Using equation (5) and $\boldsymbol{u}_f$ defined in equation (3), equation (4) can be written as

$$\dot{\boldsymbol{e}} = \boldsymbol{g}[t, \boldsymbol{e}(t)] + \boldsymbol{u}_t \qquad \textbf{(6)}$$

where the nonlinear function ***g[.]*** is appropriately defined in terms of ***f[.]***. It may be noted that when the feedforward part is incorporated into equation (4), it ensures that the origin ***e*** = ***0*** is an equilibrium point of equation (6), even if the desired orbit ***y(t)*** is not a solution of the uncontrolled version of equation (1). Of course, when ***y(t)*** is a solution of the equation (1) without the control term, the feedforward part ($\boldsymbol{u}_f$) is identically zero. In this case one recovers the formulation for stabilization of the unstable periodic orbit embedded in the basin of a chaotic attractor as considered in Shinbrot *et al.*[1993] and Chen and Dong [1993b]. At this point , we assume that equation (7) is linearizable and therefore, the local dynamics
is given by

$$\dot{\boldsymbol{e}} = \boldsymbol{A}(t)\boldsymbol{e}(t) + \boldsymbol{u}_t \; ; \; \boldsymbol{A}(t) = \boldsymbol{A} = \left[\frac{\partial \boldsymbol{g}(t,\boldsymbol{e})}{\partial \boldsymbol{e}}\right]_{\boldsymbol{e}=0} \qquad \textbf{(7)}$$

It is important to note that for time-varying systems, such as one given by equation (7), uniform asymptotic stability cannot be inferred by studying only the eigenvalues of the matrix for each fixed time instant. For a simple example to illustrate this point, one can refer to Vidyasagar [1978]. Ignoring these subtleties, stability criteria were derived by Chen and Dong [1993a, b] and Chen [1996] using the Routh-Hurwitz criteria on the time-periodic Jacobian matrix resulting from the linearization process. If $A(t)$ is periodic then one can use Floquet theory to determine stability and response.

3. Controller design via Lyapunov-Floquet transformation

Consider the general form of system (7)

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}(t)\boldsymbol{z}(t) + \boldsymbol{B}(t)\boldsymbol{u}_t \qquad \textbf{(8)}$$

where matrices $\boldsymbol{A(t)}$ and $\boldsymbol{B(t)}$ has periodic coefficients with period T, and pair [$\boldsymbol{A}$, $\boldsymbol{B}$] is controllable. In this section the idea of Lyapunov-Floquet transformation is employed to design a full state feedback control law of the type

$$\boldsymbol{u}_t = \boldsymbol{F}(t)\boldsymbol{z}(t) \tag{9}$$

where $\boldsymbol{F(t)}$ is the time varying feedback matrix. Applying the Lyapunov-Floquet (L-F) transformation (see Sinha, *et al*, 1996)

$$\boldsymbol{z}(t) = \boldsymbol{Q}(t)\boldsymbol{q}(t) \;;\; \boldsymbol{Q}(t) = \boldsymbol{Q}(t+2T) \tag{10}$$

equation (8) can be transformed to

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{R}\boldsymbol{q}(t) + \boldsymbol{Q}(t)^{-1}\boldsymbol{B}(t)\boldsymbol{u}_t(t). \tag{11}$$

Where $\boldsymbol{R}$ is a real time-invariant matrix.

In order to design a controller for system given by equation (11), an auxiliary time invariant system of the type

$$\dot{\eta}(t) = \boldsymbol{R}\eta + \boldsymbol{B}_o\boldsymbol{v}(t), \tag{12}$$

is constructed. In equation (12), $\boldsymbol{B_o}$ is a constant matrix and has the same rank as $\boldsymbol{B(t)}$, such that the pair ($\boldsymbol{R}$, $\boldsymbol{B_o}$) is controllable. The control vector $\boldsymbol{v(t)}$ is determined by designing a full state feedback controller using either the pole placement technique or the optimal control theory. Then by minimizing the performance error between systems given by equations (11) and (12) and using inverse L-F transformation we obtain [see Sinha & Joseph, 1994]

$$\boldsymbol{u}_t(t) = \boldsymbol{B}^*(t)\boldsymbol{Q}(t)\boldsymbol{B}_o\boldsymbol{F}_o\boldsymbol{Q}^{-1}(t)\boldsymbol{z}(t) \equiv \boldsymbol{F}(t)\boldsymbol{z}(t); \quad \boldsymbol{B}^*(t) = \textit{generalized inverse} \tag{13}$$

4. Controller Design via Symbolic Computation

Once again we consider equation (8) with a constant gain controller such that

$$\boldsymbol{u}_t = -\boldsymbol{k}^T\boldsymbol{z}(t), \; \boldsymbol{k}^T = \left\{ k_1, k_2, \ldots\ldots\ldots k_n \right\} \tag{14}$$

Therefore, the closed loop system is given by

$$\dot{\boldsymbol{z}} = [\boldsymbol{A}(t) - \boldsymbol{B}(t)\boldsymbol{k}^{T}]\ \boldsymbol{z} \equiv \boldsymbol{L}(t,k_i)\boldsymbol{z} \qquad \textbf{(15)}$$

Where the $\boldsymbol{L}\ (t,k_i)$ contains the unknown constants k_i.
The stability depends on the eigenvalues of the Floquet transformation matrix (FTM) $\mathsf{M}(T,k_i)$ associated with equation (15). It has been shown by Sinha and Butcher (1997) that the state transition matrix for equation (20) can be computed in symbolic form through a technique based on Chebyshev polynomials. Then by using Jury's criteria [Jury, 1974], one can select the gains k_i such that the eigenvalues of FTM, $\mathsf{M}(T,k_i)$ lie within the unit circle, thereby guaranteeing the asymptotic stability of equation (15).

5. Applications

5.1 Rossler's System

The Rossler equations are given by

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 + 0.2x_2 \\ \dot{x}_3 &= 0.2x_3(x_1 - a) \end{aligned} \qquad \textbf{(16)}$$

Where a is the critical parameter. For $a = 5.7$, equations (16) possess a familiar chaotic attractor ("Rossler bands") shown in Figures 1 and 2.

Consider an arbitrary desired periodic trajectory (also shown in Figures 1 and 2)

$$\boldsymbol{y}^{T} = \left\{ y_1,\ y_2,\ y_3 \right\}^{T} = \left\{\ 5 + \cos t,\ \sin t,\ \sin t \right\}^{T} \qquad \textbf{(17)}$$

Following the formulation of Section 2, a feedforward control is chosen as

$$\begin{aligned} u_{f,1} &= \dot{y}_1 - f_1(y_1,t) = \sin t \\ u_{f,2} &= \dot{y}_2 - f_2(y_2,t) = -5 - 0.2\sin t \\ u_{f,3} &= \dot{y}_3 - f_3(y_3,t) = -1 + \cos t + 0.1\sin t - 1.14\sin 2t \end{aligned} \qquad \textbf{(18)}$$

By including this feedforward control and linearizing about the goal trajectory, we can

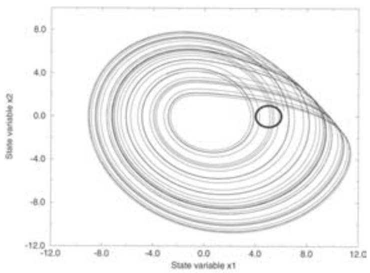

figure-1

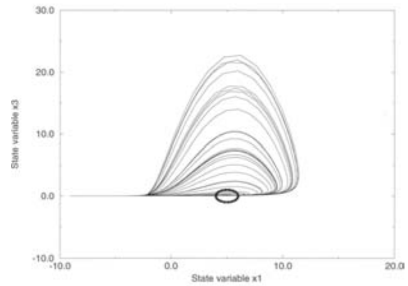

figure-2

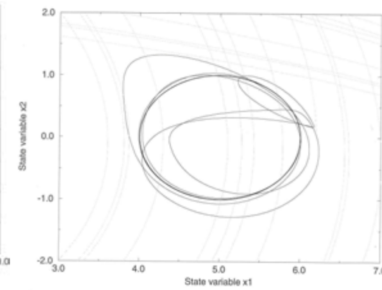

figure-3

express the linearized system as

$$\begin{Bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{Bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0.2\sin t & 0 & -0.14+0.2\cos t \end{bmatrix} \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_t \tag{19}$$

where $\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{y}$, the error between the actual trajectory and the goal trajectory. This is a . periodic system and one of these multipliers has a magnitude of 1.88, indicating an unstable error state. $\boldsymbol{u}_t$ is designed according to the procedure described in Section 3. The compound control vector $\boldsymbol{u}_t + \boldsymbol{u}_f$ drives the Rossler system to the desired periodic orbit (Figures 1, 2 & 3).

5.2 Parametrically Excited Pendulum

As an example of parametrically excited system, a base excited pendulum is considered (see figure 4). The control torque U is applied at the suspension point. The equation of motion in the state space form expanded into Taylor series about the bottom equilibrium point up to the fifth order terms is given as (David and Sinha, 2000)

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(a+b\sin 2t) & -d \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ (a+b\sin 2t)\left(\frac{x_1^{\,3}}{3!} - \frac{x_1^{\,5}}{5!}\right) + u \end{Bmatrix} \tag{20}$$

Where $\omega\tilde{t}=2t$, $a=4g/(\omega^2 L)$, $b=A/L$, $d=4c/(ML^2\omega^2)$, $u=4U/(ML^2\omega^2)$.
For a=0.1, d=0.316, T=2 and b=2.5, the system exhibits chaos (see figure-5). Following the procedure described in section 4, a linear controller is designed via symbolic computation. The control gains k_1 = -1.9184 and k_2 = 0.0891 are chosen to place the Floquet multipliers at 0.8 and 0.3. This controller drives the system to one of the periodic orbits of the system as shown in figure 6.

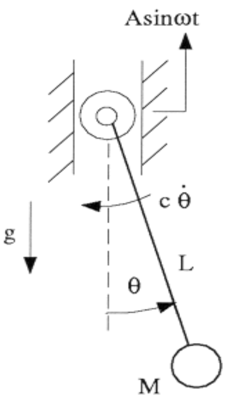

figure-4

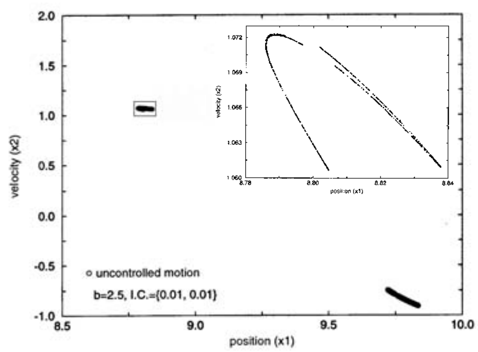

figure-5

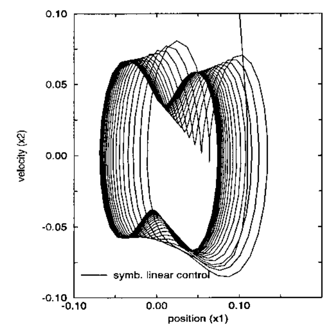

figure-6

5. Discussion and Conclusions

Two procedures for designing active controllers for nonlinear nonautonomous systems exhibiting chaos is outlined. The chaotic trajectory can be directed to any desired orbit or to a fixed point. The desired periodic motion does not have to be embedded in the chaotic attractor. The control vector consists of two parts, *viz.*, a feedforward component and a feedback component with a time-varying gain. It is shown that the linearization about the desired periodic orbit leads to a set of linear equations with periodic coefficients. In the first approach the control of the resulting periodic system is achieved through an application of Lyapunov-Floquet transformation which permits the use of standard time-invariant control techniques. In the second method, a constant gain controller is employed using a symbolic technique. Numerical results are presented for the Rossler system and a parametrically excited pendulum to show the effectiveness of the proposed technique.

Acknowledgments

Financial support provided by the National Science Foundation (Grant No. 9713971 and 0114571) is gratefully acknowledged.

References

Boghiu, Dan, Sinha, S. C. and Marghitu, Dan B. [1998] "Stability and control of a parametrically excited rotating system. Part II: Controls", *Dynamics and Control,* **8***,* pp. 19-35.

Chen, C-C. [1996] “Direct chaotic dynamics to any desired orbits via closed loop control", *Physics Letters*, **A213**, pp. 148-154.

Chen, G. [1997] “On some controllability conditions for chaotic dynamics control”, *Chaos, Solitons and Fractals*, **8**, pp. 1461-1470.

Chen, G. and Dong, X. [1993a] “From chaos to order, perspectives and methodologies in controlling chaotic nonlinear dynamical systems", *International Journal of Bifurcation Theory and Chaos*, **3**, pp. 1363-1409.

Chen, G. and Dong, X. [1993b] “On feedback control of chaotic continuous time systems", *IEEE Transaction on Circuits and Systems*, **40**, pp. 591-601.

David, A. and Sinha, S.C., [2000] “Control of chaos in nonlinear systemswith time-periodic coefficients”, Proceedings of the American Control Conference, Chicago, IL, pp. 764-768

Jury, E.I., [1974] “Inners and Stability of Dynamic Systems”, Wiley-Interscience, N. Y.

Leung, A.Y.T., Chan, C.W. and Ravindra, B. [1995] “Compensator for nonlinear dynamical systems", *Proceedings of the international conference on Structural Dynamics, Vibration, Noise and Control*, Hong Kong, December 5-7, pp. 546-549.

Ott, E., Grebogi, C. and Yorke, J.A. [1990] “Controlling chaos", *Physics Letters*, **A64**, pp. 1169-1199.

Shinbrot, T., Grebogi, C. and Yorke, J.A. [1993] “Using small perturbations to control chaos", *Nature*, **65**, pp. 3211-3214.

Sinha, S.C., and Joseph, P. [1994] “Control of general dynamic systems with periodically varying parameters via Lyapunov-Floquet transformation", *ASME Journal of Dynamic Systems, Measurement and Control*, **116**, pp. 650-658.

Sinha, S.C., Pandiyan, R. And Bibb, J.S. [1996] “Liapunov-Floquet transformation: Computation and applications to periodic systems”, *Journal of Vibration and Acoustics*, **118**, pp. 209-219.

Sinha, S.C. and Butcher, E.A. [1997] “Symbolic computation of fundamental solution matrices for linear time-periodic dynamical systems, *Journal of Sound and Vibration*, 206 (1), pp. 61-85

Vidyasagar, M.,”Nonlinear Systems Analysis”, Prentice Hall, Inc., Englewood Cliffs, N.J., 1978.