ANALYTICAL DEVELOPMENT OF A SLOTTED GRAIN SOLID ROCKET MOTOR

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Abstract

In most practical solid rocket motor design processes, final designs for grains are arrived at using computer-generated grids. This process is imminently practical for cases in which small numbers of final geometries are to be considered. However, for a grain design optimization process in which large numbers of grain configurations are to be considered, generating grids for each candidate design is often prohibitive. For such optimization processes analytical developments of burn perimeter and port area for two-dimensional grains are critically important. This paper offers a detailed development of the burn-back equations for a slotted grain with burning only on the slot faces.

Introduction

Analytical developments for solid rocket motor grains were much more prevalent in the decades before widespread use of microcomputers. Development of the burn back equations for the star grain and one type of wagon wheel can be found in Barrere. Most modern texts on solid rocket propulsion do not provide geometric details for grain design. A large number of potential grain configurations are described in NASA publications but very few geometric details are given in such publications. Recent advances in the design process for solid rocket motors involve the use of genetic algorithms to evaluate large numbers of potential grain designs to arrive at an optimum solution for the rocket motor. The evaluation of large numbers of candidate designs is much more practical if analytical expressions for burn perimeter and port as functions of burn distance are available.

This work describes an analytical development of a slotted solid rocket motor grain suitable for propelling a long, slender rod at high speeds for relatively extended times. In this configuration, the rod would extend through the center of the solid grain and the maximum diameter of the rocket is relatively small. In such configurations, a large web thickness (and burn time) is difficult to achieve with conventional grain designs. In an effort to extend the burn times, a slotted grain is considered.

Grain Development

Consider a grain situated between two cylinders with slots cut at even intervals as shown in the schematic below. The grain burns from the inside of the slots only and the slots are the only perforations. The inner cylinder radius is \( r_i \) and the outer cylinder radius is \( r_o \).
Figure 1: Schematic of a Slot Grain Design showing three slots

As the flat face of the slot burns away, the burning surface becomes a flat and a curved section. Three possible burn sequences can occur as follows:

The first phase, which is composed of the flat and initial curved sections, can end with the flat burning out. Phase II would then be only the curved section burning and would end when the burning from an adjacent slot is met at which point Phase III would occur. This is referred to as a Type I grain. Type I grains occur if the inner radius is a large portion of the outer radius and or only a few slots are present.

The first phase can end prematurely if the burning of one slot is met by the burning of an adjacent slot before the flat burns out. This means that a point is formed from the curved surfaces of the two slots and this Phase II can end by either the curved sections burning out or by the flats burning out. If the flat burns out first, the grain is referred to as a Type II. The condition for a Type II is

\[
\frac{y_o}{r_i} > \frac{\pi \varepsilon}{N}
\]

(1)

If a type II grain ends Phase II with the curves burning out, the grain is a Type III. Type II and Type III grains exist for small inner radii or large numbers of slots. The condition for a Type III is that it be a Type II and that

\[
\frac{y_o}{r_i} > \tan\left(\frac{\pi \varepsilon}{N}\right)
\]

(2)

All of the conditions for determining phase and all of the equations used to determine burn perimeter and port area are developed in this section.

Figure 2: Schematic showing burn perimeter development for Phase I
Phase I burning

Phase I is the same for all three types. Phase I ends at $y_0$ for Type I where:

$$y_0 = \sqrt{r_o^2 - r_i^2} \quad (3)$$

and Phase I ends for Type II for an angle

$$\theta = \frac{\pi \epsilon}{N} \quad (4)$$

or for a burn distance $y$ of

$$y_{phase \ II} = r_i \left( \frac{\pi \epsilon}{N} \right) \quad (5)$$

where $\epsilon$ is the percentage of the slot area filled with propellant and $N$ is the number of slots.

Burn Perimeters and Port Areas

The sections below are used to show how one half of the burn perimeter and one half of the port area for a single slot can be calculated as a function of burn distance $y$. The expressions will be multiplied by $2n$ to get the total burn perimeters and port areas.

Burn Perimeters for Phase I

The burn perimeter for phase I is composed of the flat section $s_1$ and the curved section $s_2$ which goes all the way from the end of the flat section to the inner cylindrical boundary.

$$s_1 = \sqrt{r_o^2 - y_i^2 - r_i} \quad (6)$$

To get $s_2$, an integral of $dl$ can be constructed as shown on the right side of Figure 1. The length, $dl$, can be obtained by either constructing a length $z$, differentiating, and dividing by $\cos \theta$ or it can be obtained by recognizing from the drawing that $dl$ is the length of a circular arc of radius $y - r_i \theta$ and angle $d\theta$.

$$dl = (y - r_i \theta) d\theta \quad (7)$$

This differential is a function of $\theta$ only for a given value of $y$ and a given grain design. The curve starts with $\theta = 0$ and ends when $\theta = y/r_i$ and $s_2$ can be determined by integration as follows.

$$s_2 = l = y \int_0^{y/r_i} \frac{d\theta}{r_i} - r_i \int_0^{y/r_i} \theta \, d\theta = \frac{y^2}{2r_i} - \frac{y^2}{2r_i} = \frac{y^2}{2r_i} \quad (8)$$

Figure 3: Schematic for Port Area calculation for Phase I
Port Area for Phase I

The port area associated with one half of a slot can be comprised of 4 sections as shown. The initial port area can be written as a section of a tube as follows

\[ A_p = \frac{1}{2} \left(1 - e \right) \left( \frac{\pi}{N} \right) \left( r_o^2 - r_i^2 \right) \]  

(9)

The area signified by \( A_1 \) is an arc section with a triangle subtracted. The angle for the arc can be written as

\[ \beta = \sin^{-1} \left( \frac{y}{r_o} \right) \]  

(10)

and the area \( A_1 \) is

\[ A_1 = \frac{\beta}{2} r_o^2 - \frac{y}{2} \sqrt{r_o^2 - y^2}. \]  

(11)

The area \( A_2 \) is a rectangle

\[ A_2 = s_1 y \]  

(12)

The area \( A_3 \) can be obtained by integrating the dA shown. The dA area is an arc section of radius \( y - r_i \theta \) and angle \( \theta \).

\[ dA = \frac{(y - r_i \theta)^2}{2} d\theta = \frac{1}{2} \left( y^2 - 2yr_i \theta + r_i^2 \theta^2 \right) d\theta \]  

(13)

The limits on \( \theta \) are still 0 and \( y/r_i \).

\[ A_3 = \frac{1}{2} \int_0^{y/r_i} \left( y^2 - 2yr_i \theta + r_i^2 \theta^2 \right) d\theta = \frac{1}{2} \left[ y^2 \left( \frac{y}{r_i} \right) - yr_i \left( \frac{y}{r_i} \right)^2 + r_i^2 \left( \frac{1}{3} \frac{y}{r_i} \right) \right] = \frac{y^3}{6r_i} \]  

(14)

The final port area for one half of one slot for Phase I burning is

\[ A_{\text{phase I}} = A_p + A_1 + A_2 + A_3 \]  

(15)

Figure 4: Burn Perimeter for Phase II of a Type I slot

Phase II Burn Perimeter for a Type I Grain

For Phase II burning, only the curved section burns and it burns from the outer radius all the way to the inner radius. Phase II for a Type I occurs if \( y > y_o \).

The differential length is the same as that for the \( s_2 \) calculation for Phase I. However, the initial value for \( \theta \) is no longer 0. To get the value for \( \theta_i \), refer to the triangle drawn in the figure. The Pythagorean theorem can be written for this triangle and solved for \( \theta_i \).

\[ r_o^2 = r_i^2 + (y - r_i \theta_i)^2 \]  

(16)

Or

\[ \theta_i = \frac{y - \sqrt{r_o^2 - r_i^2}}{r_i} \]  

(17)

The length of the burn perimeter \( s \) is then
\[
\int_{\theta_i}^{\pi} (y-r_i \theta) d\theta
\]

\[
l = y \left( \frac{y}{r_i} - \theta_i \right) - \frac{r_i}{2} \left( \frac{y}{r_i} \right)^2 - \theta_i^2
\]

\[
l = y \left( \frac{y}{r_i} - \frac{y}{r_i} - \frac{\sqrt{r_o^2 - r_i^2}}{r_i} \right)
\]

\[
l = y \left( \frac{y}{r_i} - \frac{\sqrt{r_o^2 - r_i^2}}{r_i} \right)
\]

\[
l = y \left( \frac{y}{r_i} - \frac{\sqrt{r_o^2 - r_i^2}}{r_i} \right)
\]

\[
l = \frac{r_o^2 - r_i^2}{2r_i^2}
\]

This is the entire burn perimeter for Phase II in a Type I grain. It is very important to note that this phase of this type is neutral regardless of the design choice.

\[
A_1 = \frac{r_o^2 \beta}{2} - \frac{r_i \sqrt{r_o^2 - r_i^2}}{2}
\]

where

\[
\beta = \cos^{-1} \left( \frac{r_i}{r_o} \right)
\]

The area \( A_2 \) can be constructed from two right triangles with a circular arc of radius \( r_i \) and angle \( \theta_i \) subtracted.

Figure 5: Port area for Phase II in a Type I slot

Port Area for Phase II burning of a Type I Grain

The port area calculation for Phase II is substantially more complex than the Phase I port area calculation. Five separate geometric areas are considered and added to the initial port area. The initial port area is unchanged. For this diagram, the area \( A_1 \) is an arc section with a triangle subtracted.
The determination of $\theta_i$ is the same for the area calculation and will be the same for phase III burning. $A_3$ is a triangle with area of 1/2 bh.

$$A_3 = \frac{1}{2} \left( (y - r_i \theta_i) + r_i \tan \left( \frac{\theta_i}{2} \right) \right) x$$

$$= \left( y_o - r_i \tan \left( \frac{\theta_i}{2} \right) \right) \sin \theta_i$$

$A_4$ is an arc section with a triangle subtracted from it.

$$A_4 = \frac{1}{2} \left( r_o^2 \theta_i \right) - \left( r_o \sin \left( \frac{\theta_i}{2} \right) \right) \left( r_o \cos \frac{\theta_i}{2} \right)$$

It can be shown that the areas $A_2$, $A_3$ and $A_4$ can be combined into the following compact relationship.

$$A_2 + A_3 + A_4 = \frac{1}{2} y_o^2 \theta_i$$

$A_5$ is an area that is calculated using the same pie-shaped differential area as that used in the Phase I $A_3$ calculation but the initial angle for $\theta$ is $\theta$, and not 0.

$$A_5 = \frac{1}{2} \int_{\theta}^{\pi} \left[ (y^2 - 2y_r \theta + r_r^2 \theta^2) d\theta \right]$$

$$= \frac{1}{2} \left[ y^2 \left( \frac{y}{r_r} \right) - y_r \left( \frac{y}{r_r} \right)^2 + r_r^2 \left( \frac{1}{3} \right) \left( \frac{y}{r_r} \right)^3 \right]$$

$$- \frac{1}{2} \left[ y^2 (\theta) - y_r (\theta) \right] + r_r^2 \left( \frac{1}{3} \right) (\theta)^3$$

$$= \frac{y^3}{6r_r} - \frac{1}{2} \left[ y^2 (\theta) - y_r (\theta) \right] + r_r^2 \left( \frac{1}{3} \right) (\theta)^3$$

The entire burn area for 1/2 of 1 slot for a Type I grain in Phase II is

$$A_{\text{phase II}} = A_p + A_1 + A_2 + A_3 + A_4 + A_5$$

Figure 6: Phase III slot geometry for Type I
area where \( n \) is the number of slots. The condition to start Phase III for a Type I grain is \( y/r_i > \theta_i/N \). For a Type II grain, the condition is \( y > y_o \). If \( \theta_i > \theta_i/N \), burnout has occurred.

The final angle \( \theta_f \) can be determined from the triangle using the expression

\[
\tan\left(\frac{\pi \varepsilon}{N} - \theta_f\right) = \frac{y - r_i \theta_f}{r_i}
\]  

This equation will need to be solved iteratively for \( \theta_f \).

The burn perimeter can then be solved from

\[
l = \int_{\theta_i}^{\theta_f} (y - r_i \theta) \, d\theta
\]  

The areas for Phase III are the same as the areas for Phase II Type I except that the limits on the integration for \( A_5 \) are modified and \( A_6 \) is added. \( A_5 \) becomes

\[
A_5 = \frac{1}{2} \int_{\theta_f}^{\theta_i} \left( y^2 - 2yr_i \theta + r_i^2 \theta^2 \right) d\theta
\]

This concludes all three phases for a Type I grain. The only remaining cases are Type II and III Phase II and Type III Phase III.

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**Figure 7: Phase II geometry for types II and III**

Types II and III, Phase II Burn Perimeter and Port Area

If a grain is in Phase I and \( y/r_i \) exceeds \( \pi/n - \varepsilon/2 \), Phase II will ensue.

**Burn perimeter**

The length \( s_1 \) is calculated exactly as it was in Phase I. The distance \( s_2 \) is calculated using the same integrand that has been used for the curved section with

\[
s_2 = \int_{0}^{\theta_i} \left( y - r_i \theta \right) \, d\theta = y \theta - \frac{r_i \theta_f^2}{2}
\]

**Port Area**

The areas \( A_1 \) and \( A_2 \) are calculated exactly as they were in Phase I. The area \( A_4 \) is calculated using the same equation as
was used for $A_5$ in Phase III of Types I and II. The area $A_3$ is calculated using the same integrand as was used for the $A_5$ calculation in Phase III but with the lower limit on $\theta$ set to 0.

Type III, Phase III burn perimeter and port area

For a Type III Grain, the only burning surface is the flat. A Type III rocket exist if $y_o$ calculated in Phase I is larger than $r_i \tan(\pi \epsilon / N)$. From the diagram shown in Figure 8, the burn perimeter is

$$s = \left[ \sqrt{r_o^2 - y^2} - r_i \right] \frac{y - r_i \tan \left( \frac{\pi \epsilon}{N} \right)}{\tan \left( \frac{\pi \epsilon}{N} \right)}$$  \hspace{1cm} (37)$$

The port area can be considered as an arc section of radius $r_o$ and angle $\beta$ (where $\beta$ is

$$A_3 = \frac{1}{2} \int_0^{\theta_f} \left( y^2 - 2yr_i \theta + r_i^2 \theta^2 \right) d\theta$$

$$= \frac{1}{2} \left( y_f^2 \left( \theta_f \right)^2 - 2yr_i \left( \frac{\theta_f^2}{2} + r_i^2 \left( \frac{\theta_f^3}{3} \right) \right) \right)$$  \hspace{1cm} (36)$$

calculated exactly as in Phase I) and a triangle with an arc section of radius $r_i$ and angle $\pi \epsilon / N$ removed.

$$A_i + A_3 = \frac{r_i^2}{2} \left( \beta \right) - \frac{1}{2} s r_i \left( \tan(\beta) \right) - \frac{1}{2} r_i^2 \left( \frac{\pi \epsilon}{N} \right)$$  \hspace{1cm} (38)$$

For this mode, burnout occurs for $s_1 = 0$. This leads to a burnout condition as Follows:

$$y = r_o \sin \left( \frac{\pi \epsilon}{N} \right)$$  \hspace{1cm} (39)$$

Figure 8: Phase III geometry for a type III

Web Thickness

The web thickness for this slotted grain design can be calculated for types I and II using an arc for an angle $\theta_{End Burn}$ and a straight section is $y_o$ where $\theta_{End Burn}$ can be determined using the equation

$$\frac{\pi \epsilon}{N} - \theta_{End Burn} = \cos^{-1} \left( \frac{r_i}{r_o} \right)$$  \hspace{1cm} (40)$$

or

$$\theta_{End Burn} = \frac{\pi \epsilon}{N} - \cos^{-1} \left( \frac{r_i}{r_o} \right)$$  \hspace{1cm} (41)$$

The equation for the web thickness is then

$$\text{web} = y_o + r_i \theta_{End Burn}$$  \hspace{1cm} (42)$$

For a Type III grain, the web thickness is given by
To illustrate the idea that the slot can potentially provide a larger web thickness than those possible with conventional cp star or wagon wheel grains, the following plots of burn area versus burn distance have been constructed. Notice that the burn area is substantial for the slotted grain, well beyond the maximum web thickness for a star grain geometry, for this type of a rocket motor configuration. The longest possible burn distance occurs for a minimum number of slots.

\[ web = r_n \sin \left( \frac{\pi \varepsilon}{N} \right) \]  

Figure 9
Figure 10

References


4. “Solid Propellant Grain Design and Internal Ballistics”, NASA SP 8076 (library.msfc.nasa.gov/cgi-bin/lsp8000)

5. “Solid Rocket Motor Performance Analysis and Prediction”, NASA SP 8039 (library.msfc.nasa.gov/cgi-bin/lsp8000)

