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## 2 Fatigue

A periodic stress oscillating between some limits applied to a machine member is called *repeated*, *alternating*, or *fluctuating* stress. The failure of the machine members under the action of these stresses is called *fatigue failure*. A small crack is enough to initiate the fatigue failure. The crack progresses rapidly since the stress concentration effect becomes greater around it. If the stressed area decreases in size, the stress increases in magnitude and if the remaining area is small, the member can fail. A member failed because of fatigue shows two distinct regions. The first one is due to the progressive development of the crack, while the other one is due to the sudden fracture.

### 2.1 Endurance Limit

The strength of materials acted upon by fatigue loads can be determined by performing a fatigue test provided by R. R. Moore's high-speed rotating beam machine (Fig. 2.1). During the test, the specimen is subjected to pure bending by using weights and rotated with constant velocity. For a particular magnitude of the weights, one records the number of revolutions at which the specimen fails. Then, a second test is performed for a specimen identical with the first one, but the magnitude of the weight is reduced. Again, the number of revolutions at which the fatigue failure occurs is recorded. The process is repeated several times. The intensity of the reversed stress causing failure after a given number of cycles is the *fatigue strength* corresponding to that number of loading cycles. Finally, the fatigue strengths considered for each test are plotted against the corresponding number of revolutions. The resulting chart is called the S-N *diagram*. The S-N curves are plotted on log-log coordinates.

Numerous tests have established that the ferrous materials have an *endurance limit* defined as the highest level of alternating stress that can be withstood indefinitely by a test specimen without failure. The symbol for endurance limit is  $S'_e$ . The endurance limit can be related to the tensile strength through some relationships. For example, for steel, Mischke predicted the following relationships [11]:

$$S'_e = \begin{cases} 0.50 S_{ut}, & S_{ut} \leq 200 \text{ kpsi (1400 MPa)} \\ 100 \text{ kpsi}, & S_{ut} > 200 \text{ kpsi} \\ 700 \text{ MPa}, & S_{ut} > 1400 \text{ MPa} \end{cases} \quad (2.1)$$

where  $S_{ut}$  (or  $S_u$ ) is the ultimate strength in tension. Table 2.1 lists the values of the endurance limit for various classes of cast iron that is polished or machined. The symbol  $S'_e$  refers to the endurance limit of the test specimen that can be significantly different from the endurance limit  $S_e$  of any machine element subjected to any kind of loads. The endurance limit  $S_e$  can be affected by several factors called *modifying factors*. Some of these factors are the surface factor  $k_S$ , the gradient (size) factor  $k_G$ , or the load factor  $k_L$ . Thus, the endurance limit of a member can be related to the endurance limit of the test specimen by the following relationship:

$$S_e = k_S k_G k_L S'_e. \quad (2.2)$$

#### Surface factor $k_S$

The influence of the surface of the specimen is described by the modification factor  $k_S$  which depends upon the quality of the finishing. The following formula describes the surface factor [20]:

$$k_S = aS_{ut}^b, \quad (2.3)$$

where  $S_{ut}$  is the ultimate tensile strength. Some values for  $a$  and  $b$  are listed in Table 2.2.

#### Gradient (size) factor $k_G$

The results of the tests performed to evaluate the size factor in the case of bending and torsion are as follows [20]:

$$k_G = \begin{cases} \left(\frac{d}{0.3}\right)^{-0.1133} & \text{in.} \quad 0.11 \leq d \leq 2 \text{ in.} \\ \left(\frac{d}{7.62}\right)^{-0.1133} & \text{mm} \quad 2.79 \leq d \leq 51 \text{ mm,} \end{cases} \quad (2.4)$$

where  $d$  is the diameter of the test bar. To apply Eq. (2.4) for a nonrotating round bar in bending or for a noncircular cross section, an effective dimension  $d_e$  is introduced [20]. This dimension is obtained by considering the volume of material stressed at and above 95% of the maximum stress and a similar volume in the rotating beam specimen. When these two volumes are equated, the lengths cancel each other out and only the areas have to be considered.

Some recommended values for  $k_G$  are given in reference [27]:

$$\begin{aligned}
 & \bullet \text{ for bending and torsion :} \\
 & d \leq 8 \text{ mm} \quad k_G = 1, \\
 & 8 \text{ mm} < d \leq 250 \text{ mm} \quad k_G = 1.189 d_e^{-0.097}, \\
 & d > 250 \text{ mm} \quad 0.6 \leq k_G \leq 0.75, \\
 & \bullet \text{ for axial loading :} \quad k_G = 1.
 \end{aligned} \tag{2.5}$$

### Load factor $k_L$

Tests revealed that the load factor has the following values [20]:

$$k_L = \begin{cases} 0.923, & \text{axial loading, } S_{ut} \leq 220 \text{ kpsi (1520 MPa),} \\ 1, & \text{axial loading, } S_{ut} > 220 \text{ kpsi (1520 MPa),} \\ 1, & \text{bending,} \\ 0.577 & \text{torsion and shear.} \end{cases} \tag{2.6}$$

Juvinall and Marshek present a summary of all modifying factors for bending, axial loading, and torsion used for fatigue of ductile materials and listed in Table 2.3 [7].

## 2.2 Fluctuating Stresses

In design problems the stress frequently fluctuates without passing through zero. The components of the stresses are depicted in Fig. 2.3(a), where  $\sigma_{min}$  is minimum stress,  $\sigma_{max}$  the maximum stress,  $\sigma_a$  the stress amplitude or the alternating stress,  $\sigma_m$  the midrange or the mean stress,  $\sigma_r$  the stress range. The steady stress or static stress,  $\sigma_s$ , can have any value between  $\sigma_{min}$  and  $\sigma_{max}$  and exists because of a fixed load. It is usually independent of the varying portion of the load. The following relations between the stress components exist as

$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}, \tag{2.7}$$

$$\sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2}. \tag{2.8}$$

The fluctuating stresses are described by the the stress ratios

$$R = \frac{\sigma_{min}}{\sigma_{max}}, \quad A = \frac{\sigma_a}{\sigma_m}. \tag{2.9}$$

A fluctuating stress is a combination of static plus completely reversed stress. Figure 2.3(a) shows a sinusoidal fluctuating stress, Fig. 2.3(b) represents a repeated stress, and Fig. 2.3(c) is a completely reversed sinusoidal stress.

### 2.3 Constant Life Fatigue Diagram

Figure 2.4 illustrates the graphical representation of various combinations of mean and alternating stress in relation to yielding and various fatigue life [7]. This diagram is called the *constant life fatigue diagram* because it has lines corresponding to a constant  $10^6$  cycle or “infinite” life, constant  $10^5$  cycle, constant  $10^4$  cycle, and so forth. The horizontal axis ( $\sigma_a = 0$ ) corresponds to static loading. The point  $A(\sigma_m = S_y, \sigma_a = 0)$  represents the yield strength. For ductile materials the point  $A'(-S_y, 0)$  represents the compressive yield strength. The point  $B(S_u, 0)$  represents the ultimate tensile strength. At the point  $A''(\sigma_m = 0, \sigma_a = S_y)$  the stress fluctuates between  $+S_y$  and  $-S_y$ . The points on line  $AA''$  correspond to fluctuations having a tensile peak of  $S_y$ . The points on line  $A'A''$  correspond to fluctuations having a compressive peak of  $S_y$ . Within the triangle  $AA'A''$  there are all the combinations with no yielding. The points  $C$ ,  $D$ ,  $E$ , and  $F$  correspond to  $\sigma_m = 0$  for various values of fatigue life and are obtained from the S-N diagram. The lines  $CB$ ,  $DB$ ,  $EB$  and  $FB$  are the estimated lines of constant life. These lines are called the Goodman lines.

The area  $A'HCGA$  corresponds to a life of at least  $10^6$  cycles and no yielding. For a life of at least  $10^6$  cycles and yielding, in addition to the area  $A'HCGA$ , the area  $AGB$  and the area to the left of the line  $A'H$  may be used. The area  $HCGA''H$  corresponds to less than  $10^6$  cycles of life and no yielding.

### 2.4 Fatigue Life for Randomly Varying Loads

For most mechanical parts acted upon by randomly varying stresses, the prediction of fatigue life is not an easy task. Instead of a single reversed stress  $\sigma$  for  $n$  cycles, a part is subjected to  $\sigma_1$  for  $n_1$  cycles,  $\sigma_2$  for  $n_2$  cycles, and so forth, and the problem is to estimate the fatigue life of the part. The procedure for dealing with this situation is the *linear cumulative damage rule*

(or Miner's rule) and can be expressed by the following equation [7, 20]:

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} + \dots + \frac{n_k}{N_k} = 1 \quad \text{or} \quad \sum_{j=1}^{j=k} \frac{n_j}{N_j} = 1 \quad (2.10)$$

where  $n_1, n_2, \dots, n_k$  represent the number of cycles at specific overstress levels  $\sigma_1, \sigma_2, \dots, \sigma_k$  and  $N_1, N_2, \dots, N_k$  represent the life (in cycles) at these overstress levels, as taken from the appropriate S-N curve. Fatigue failure is predicted when the previous equation holds.

## 2.5 Variable Loading Failure Theories

There are various techniques for plotting the results of the fatigue failure test of a part subjected to fluctuating stress. One of them is called the *modified Goodman diagram* and is shown in Fig. 2.5, [20]. For this diagram the mean stress  $\sigma_m$  is plotted on abscissa and the other stress components ( $S_e, S_y, S_u$ ) on the ordinate (tension is the positive direction). The mean stress line makes a  $45^\circ$  angle with the abscissa from the origin to the tensile strength. Lines are constructed to  $S_e$  (above the origin) and to  $-S_e$  (below the origin) as shown in Fig. 2.5. Yielding can be considered as a criterion of failure if  $\sigma_{max} > S_y$ .

Another way to display the results is shown in Fig. 2.6 using the strengths [20]. The fatigue limit  $S_e$  (or the finite life strength  $S_f$ ) is plotted on the ordinate. The tensile yield strength  $S_{yt}$  is plotted on both coordinate axes. The ultimate tensile strength  $S_{ut}$  is plotted on the abscissa. The alternating strength is  $S_a$  as a limiting value of  $\sigma_a$  and is plotted on the ordinate. The mean strength is  $S_m$  as a limiting value of  $\sigma_m$  and is plotted on the abscissa.

Four criteria of failure are shown in the diagram in Fig. 2.6, that is, Soderberg, the modified Goodman, Gerber, and yielding. Only the Soderberg criterion guards against yielding as shown in Fig. 2.6.

The Soderberg, Goodman, and yield criterion are described by the equation of a straight line in intercept form:

$$\frac{S_m}{a} + \frac{S_a}{b} = 1, \quad (2.11)$$

where  $a$  and  $b$  are the  $S_m$  and  $S_a$  intercepts, respectively. The equation for the Soderberg line is

$$\frac{S_m}{S_{yt}} + \frac{S_a}{S_e} = 1. \quad (2.12)$$

Similarly, the equation for the modified Goodman line is

$$\frac{S_m}{S_{ut}} + \frac{S_a}{S_e} = 1. \quad (2.13)$$

The yielding line is described by the equation

$$\frac{S_m}{S_{yt}} + \frac{S_a}{S_y} = 1. \quad (2.14)$$

The Gerber criterion is also called the *Gerber parabolic relation* because the curve can be modeled by a parabolic equation of the form

$$\frac{S_a}{S_e} + \left( \frac{S_m}{S_{yt}} \right)^2 = 1. \quad (2.15)$$

The curve representing the Gerber theory is a better predictor since it passes through the central region of the failure points.

If each strength in Eqs. (2.12) to (2.15) is divided by a safety factor  $SF$ , the stresses  $\sigma_a$  and  $\sigma_m$  can replace  $S_a$  and  $S_m$ . Therefore, the Soderberg equation becomes

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} = \frac{1}{SF}, \quad (2.16)$$

the modified Goodman equation becomes

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}} = \frac{1}{SF}, \quad (2.17)$$

and the Gerber equation becomes

$$\frac{SF \sigma_a}{S_e} + \left( \frac{SF \sigma_m}{S_{ut}} \right)^2 = 1. \quad (2.18)$$

Figure 2.7 shows the explanation of Eq. (2.17) [20]. A safe stress line through point  $A$  of coordinates  $\sigma_m$ ,  $\sigma_m$  is drawn parallel to the modified Goodman line. The safe stress line is the locus of all points of coordinates  $\sigma_m$ ,  $\sigma_a$  for which the same safety factor  $SF$  is considered, that is,  $S_m = SF \sigma_m$  and  $S_a = SF \sigma_a$ .

Table 2.4 lists the values of the tensile strength and Table 2.5 gives the yield strength for various materials.

## 2.6 Examples

**Example 2.1.** Estimate the S-N curve for a precision steel part for torsional loading. The part has the cross-section diameter under 2 in. and has a fine ground surface. The material has the ultimate tensile strength  $S_u = 110$  kpsi and the yield strength  $S_y = 77$  kpsi. Use the empirical relationships given in Table 2.3.

Solution.

According to Table 2.3, the  $10^3$ -cycle peak alternating strength for torsional loaded material is  $S = 0.9 S_{us}$ , and for steel is  $S_{us} = 0.8 S_u$ . It results:

$$S = 0.9 S_{us} = 0.9 (0.8) S_u = 0.9 (0.8) (110) = 79.2 \text{ kpsi for } N = 10^3 \text{ cycles.}$$

The  $10^6$ -cycle peak alternating strength (endurance limit) for torsional loaded ductile material is  $S_e = k_S k_G k_L S'_e$ . The endurance limit of the test specimen, for  $S_u=110$  kpsi < 200 kpsi, is given by Eq. (2.1):

$$S'_e = 0.5 S_u = 0.5 (110) = 55 \text{ kpsi.}$$

The surface factor is found from Fig. 2.2, for fine ground surface,  $k_S = 0.9$ . The other modifying factors for the endurance limit are given in Table 2.3. The gradient (size) factor is  $k_G = 0.9$  for  $d < 2$  in. The load factor for torsional load is  $k_L = 0.58$ . The endurance limit is

$$S_e = k_S k_G k_L S'_e = 0.9 (0.9) (0.58) (55) = 25.839 \text{ kpsi for } N = 10^6 \text{ cycles.}$$

The S-N diagram is plotted on log - log coordinates.

For  $\log 10^3=3$  it results  $\log S = \log 79.2 = 1.898$ .

For  $\log 10^6=6$  it results  $\log S_e = \log 25.839 = 1.412$ .

The estimated S-N curve is plotted in Fig. 2.8, and the *Mathematica*<sup>TM</sup> program is given in Program 2.1.

**Example 2.2.** A precision steel part is subjected to fluctuating axial loading. The part has the cross-section diameter under 8 mm and has a fine ground surface. The material has the ultimate tensile strength  $S_u = 1100$  MPa and the yield strength  $S_y = 715$  MPa. Find the  $10^6$  cycle strength (endurance limit).

Solution.

The endurance limit of the test specimen, for  $S_u = 1100 \text{ MPa} < 1400 \text{ MPa}$ , is given by Eq. (2.1):

$$S'_e = 0.5 S_u = 0.5 (1100) = 550 \text{ MPa.}$$

Equation (2.3) gives the surface factor

$$k_S = aS_{ut}^b = 1.58(1100)^{-0.085} = 0.871,$$

where  $a = 1.58$  and  $b = -0.085$  are obtained from Table 2.2. The gradient (size) factor is  $k_G = 1$  for axial loading from Eq. (2.5). Equation (2.6) gives the the load factor  $k_L = 0.923$  for axial loading and  $S_u=1100 \text{ MPa} < 1520 \text{ MPa}$ . The endurance limit is

$$S_e = k_S k_G k_L S'_e = 0.871 (1) (0.923) (550) = 442.286 \text{ MPa for } N = 10^6 \text{ cycles.}$$

The *Mathematica*<sup>TM</sup> program is given in Program 2.2.

**Example 2.3.** A 15 mm diameter steel bar has a fine ground surface with the ultimate strength  $S_u = 1100 \text{ MPa}$  and the yield strength  $S_y = 715 \text{ MPa}$ . a) Using Table 2.3, estimate the S-N curve and the family of constant life fatigue curves for bending load. Estimate the bending fatigue life for  $5 \times 10^4$  cycles; b) Determine the fatigue strength corresponding to  $10^6$  cycles and to  $5 \times 10^4$  cycles for the case of zero-to-maximum (rather than completely reversed) load fluctuations for bending and no yielding.

Solution.

a) According to Table 2.3, the  $10^3$ -cycle peak alternating strength for bending load is

$$S = 0.9 S_u = 0.9 (1100) = 990 \text{ MPa.}$$

The  $10^6$ -cycle peak alternating strength (endurance limit) is  $S_e = k_S k_G k_L S'_e$ . The endurance limit of the test specimen for  $S_u=1100 \text{ MPa} < 1400 \text{ MPa}$  is given by Eq. (2.1):

$$S'_e = 0.5 S_u = 0.5 (1100) = 550 \text{ MPa.}$$

The surface factor is found from Fig. 2.2, for fine ground surface and  $S_u = 1100 \text{ MPa}$ ,  $k_S = 0.89$ . The gradient (size) factor is  $k_G = 0.9$  for  $10 \text{ mm} <$

$d = 15 \text{ mm} < 50 \text{ mm}$  from Table 2.3. The load factor for bending load is  $k_L = 1$  from Table 2.3. The endurance limit is

$$S_e = k_S k_G k_L S'_e = 0.89 (0.9) (1) (550) = 440.55 \text{ MPa.}$$

The S-N diagram is plotted on log - log coordinates.

For  $\log 10^3=3$  it results  $\log S = \log 990 = 2.995$ .

For  $\log 10^6=6$  it results  $\log S_e = \log 440.55 = 2.644$ .

The estimated S-N line is plotted in Fig. 2.6. The equation of the S-N line on log-log coordinates is

$$y = m x + b = -0.117 x + 3.347,$$

where the slope  $m$  is

$$m = \frac{\log S_e - \log S}{6 - 3} = \frac{2.644 - 2.995}{6 - 3} = -0.117,$$

and the  $y$ -intercept  $b$  is

$$b = \log S - 3 m = 2.995 - 3(-0.117) = 3.347.$$

The log of  $10^4$ -cycle peak alternating strength is

$$\log S_4 = 4 m + b = 4(-0.117) + 3.347 = 2.878,$$

and the  $10^4$ -cycle peak alternating strength is

$$S_4 = 10^{\log S_4} = 10^{2.878} = 755.826 \text{ MPa.}$$

The log of  $10^5$ -cycle peak alternating strength is

$$\log S_5 = 5 m + b = 5(-0.117) + 3.347 = 2.761,$$

and the  $10^5$ -cycle peak alternating strength is

$$S_5 = 10^{\log S_5} = 10^{2.761} = 577.043 \text{ MPa.}$$

The log of fatigue life for  $N = 5 \times 10^4$  cycles is

$$\log S_N = m \log(5 \times 10^4) + b = (-0.117)(4.698) + 3.347 = 2.796,$$

and the fatigue life for  $N = 5 \times 10^4$  cycles is

$$S_N = 10^{\log S_N} = 10^{2.796} = 625.883 \text{ MPa.}$$

The estimated S-N curve and the  $\sigma_m - \sigma_a$  curves for  $10^3, 10^4, 5 \times 10^4, 10^5$ , and  $10^6$  cycles of life are plotted on Fig. 2.9.

b) The case of zero-to-maximum load fluctuations corresponds to  $\sigma_m = \sigma_a$ . This is represented by the line  $OA$  on Fig. 2.10. The  $\sigma_m - \sigma_a$  curves for  $10^6$  and  $5 \times 10^4$  cycles of life are plotted on Fig. 2.10. The line equation corresponding to  $10^6$  cycles of life is

$$\sigma_a = S_e \left( 1 - \frac{\sigma_m}{S_u} \right) = 440.55 \left( 1 - \frac{\sigma_m}{1100} \right).$$

The intersection of the line  $OA$  with the  $10^6$  cycles line is the point  $B$  of coordinates  $\sigma_m = \sigma_a = 314.566$  MPa. For infinite life ( $10^6$  cycles of life)  $\sigma_{max} = \sigma_a + \sigma_m = 629.132$  MPa.

The line equation corresponding to  $5 \times 10^4$  cycles of life is

$$\sigma_a = S_e \left( 1 - \frac{\sigma_m}{S_u} \right) = 625.883 \left( 1 - \frac{\sigma_m}{1100} \right).$$

The intersection of the line  $OA$  with the  $5 \times 10^4$  cycles line is the point  $C$  of coordinates:  $\sigma_m = \sigma_a = 398.91$  MPa. For this case  $\sigma_{max} = \sigma_a + \sigma_m = 797.819$  MPa greater than the yield strength  $S_y = 715$  MPa and this is not permitted. The line equation corresponding to the  $S_y - S_y$  line is

$$\sigma_a = S_y \left( 1 - \frac{\sigma_m}{S_y} \right) = 715 \left( 1 - \frac{\sigma_m}{715} \right).$$

The intersection of the line  $OA$  with the  $S_y - S_y$  line is the point  $D$  of coordinates  $\sigma_m = \sigma_a = 357.5$  MPa. If no yielding is permitted the point  $D$  is selected and  $\sigma_{max} = \sigma_a + \sigma_m = 715$  MPa  $\leq S_y = 715$  MPa. The *Mathematica*<sup>TM</sup> program is given in Program 2.3.

**Example 2.4.** A round steel part with the ultimate strength  $S_u = 110$  kpsi and the yield strength  $S_y = 77$  kpsi has average machined surfaces. The diameter of the part is less than 1 in. The part is subjected to an axial load fluctuating between 1000 and 6000 lb. A safety factor of 2 is applied to the loads. Determine the required diameter for infinite life ( $10^6$  cycles of life) and for  $10^3$  cycles of life. No yielding is permitted.

Solution.

According to Table 2.3, the  $10^3$ -cycle peak alternating strength is

$$S = 0.75 S_u = 0.75 (110) = 82.5 \text{ kpsi.}$$

The surface factor is  $k_S = 0.74$  (from Fig. 2.2 for machined surfaces and  $S_u = 110$  kpsi). The gradient (size) factor is  $k_G = 0.8$  (between 0.7 and 0.9 from Table 2.3). The load factor for axial load is  $k_L = 1$  from Table 2.3. The endurance limit of the test specimen, for  $S_u = 110$  kpsi < 200 kpsi is

$$S'_e = 0.5 S_u = 0.5 (110) = 55 \text{ kpsi.}$$

The endurance limit ( $10^6$ -cycle peak alternating strength) is

$$S_e = k_S k_G k_L S'_e = 0.74 (0.8) (1) (55) = 32.55 \text{ kpsi.}$$

The  $\sigma_m - \sigma_a$  curves for  $10^3$  and  $10^6$  cycles of life are plotted on Fig. 2.8.

The following relations between the stress components exist:

$$\sigma_m = SF \frac{F_m}{A}, \quad \sigma_a = SF \frac{F_a}{A},$$

where  $A$  is the unknown area,  $SF = 2$  is the safety factor, and

$$F_m = \frac{F_{max} + F_{min}}{2}, \quad F_a = \frac{F_{max} - F_{min}}{2},$$

with  $F_{max} = 6$  kip and  $F_{min} = 1$  kip.

The ratio between the alternating stress and the mean stress is  $\sigma_a/\sigma_m = 0.714$  and the equation of the line  $OA$  on Fig. 2.11 is

$$\sigma_a = 0.714 \sigma_m.$$

At the point  $O$ ,  $\sigma_a = \sigma_m = 0$ . Moving out along line  $OA$ , the area  $A$  of the part is decreasing.

The intersection of the line  $OA$  with the  $10^6$  cycles line (infinite life) is the point  $B$  of coordinates  $\sigma_a = 23.020$  kpsi,  $\sigma_m = 32.228$  kpsi. At this point  $\sigma_a = SF F_a/A$  and the area  $A$  is determined as

$$A = \frac{\pi d^2}{4} = SF \frac{F_a}{\sigma_a} = \frac{2(2.5)}{23.020}.$$

The required diameter for infinite life is  $d = 0.525$  in. This diameter is within the range for the gradient factor  $k_G = 0.8$ .

The intersection of the line  $OA$  with the  $10^3$  cycles line is the point  $C$  of coordinates  $\sigma_a = 40.243$  kpsi,  $\sigma_m = 56.341$  kpsi, and  $\sigma_{max} = 96.585$  kpsi  $>$   $S_y = 77$  kpsi. This is not permitted because yielding is unacceptable. The line equation corresponding to the  $S_y - S_y$  line is

$$\sigma_a = S_y \left( 1 - \frac{\sigma_m}{S_y} \right) = 77 \left( 1 - \frac{\sigma_m}{77} \right).$$

The intersection of the line  $OA$  with the  $S_y - S_y$  line is the point  $D$  of coordinates  $\sigma_a = 32.083$  kpsi,  $\sigma_m = 44.916$  kpsi. If no yielding is permitted the point  $D$  is selected and the diameter is selected based on this point. At point  $D$  the area  $A$  is

$$A = \frac{\pi d^2}{4} = SF \frac{F_a}{\sigma_a} = \frac{2(2.5)}{32.083},$$

and the required diameter is  $d = 0.445$  in. A diameter smaller than 0.445 in. would cause yielding. The *Mathematica*<sup>TM</sup> program is given in Program 2.4.

**Example 2.5.** Consider a 30 mm diameter steel bar having the ultimate tensile strength  $S_u = 950$  MPa and the yield strength  $S_y = 600$  MPa. The part has a hot rolled surface finish and is subjected to axial loading. The fluctuating stress of the bar for a typical  $t = 5$  seconds of operation includes, in order, two cycles with minimum stress  $\sigma_{minI} = -100$  MPa and maximum stress  $\sigma_{maxI} = 300$  MPa, and three cycles with minimum stress  $\sigma_{minII} = -100$  MPa and maximum stress  $\sigma_{maxII} = 400$  MPa (Fig. 2.12). For the axial loading of the part the  $10^3$  cycle peak strength is  $S = 712.5$  MPa and the  $10^6$  cycle peak strength (endurance limit) is  $S_e = 180.5$  MPa. Estimate the life of the part when operating continuously.

Solution.

For the first two cycles of fluctuation ( $n_I = 2$ ), with minimum stress  $\sigma_{minI} = -100$  MPa and maximum stress  $\sigma_{maxI} = 300$  MPa, the mean stress and the alternating stress are

$$\begin{aligned} \sigma_{mI} &= \frac{\sigma_{maxI} + \sigma_{minI}}{2} = \frac{300 + (-100)}{2} = 100 \text{ MPa,} \\ \sigma_{aI} &= \frac{\sigma_{maxI} - \sigma_{minI}}{2} = \frac{300 - (-100)}{2} = 200 \text{ MPa.} \end{aligned}$$

The point  $A$  of coordinates  $\sigma_{mI} = 100$  MPa,  $\sigma_{aI} = 200$  MPa on the  $\sigma_m - \sigma_a$  plot in Fig. 2.13(a) is connected by a straight line to the point  $\sigma_m = S_u = 950$  MPa on the horizontal axis. The slope of this line equation (line  $AS_u$ ) is

$$m_I = \frac{\sigma_{aI}}{\sigma_{mI} - S_u} = \frac{200}{100 - 950} = -0.235.$$

The intersection of the line  $AS_u$  with the vertical axis ( $\sigma_m = 0$ ) is the  $y$ -intercept

$$S_I = -m_I S_u = (-0.235)(950) = 223.529 \text{ MPa.}$$

For the second three cycles of fluctuation ( $n_{II} = 3$ ), with minimum stress  $\sigma_{minII} = -100$  MPa and maximum stress  $\sigma_{maxII} = 400$  MPa, the mean stress and the alternating stress are

$$\begin{aligned} \sigma_{mII} &= \frac{\sigma_{maxII} + \sigma_{minII}}{2} = \frac{400 + (-100)}{2} = 150 \text{ MPa,} \\ \sigma_{aII} &= \frac{\sigma_{maxII} - \sigma_{minII}}{2} = \frac{400 - (-100)}{2} = 250 \text{ MPa.} \end{aligned}$$

The point  $B$  of coordinates  $\sigma_{mII} = 150$  MPa,  $\sigma_{aII} = 250$  MPa on the  $\sigma_m - \sigma_a$  plot in Fig. 2.13(a) is connected by a straight line to the point  $\sigma_m = S_u = 950$  MPa on the horizontal axis. The slope of this line equation (line  $BS_u$ ) is

$$m_{II} = \frac{\sigma_{aII}}{\sigma_{mII} - S_u} = \frac{250}{150 - 950} = -0.312.$$

The intersection of the line  $BS_u$  with the vertical axis ( $\sigma_m = 0$ ) is the  $y$ -intercept

$$S_{II} = -m_{II} S_u = (-0.312)(950) = 296.875 \text{ MPa.}$$

The lines  $S_I AS_u$  and  $S_{II} BS_u$  are Goodman lines (constant life) and the points  $A$  and  $B$  correspond to the same fatigue lives as the points  $S_I$  and  $S_{II}$ . These fatigue lives are determined from the S-N diagram (log-log coordinates) in Fig. 2.13(b). For  $\log 10^3 = 3$  it results  $\log S = \log 712.5 = 2.852$ .

For  $\log 10^6 = 6$  it results  $\log S_e = \log 180.5 = 2.256$ .

The estimated S-N line is plotted in Fig. 2.13.b. The equation of the S-N line on log - log coordinates is

$$y = m x + b = -0.198 x + 3.449.$$

For  $\log S_I = \log 223.529 = 2.349$  it results:

$$\log N_I = \frac{\log S_I - b}{m} = \frac{2.349 - 3.449}{-0.198} = 5.532,$$

and the number of cycles is

$$N_I = 10^{\log N_I} = 10^{5.532} = 341065 \text{ cycles.}$$

For  $\log S_{II} = \log 296.875 = 2.472$  it results:

$$\log N_{II} = \frac{\log S_{II} - b}{m} = \frac{2.472 - 3.449}{-0.198} = 4.912,$$

and the number of cycles is

$$N_{II} = 10^{\log N_{II}} = 10^{4.912} = 81814 \text{ cycles.}$$

Adding the portions of life consumed by cycles  $I$  and  $II$  gives

$$C = \frac{n_I}{N_I} + \frac{n_{II}}{N_{II}} = \frac{2}{341065} + \frac{3}{81814} = 0.0000425325.$$

This means that the estimated life corresponds to  $1/C$  of  $t = 5$ -second duration. The life of the part is

$$l = t/C = 117557 \text{ s} = 1959.28 \text{ min} = 32.6547 \text{ h}$$

The *Mathematica*<sup>TM</sup> program for this example is given in Program 2.5.

**Example 2.6.** A 2 in. tension bar is machined from a material with the ultimate tensile strength  $S_u = 97$  kpsi and the yield strength  $S_y = 68$  kpsi. This part is to withstand a fluctuating tensile load varying from 3 to 60 kip. The endurance limit is  $S_e = 29.488$  kpsi. Using the modified Goodman theory, determine the safety factor under the assumption that a)  $\sigma_m$  remains fixed; b)  $\sigma_a$  remains fixed; and c) the ratio  $\sigma_a/\sigma_m$  is constant.

Solution.

The equation for the modified Goodman line is

$$\frac{S_m}{S_u} + \frac{S_a}{S_e} = 1, \quad \text{or} \quad \frac{S_m}{97} + \frac{S_a}{29.488} = 1.$$

This equation is plotted in Fig. 2.14. The alternating and mean loads are

$$F_a = \frac{F_{max} - F_{min}}{2} = \frac{60 - 3}{2} = 28.5 \text{ kip},$$

$$F_m = \frac{F_{max} + F_{min}}{2} = \frac{60 + 3}{2} = 31.5 \text{ kip}.$$

The alternating and mean stresses are found to be

$$\sigma_a = \frac{4 F_a}{\pi d^2} = \frac{4(28.5)}{3.141(2)^2} = 9.071 \text{ kpsi},$$

$$\sigma_m = \frac{4 F_m}{\pi d^2} = \frac{4(31.5)}{3.141(2)^2} = 10.026 \text{ kpsi}.$$

These data are shown in Fig. 2.14.

a) Using  $S_m = \sigma_m = 10.026$  kpsi, Eq. (2.13) yields

$$S_a = S_e \left(1 - \frac{S_m}{S_u}\right) = 29.488 \left(1 - \frac{10.026}{97}\right) = 26.439 \text{ kpsi}.$$

The safety factor is

$$SF = \frac{S_a}{\sigma_a} = \frac{26.439}{9.071} = 2.914.$$

b) Using  $S_a = \sigma_a = 9.071$  kpsi, Eq. (2.13) yields

$$S_m = S_u \left(1 - \frac{S_a}{S_e}\right) = 97 \left(1 - \frac{9.071}{29.488}\right) = 67.158 \text{ kpsi}.$$

The safety factor is

$$SF = \frac{S_m}{\sigma_m} = \frac{67.158}{10.026} = 6.697.$$

c) From Eq. (2.9):

$$A = \frac{S_a}{S_m} = \frac{\sigma_a}{\sigma_m} = \frac{9.071}{10.026} = 0.904.$$

Equation (2.13) yields

$$S_m = \frac{S_e S_u}{S_e + A S_u} = \frac{29.488 (97)}{29.488 + 0.904 (97)} = 24.395 \text{ kpsi},$$

and the safety factor is

$$SF = \frac{S_m}{\sigma_m} = \frac{24.395}{10.026} = 2.433.$$

These data are plotted on Fig. 2.14. The *Mathematica*<sup>TM</sup> program for this example is given in Program 2.6.

## 2.7 Problems

- 2.1 Using the empirical relationships given in Table 2.3, estimate the S-N curve for a precision steel part for axial and bending loading. The part has the cross-section diameter under 2 in. and has a machined surface. The material has the ultimate tensile strength  $S_u = 110$  kpsi and the yield strength  $S_y = 77$  kpsi.
- 2.2 A precision steel part with the diameter under 8 mm is subjected to fluctuating bending and torsional loading. The part has a hot rolled surface. The material has the ultimate tensile strength  $S_u = 1100$  MPa and the yield strength  $S_y = 715$  MPa. Find the  $10^6$  cycle strength (endurance limit).
- 2.3 Plot the S-N curves for bending, axial, and torsional loading of a steel shaft with the diameter  $d = 1.5$  in. The shaft was machined from steel having tensile properties  $S_u = 90$  kpsi and  $S_y = 75$  kpsi. Find the fatigue strength for  $6 \times 10^4$  cycles
- 2.4 A steel bar having the ultimate strength  $S_u = 950$  MPa and the yield strength  $S_y = 600$  MPa has a hot rolled surface finish. The surface factor is  $k_S = 0.475$ , the gradient factor is  $k_G = 0.8$ , and the load factor is  $k_L = 1$ . Determine the fatigue strength at  $5 \times 10^4$  cycles for reversed axial loading.
- 2.5 A steel round link has the diameter  $d = 25$  mm is subjected to an axial load fluctuating between 1000 and 6000 lb. The link has a hot rolled surface finish. The ultimate tensile strength of the material is  $S_u = 950$  MPa and the yield strength is  $S_y = 600$  MPa. Find the fatigue strength corresponding to  $10^6$  cycles.
- 2.6 A 2 in. diameter shaft is machined from AISI 4320 steel having  $S_u = 140$  kpsi and  $S_y = 90$  kpsi. Estimate the  $\sigma_m - \sigma_a$  curves for bending load.
- 2.7 A 15 mm diameter steel bar has a forged surface with the ultimate strength  $S_u = 1100$  MPa and the yield strength  $S_y = 715$  MPa. a) Using Table 2.3 estimate the S-N curve and the family of constant life fatigue curves for axial load. Estimate the fatigue life for  $4 \times 10^5$  cycles. b) Determine the fatigue strength corresponding to  $10^6$  cycles and to

$4 \times 10^4$  cycles for the case of zero-to-maximum (rather than completely reversed) load fluctuations for bending and no yielding.

- 2.8 A round steel link with the ultimate strength  $S_u = 110$  kpsi and the yield strength  $S_y = 77$  kpsi has ground surfaces. The diameter of the link is less than 2 in. The link is subjected to a bending load fluctuating between 1000 and 6000 lb. A safety factor of 1.5 is applied to the loads. Determine the required diameter for infinite life ( $10^6$  cycles of life) and for  $10^3$  cycles of life.
- 2.9 Consider a 30 mm diameter steel bar having the ultimate tensile strength  $S_u = 950$  MPa and the yield strength  $S_y = 600$  MPa. The part has a hot rolled surface finish and is subjected to axial loading. The fluctuating stress of the bar for a typical  $t = 6$  seconds of operation includes, in order, two cycles with minimum stress of -100 MPa and maximum stress of 300 MPa, three cycles with minimum stress of -150 MPa and maximum stress of 400 MPa, and four cycles with minimum stress of -200 MPa and maximum stress of 600 MPa. Estimate the life of the part when operating continuously.
- 2.10 A steel part with the diameter 20 mm has the ultimate tensile strength  $S_u = 1100$  MPa and the yield strength  $S_y = 715$  MPa. The part has a hot rolled surface finish and is subjected to bending loading. The fluctuating stress of the bar for 10 seconds of operation includes, in order, two cycles with zero minimum stress and maximum stress equal to 100 MPa, two cycles with minimum stress equal to -50 MPa and maximum stress equal to 300 MPa, and two cycles with minimum stress equal to 100 MPa and maximum stress equal to 500 MPa. Determine the fatigue life of the part.
- 2.11 A tension bar is machined from AISI 1050 ( $S_u = 100$  kpsi and  $S_y = 84$  kpsi). This bar has the diameter  $d = 1.5$  in. and is subjected to a fluctuating tensile load varying from 0 to 50 kip. The endurance limit is  $S_e = 25$  kpsi. Using the modified Goodman theory, determine the safety factor under the assumption that  $\sigma_m$  and  $\sigma_a$  remain constant.
- 2.12 Repeat the previous problem considering that the ratio  $\sigma_a/\sigma_m$  is constant.

## **2.8 Programs**

**Program 2.1**

**Program 2.2**

**Program 2.3**

**Program 2.4**

**Program 2.5**

**Program 2.6**

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## Figure captions

Figure 2.1. Rotating beam fatigue testing machine.

Figure 2.2. Surface factor  $k_S$ .

Figure 2.3. Time varying stresses: a) sinusoidal fluctuating stress; (b) repeated stress; and (c) reversed sinusoidal stress.

Figure 2.4. Constant life fatigue diagram.

Figure 2.5. Modified Goodman diagram.

Figure 2.6. Various failure theories.

Figure 2.7. Safe stress line.

Figure 2.8. S-N diagram for Example 2.1.

Figure 2.9. S-N diagram and constant life fatigue diagram for Example 2.3(a).

Figure 2.10.  $\sigma_m - \sigma_a$  curves for Example 2.3(b).

Figure 2.11.  $\sigma_m - \sigma_a$  curves for Example 2.4.

Figure 2.12. Stress time plot for Example 2.5.

Figure 2.13. (a)  $\sigma_m - \sigma_a$  curves and (b) S-N diagram for Example 2.5.

Figure 2.14.  $\sigma_m - \sigma_a$  curves for Example 2.6.