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1 Stress and Deflection

1.1 Stress

In the design process, an important problem is to ensure that the strength of the mechanical element to be designed always exceeds the stress due to any load exerted on it.

Uniform distribution of stresses is an assumption that is frequently considered in the design process. Depending upon the way the force is applied to an element, the result is called *pure tension* (*compression*) or *pure shear*, respectively.

A tension load F is applied to the ends of a bar. If the bar is cut at a section remote from the ends and one piece is removed, the effect of the removed part can be replaced by applying a uniformly distributed force of magnitude σA to the cut end, where σ is the *normal stress* and A the cross section area of the bar. The stress σ is given by the following expression:

$$\sigma = \frac{F}{A}. \tag{1.1}$$

This uniform stress distribution requires that

- the bar be straight and made of a homogeneous material, and
- the line of action of the force contains the centroid of the section.

Equation (1.1) and the previous assumptions also hold for pure compression.

If a body is in shear (uniform stress distribution), the following equation can be used:

$$\tau = \frac{F}{A}, \tag{1.2}$$

where τ is the *shear stress*.

Stress Components

A general three-dimensional *stress element* is illustrated in Fig. 1.1(a). Three normal positive stresses, σ_x, σ_y, and σ_z, and six positive shear stresses, τ_{xy}, τ_{yx}, τ_{yz}, τ_{zy}, τ_{zx}, and τ_{xz} are shown. Static equilibrium requires

$$\tau_{xy} = \tau_{yx}, \quad \tau_{yz} = \tau_{zy}, \quad \tau_{xz} = \tau_{zx}. \tag{1.3}$$

The normal stresses σ_x, σ_y, and σ_z are called *tension* or *tensile stresses* and considered positive if they are oriented in the direction shown in the figure.

Shear stresses on a positive face of an element are positive if they act in the positive direction of the reference axis. The first subscript of any shear stress component denotes the axis to which it is perpendicular. The second subscript denotes the axis to which the shear stress component is parallel.

A general two-dimensional stress element is shown in Fig. 1.1(b). The two normal stresses, σ_x and σ_y, respectively, are in the positive direction. Shear stresses are positive when they are in the clockwise (cw) direction, and negative when they are in the counterclockwise (ccw) direction. Thus, τ_{yx} is positive (cw) and τ_{xy} is negative (ccw).

Mohr's Circle

The element in Fig. 1.1(b) is considered cut by an oblique plane at angle ϕ as shown in Fig. 1.2. The stresses σ and τ act on this oblique plane. The stresses σ and τ can be calculated with the formulas

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\phi + \tau_{xy} \sin 2\phi, \tag{1.4}$$

$$\tau = -\frac{\sigma_x - \sigma_y}{2} \sin 2\phi + \tau_{xy} \cos 2\phi. \tag{1.5}$$

Differentiating Eq. (1.4) with respect to the angle ϕ and setting the result equal to zero yields

$$\tan 2\phi = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}. \tag{1.6}$$

The solution of Eq. (1.6) gives two values for the angle 2ϕ defining the maximum normal stress σ_1 and the minimum normal stress σ_2. These minimum and maximum normal stresses are called the *principal stresses*. The corresponding directions are called the *principal directions*. The angle between the principal directions is 90°.

Similarly, differentiating Eq. (1.5) and setting the result to zero will result in the following relation:

$$\tan 2\phi = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}. \tag{1.7}$$

The solutions of Eq. (1.7) define the angles 2ϕ at which the shear stress τ reaches extreme values.

Equation (1.6) can be rewritten as

$$2\tau_{xy} \cos 2\phi = (\sigma_x - \sigma_y) \sin 2\phi,$$

or

$$\sin 2\phi = \frac{2\tau_{xy} \cos 2\phi}{\sigma_x - \sigma_y}. \tag{1.8}$$

Substituting Eq. (1.8) into Eq. (1.5) gives

$$\tau = -\frac{\sigma_x - \sigma_y}{2} \frac{2\tau_{xy} \cos 2\phi}{\sigma_x - \sigma_y} + \tau_{xy} \cos 2\phi = 0. \tag{1.9}$$

From Eq. (1.9) it results that the shear stress associated with both principal directions is zero.

Substituting Eq. (1.7) into Eq. (1.4) yields

$$\sigma = \frac{\sigma_x + \sigma_y}{2}. \tag{1.10}$$

The Eq. (1.10) states that the two normal stresses associated with the directions of the two maximum shear stresses are equal.

The analytical expressions for the two principal stresses can be obtained by manipulating Eqs. (1.6) and (1.4):

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}. \tag{1.11}$$

Similarly, the maximum and minimum values of the shear stresses are obtained:

$$\tau_1, \tau_2 = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}. \tag{1.12}$$

The *Mohr's circle diagram* (Fig. 1.3) is a graphical method to visualize the stress state. The normal stresses are plotted along the abscissa axis of the coordinate system and the shear stresses along the ordinate axis. Tensile normal stresses are considered positive (σ_x and σ_y are positive in Fig. 1.3) and compressive normal stresses are negative. Clockwise (cw) shear stresses are considered positive, while counterclockwise (ccw) shear stresses are negative.

The following notation is used: OA as σ_x, AB as τ_{xy}, OC as σ_y, and CD as τ_{yx}. The center of the Mohr's circle is at point E on the σ-axis. Point B has the stress coordinates σ_x, τ_{xy} on the x-faces and point D the stress coordinates σ_y, τ_{yx} on the y-faces. The angle 2ϕ between EB and ED is

180°, hence the angle between x and y on the stress element is $\phi = 90°$. The maximum principal normal stress is σ_1 at point F, and the minimum principal normal stress is σ_2 at point G. The two extreme values of the shear stresses are plotted at points I and H, respectively. Thus, the Mohr's diagram is a circle of center E and diameter BD.

For three-dimensional stress elements it is considered a particular orientation when all shear stress components are zero. The principal directions are the normals to the faces for this particular orientation. Since the stress element is three-dimensional, there are three principal directions and three principal stresses σ_1, σ_2, and σ_3 associated with the principal directions. In three dimensions, only six components of stress are required to specify the stress state, namely, σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{zx}.

To plot Mohr's circles for triaxial stress, the principal normal stresses are ordered so that $\sigma_1 > \sigma_2 > \sigma_3$. The result is shown in Fig. 1.4(a). The three *principal shear stresses* $\tau_{1/2}, \tau_{2/3}$, and $\tau_{1/3}$ are also shown in Fig. 1.4(a). Each of these shear stresses occurs on two planes, one of the planes being shown in Fig. 1.4(b). The principal shear stresses are

$$\tau_{1/2} = \frac{\sigma_1 - \sigma_2}{2}; \quad \tau_{2/3} = \frac{\sigma_2 - \sigma_3}{2}; \quad \tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2}. \tag{1.13}$$

If $\tau_{max} = \tau_{1/3}$, then $\sigma_1 > \sigma_2 > \sigma_3$.

Elastic Strain

If a tensile load is applied to a straight bar, it becomes longer. The amount of elongation is called the *total strain.* The elongation per unit length of the bar, ϵ, is called *strain.* The strain is

$$\epsilon = \frac{\delta}{l} \tag{1.14}$$

where δ is the total elongation (total strain) of the bar of length l.

Shear strain γ is the change in a right angle of an element subjected to pure shear stresses.

Elasticity is a property of materials that allows them to regain the original geometry when the load is removed. The elasticity of a material can be expressed in terms of Hooke's law: the stress in a material is proportional to the strain that produced it, within certain limits

$$\sigma = E\,\epsilon, \qquad \tau = G\,\gamma, \tag{1.15}$$

where E and G are constants of proportionality. The constant E is called the *modulus of elasticity* and the constant G is called the *shear modulus of elasticity* or the *modulus of rigidity*. A material that obeys Hooke's law is called elastic.

Substituting $\sigma = F/A$ and $\epsilon = \delta/l$ into Eq. (1.15) the expression for the total deformation δ of a bar loaded in axial tension or compression is

$$\delta = \frac{F\,l}{A\,E}. \tag{1.16}$$

When a tension load is applied to an elastic body, not only does an axial strain occur, but also a lateral strain occurs and the two strains are proportional to each other. This proportionality constant is called *Poisson's ratio* and is given by

$$\nu = \frac{\text{lateral strain}}{\text{axial strain}}. \tag{1.17}$$

The elastic constants are related by the following expression:

$$E = 2\,G\,(1 + \nu). \tag{1.18}$$

The *principal strains* are defined as the strains in the direction of the principal stresses. The shear strains are zero on the faces of an element aligned along the principal directions. Table 1.1 lists the relationships for all types of stress. The values of Poisson's ratio ν for various materials are listed in Table 1.2.

Shear and Moment

A beam supported by the reactions R_1 and R_2 and loaded by the transversal forces F_1, F_2 is shown in Fig. 1.5(a). The reactions R_1 and R_2 are considered positive since they act in the positive direction of the y-axis. The concentrated forces F_1 and F_2 are considered negative since they act in the negative y-direction. A cut is considered at a section located at x. Only the left-hand part of the beam with respect to the cut is taken as a free-body. To ensure equilibrium, an internal shear force V and an internal bending moment M must act on the cut surface [Fig. 1.5(b)]. The shear force is obtained by summing the forces to the left of the cut section. The bending moment is obtained by summing the moments of the forces to the left of the section with respect to an axis through the section. The shear force and the bending moment are related by the following expression:

$$V = \frac{dM}{dx}. \tag{1.19}$$

If bending is caused by a uniformly distributed load w (acting downward), then the relation between shear force and bending moment is

$$\frac{dV}{dx} = \frac{d^2M}{dx^2} = -w. \tag{1.20}$$

The units for w are units of force per unit of length. A general force distribution called *load intensity*, q, can be expressed as

$$q = \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x}.$$

Integrating Eqs. (1.19) and (1.20) between two points on the beam of coordinates x_A and x_B yields

$$\int_{V_A}^{V_B} dV = \int_{x_A}^{x_B} q\, dx = V_B - V_A. \tag{1.21}$$

The above equation states that the changes in shear force from A to B are equal to the area of the loading diagram between x_A and x_B. Similarly,

$$\int_{M_A}^{M_B} dM = \int_{x_A}^{x_B} V\, dx = M_B - M_A, \tag{1.22}$$

which states that the changes in moment from A to B is equal to the area of the shear force diagram between x_A and x_B.

Table 1.3 lists a set of five *singularity functions* that are useful in developing the general expressions for the shear force and the bending moment in a beam when it is loaded by concentrated forces or moments.

Normal Stresses in Pure Bending

The relationships for the normal stresses in beams are derived by considering the following: the beam is subjected to pure bending, the material is isotropic and homogeneous and obeys Hooke's law, the beam is initially straight with a constant cross section throughout the all length, the beam axis of symmetry is in the plane of bending, and the beam cross sections remain plane during bending.

Figure 1.6 shows a part of a beam on which a positive bending moment $\mathbf{M}_z = M\mathbf{k}$ ($\mathbf{k}$ being the unit vector associated with z-axis) is applied. A *neutral plane* is a plane that is coincident with the elements of the beam of zero strain. The xz plane is considered as the neutral plane. The x-axis is coincident with the *neutral axis* of the section and the y-axis is coincident with the axis of symmetry of the section.

Applying a positive moment on the beam, the upper surface will bend downward and, therefore, the neutral axis will also bend downward (Fig. 1.6). Because of this fact, the section PQ initially parallel to RS will rotate through the angle $d\,\phi$ to $P'Q'$. In Fig. 1.6, ρ is the radius of curvature of the neutral axis, $d\,s$ is the length of a differential element of the neutral axis, and $d\,\phi$ is the angle between the two adjacent sides, RS and $P'Q'$. The definition of the curvature is

$$\frac{1}{\rho} = \frac{d\phi}{ds}. \tag{1.23}$$

The deformation of the beam at distance y from the neutral axis is

$$dx = y\,d\phi, \tag{1.24}$$

and the strain is

$$\epsilon = -\frac{dx}{ds}, \tag{1.25}$$

where the negative sign suggests that the beam is in compression. Equations (1.23), (1.24), and (1.25) give

$$\epsilon = -\frac{y}{\rho}. \tag{1.26}$$

Since $\sigma = E\epsilon$, the expression for stress is

$$\sigma = -\frac{Ey}{\rho}. \tag{1.27}$$

The force acting on an element of area dA is σdA and integrating this force

$$\int \sigma\,dA = -\frac{E}{\rho}\int y\,dA = 0. \tag{1.28}$$

Since the x-axis is the neutral axis, the previous equation states that the moment of the area about the neutral axis is zero. Thus, Eq. (1.28) defines the location of the neutral axis, that is, the neutral axis passes through the centroid of the cross-sectional area.

For equilibrium, the internal bending moment created by the stress σ must be the same as the external moment $\mathbf{M}_z = M\mathbf{k}$, namely

$$M = \int y\sigma \, dA = \frac{E}{\rho} \int y^2 \, dA, \tag{1.29}$$

where the second integral is the second moment of area I about the z-axis

$$I = \int y^2 \, dA. \tag{1.30}$$

From Eqs. (1.29) and (1.30)

$$\frac{1}{\rho} = \frac{M}{EI} \tag{1.31}$$

is obtained. Eliminating ρ from Eqs. (1.27) and (1.31) yields

$$\sigma = -\frac{My}{I}. \tag{1.32}$$

Equation (1.32) states that the stress σ is directly proportional to the bending moment M and the distance y from the neutral axis (Fig. 1.7). The maximum stress is

$$\sigma_{max} = \frac{Mc}{I}, \tag{1.33}$$

where $c = y_{max}$. Equation (1.33) can also be written in the following two forms

$$\sigma_{max} = \frac{M}{I/c} \quad \text{and} \quad \sigma_{max} = \frac{M}{Z}, \tag{1.34}$$

where $Z = I/c$ is called the *section modulus*.

Normal Stresses in Beams with Asymmetrical Sections

If the plane of bending coincides with one of the two principal axes of the section, the results of the previous section can be applied to beams with asymmetrical sections.

From Eq. (1.28), the stress at a distance y from the neutral axis is

$$\sigma = -\frac{Ey}{\rho}. \tag{1.35}$$

The force on the element of area dA shown in Fig. 1.8 is

$$dF = \sigma\, dA = -\frac{Ey}{\rho} dA.$$

The moment of this force about the y-axis gives

$$M_y = \int z\, dF = \int \sigma z\, dA = -\frac{E}{\rho} \int yz\, dA, \tag{1.36}$$

where the integral is across the section. The last integral in Eq. (1.36) is the product of inertia I_{yz}. If the bending moment on the beam is in the plane of one of the principal axes then

$$I_{yz} = \int yz\, dA = 0. \tag{1.37}$$

Hence, the relations developed in the preceding section can be applied to beams having asymmetrical sections only if $I_{yz} = 0$.

Shear Stresses in Beams

In the general case, beams have both shear forces and bending moments acting upon them. A beam of constant cross section subjected to a shear force $\mathbf{V} = V\mathbf{j}$ and a bending moment $\mathbf{M}_z = M\mathbf{k}$ ($\mathbf{j}$ and $\mathbf{k}$ are the unit vectors corresponding to the y- and z-axes) is considered in Fig. 1.9. The relationship between V and M is

$$V = \frac{dM}{dx}. \tag{1.38}$$

An element of length dx located at a distance y_1 above the neutral axis is considered. Because of the shear force, the bending moment is not constant along the x-axis. The bending moment M on the beginning side of the section produces the normal stress σ and the bending moment $M + dM$ on the end side of the section produces the normal stress $\sigma + d\sigma$. The normal stress σ generates the force $\mathbf{F}_b = F_b\, \mathbf{i}$ and the the normal stress $\sigma + d\sigma$ generates the force $\mathbf{F}_e = -F_e\, \mathbf{i}$. Since $F_e > F_b$, the resultant of these forces would cause the element to slide in the $-x$ direction. To ensure equilibrium, the resultant must be balanced by a shear force acting in the $+x$ direction on the bottom of the section. A shear stress τ generates the shear force $\mathbf{F}_s = F_s\,\mathbf{\imath}$.

The force on the beginning side is

$$F_b = \int_{y_1}^{c} \sigma \, dA, \tag{1.39}$$

where the integration is from the bottom of the element $y = y_1$ to the top $y = c$ and dA is a small element of area on the face. Using the expression $\sigma = \dfrac{My}{I}$, the previous equation yields

$$F_b = \frac{M}{I} \int_{y_1}^{c} y \, dA. \tag{1.40}$$

The force acting on the end face is calculated in a similar way:

$$F_e = \int_{y_1}^{c} (\sigma + d\sigma) \, dA = \frac{M + dM}{I} \int_{y_1}^{c} y \, dA. \tag{1.41}$$

The force on the bottom face is

$$F_s = \tau b \, dx, \tag{1.42}$$

where b is the width of the element and $b\,dx$ is the area of the bottom face.

The sum of all the forces on x direction gives

$$\sum F_x = F_b - F_e + F_s = 0, \tag{1.43}$$

or

$$F_s = F_e - F_b = \frac{M + dM}{I} \int_{y_1}^{c} y \, dA - \frac{M}{I} \int_{y_1}^{c} y \, dA = \frac{dM}{I} \int_{y_1}^{c} y \, dA. \tag{1.44}$$

Substituting Eq. (1.42) for F_e and solving for shear stress gives

$$\tau = \frac{dM}{dx} \frac{1}{Ib} \int_{y_1}^{c} y \, dA. \tag{1.45}$$

Using Eq. (1.38), the shear stress formula becomes

$$\tau = \frac{V}{Ib} \int_{y_1}^{c} y \, dA. \tag{1.46}$$

The integral

$$Q = \int_{y_1}^{c} y \, dA \tag{1.47}$$

is the first moment of area of the vertical face about the neutral axis. Therefore, Eq. (1.46) can be rewritten as

$$\tau = \frac{VQ}{Ib}, \tag{1.48}$$

where I is the second moment of area of the section about the neutral axis.

Figure 1.10 shows a part of a beam with a rectangular cross section. A shear force $\mathbf{V} = V\,\mathbf{J}$ and a bending moment $\mathbf{M}_z = M\mathbf{k}$ act on the beam. Due to the bending moment, a normal stress σ is produced on a cross section of the beam, such as $A-A$. The beam is in compression above the neutral axis and in tension below. An element of area dA located a distance y above the neutral axis is considered. With $dA = b\,dy$, Eq. (1.47) becomes

$$Q = \int_{y_1}^{c} y \, dA = b\int_{y_1}^{c} y \, dy = \left.\frac{by^2}{2}\right|_{y_1}^{c} = \frac{b}{2}(c^2 - y_1^2). \tag{1.49}$$

Substituting Eq. (1.49) into Eq. (1.48) gives

$$\tau = \frac{V}{2I}(c^2 - y_1^2). \tag{1.50}$$

The previous equation represents the general equation for shear stress in a beam of rectangular cross section. The expression for the second moment of area I for a rectangular section is

$$I = \frac{bh^2}{12}.$$

and, substituting $h = 2c$ and $A = bh = 2bc$, the expression for I becomes

$$I = \frac{Ac^2}{3}. \tag{1.51}$$

Substituting Eq. (1.51) into Eq. (1.50) yields

$$\tau = \frac{3V}{2A}\left(1 - \frac{y_1^2}{c^2}\right) = C\frac{V}{A}. \tag{1.52}$$

The values C versus y_1 are listed in Table 1.4[20]. The maximum shear stress is obtained for $y_1 = 0$, that is

$$\tau_{max} = \frac{3V}{2A}, \tag{1.53}$$

and the zero shear stress is obtained at the outer surface where $y_1 = c$. Formulas for the maximum flexural shear stress for the most commonly used shapes are listed in Table 1.5.

Torsion

A *torque vector* is a moment vector collinear with an axis of a mechanical element, causing the element to twist about that axis. A torque $\mathbf{T}_x = T\,\mathbf{\imath}$ applied to a solid round bar is shown in Fig. 1.11. The angle of twist is given by the following relation:

$$\theta = \frac{T\,l}{G\,J}, \tag{1.54}$$

where T is the torque, l the length, G the modulus of rigidity, and J the polar second moment of area. Since the shear stress is zero at the center and maximum at the surface for a solid round bar, the shear stress is proportional to the radius ρ, namely

$$\tau = \frac{T\,\rho}{J}. \tag{1.55}$$

If r is the radius to the outer surface, then

$$\tau_{max} = \frac{T\,r}{J}. \tag{1.56}$$

For a solid round section with the diameter d, the polar second moment of area is

$$J = \frac{\pi\,d^4}{32},$$

and for a hollow round section with the outside diameter d_o and inside diameter d_i, it is

$$J = \frac{\pi\,(d_o^4 - d_i^4)}{32}.$$

For a rotating shaft, the torque T can be expressed in terms of power and speed:

$$H = \frac{2\pi T n}{33000(12)} = \frac{FV}{33000} = \frac{Tn}{63000}, \tag{1.57}$$

where H is the power in hp, T is the torque in lb·in, n is the shaft speed in rpm, F is the force in lb, and V is the velocity in ft/min. For SI units the power is

$$H = T\,\omega, \tag{1.58}$$

where H is the power in W, T is the torque in N·m, and ω is the angular velocity in rad/s. The torque T can be approximated by

$$T = 9.55\frac{H}{n}, \tag{1.59}$$

where H is in W and n is in rpm.

For rectangular sections, the following approximate formula applies [20]:

$$\tau_{max} = \frac{T}{w\,t^2}\left(3 + 1.8\frac{t}{w}\right), \tag{1.60}$$

where w and t are the width and the thickness of the bar, respectively ($t < w$).

1.2 Deflection

A *rigid* element does not bend, deflect, or twist when an external force or moment is exerted on it. Conversely, a *flexible* element changes its geometry when an external force, moment, or torque is applied. Therefore, *rigidity* and *flexibility* are terms that apply to particular situations.

The property of a material that enables it to regain its original geometry after having been deformed is called *elasticity*. A straight beam of length l, which is simply supported at the ends and loaded by the transversal force F, is considered in Fig. 1.12(a). If the elastic limit of the material is not exceeded (as indicated by the graph), the deflection y of the beam is linearly related to the force and, therefore, the beam can be described as a *linear spring*.

The case of a straight beam supported by two cylinders is illustrated in Fig. 1.12(b). As the force F is applied to the beam, the length between the supports decreases and, therefore, a larger force is needed to deflect a short beam than that required for a long one. Hence, the more this beam is deflected, the stiffer it becomes. The force is not linearly related to the deflection, and, therefore, the beam can be described as a *nonlinear stiffening spring*.

The *spring rate* is defined as

$$k(y) = \lim_{\Delta y \to 0} \frac{\Delta F}{\Delta y} = \frac{dF}{dy}, \tag{1.61}$$

where y is measured at the point of application of F in the direction of F ($F = F(y)$). For a linear spring, k is a constant called the *spring constant*, and Eq. (1.61) becomes

$$k = \frac{F}{y}. \tag{1.62}$$

The total extension or deformation of a uniform bar is

$$\delta = \frac{Fl}{AE}, \tag{1.63}$$

where F is the force applied on the bar, l the length of the bar, A the cross-section area, and E the modulus of elasticity. From Eqs. (1.62) and (1.63), the spring constant of an axially loaded bar is obtained:

$$k = \frac{AE}{l}. \tag{1.64}$$

If a uniform round bar is subjected to a torque T, the angular deflection is

$$\theta = \frac{Tl}{GJ}, \tag{1.65}$$

where T is the torque, l the length of the bar, G the modulus of rigidity, and J the polar moment of inertia. Multiplying Eq. (1.65) by $180/\pi$ and substituting $J = \pi d^4/32$ (for a solid round bar), the expression for θ becomes

$$\theta = \frac{583.6Tl}{Gd^4}, \tag{1.66}$$

where θ is in degrees and d is the diameter of the round cross section. The torsional spring rate is defined as

$$k = \frac{T}{\theta} = \frac{GJ}{l}. \tag{1.67}$$

If a beam is subjected to a positive bending moment M, the beam will deflect downward. The relationship between the curvature of the beam and the external moment M is

$$\frac{1}{\rho} = \frac{M}{EI}, \tag{1.68}$$

where ρ is the radius of curvature, E the modulus of elasticity, and I the second moment of area. The curvature of a plane curve is

$$\frac{1}{\rho} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}, \tag{1.69}$$

where y is the deflection of the beam at any point of coordinate x along its length. The slope of the beam at point x is

$$\theta = \frac{dy}{dx}. \tag{1.70}$$

If the slope is very small, that is, $\theta \approx 0$, then the denominator of Eq. (1.69) is expressed as

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = [1 + \theta^2]^{3/2} \approx 1.$$

Hence, Eq. (1.68) yields

$$\frac{M}{EI} = \frac{d^2y}{dx^2}. \tag{1.71}$$

Differentiating successively Eq. (1.71) two times gives

$$\frac{V}{EI} = \frac{d^3y}{dx^3}, \tag{1.72}$$

$$\frac{q}{EI} = \frac{d^4y}{dx^4}, \tag{1.73}$$

where q is the load intensity and V the shear force:

$$V = \frac{dM}{dx} \quad \text{and} \quad \frac{dV}{dx} = \frac{d^2M}{dx^2} = q.$$

The above relations can be arranged as follows:

$$\frac{q}{EI} = \frac{d^4y}{dx^4}, \tag{1.74}$$

$$\frac{V}{EI} = \frac{d^3y}{dx^3}, \tag{1.75}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}, \tag{1.76}$$

$$\theta = \frac{dy}{dx}, \tag{1.77}$$

$$y = f(x). \tag{1.78}$$

Figure 1.13 shows a beam of length $l = 10$ in. loaded by the uniform load $w = 10$ lb/in. All quantities are positive if upward and negative if downward. Figure 1.13 also shows the shear force, bending moment, slope and deflection diagrams. The values of these quantities at the ends of the beam, that is at $x = 0$ and $x = l$, are called *boundary values*. For example, the bending moment and the deflection are zero at each end because the beam is simply supported.

Deflections Analysis using Singularity Functions

A simply supported beam acted upon by a concentrated load at a distance a from the origin of the xy coordinate system is shown in Fig. 1.14. The analytical expression for the deflection of the beam will be calculated using the singularity functions. The deflection of the beam in between the supports $(0 < x < l)$ will be determined. Thus, Eq. (1.74) yields

$$EI\frac{d^4y}{dx^4} = q = -F\langle x - a\rangle^{-1}. \tag{1.79}$$

Due to the range chosen for x, the reactions R_1 and R_2 do not appear in the above equation. Integrating from 0 to x Eq. (1.79) and using Eq. (1.75) gives

$$EI\frac{d^3y}{dx^3} = V = -F\langle x - a\rangle^0 + C_1, \tag{1.80}$$

where C_1 is an integration constant. Using Eq. (1.76) and integrating again we obtain

$$EI\frac{d^2y}{dx^2} = M = -F\langle x-a\rangle^1 + C_1x + C_2, \tag{1.81}$$

where C_2 is also an integration constant. We can determine the constants C_1 and C_2 by considering two boundary conditions. The boundary condition can be $M = 0$ at $x = 0$ applied to Eq. (1.81) which gives $C_2 = 0$ and $M = 0$ at $x = l$ also applied to Eq. (1.81) which gives

$$C_1 = \frac{F(l-a)}{l} = \frac{Fb}{l}.$$

Substituting C_1 and C_2 in Eq. (1.81) gives

$$EI\frac{d^2y}{dx^2} = M = \frac{Fbx}{l} - F\langle x-a\rangle^1. \tag{1.82}$$

Integrating Eq. (1.82) twice accordingly to Eqs. (1.77) and (1.78) yields

$$EI\frac{dy}{dx} = EI\theta = \frac{Fbx^2}{2l} - \frac{F\langle x-a\rangle^2}{2} + C_3, \tag{1.83}$$

$$EIy = \frac{Fbx^3}{6l} - \frac{F\langle x-a\rangle^3}{6} + C_3x + C_4. \tag{1.84}$$

The integration constants C_3 and C_4 in the above equations can be evaluated by considering the boundary conditions $y = 0$ at $x = 0$ and $y = 0$ at $x = l$. Substituting the first boundary condition in Eq. (1.84) yields $C_4 = 0$. The second condition substituted in Eq. (1.84) yields

$$0 = \frac{Fbl^2}{6} - \frac{Fb^3}{6} + C_3l,$$

or

$$C_3 = -\frac{Fb}{6l}(l^2 - b^2).$$

Substituting C_3 and C_4 in Eq. (1.84), the analytical expression for the deflection y is obtained:

$$y = \frac{F}{6EIl}[bx(x^2 + b^2 - l^2) - l\langle x-a\rangle^3]. \tag{1.85}$$

The shear force and bending moment diagrams are shown in Fig. 1.14.

Strain energy

The work done by the external forces on a deforming elastic member is transformed into *strain*, or *potential energy*. If y is the distance a member is deformed, then the strain energy is

$$U = \frac{F}{2} y = \frac{F^2}{2k}. \tag{1.86}$$

where $y = \frac{F}{k}$. In the above equation, F can be a force, moment or torque.

For tension (compression) and torsion, the potential energy is, respectively,

$$U = \frac{F^2 l}{2AE}, \tag{1.87}$$

$$U = \frac{T^2 l}{2GJ}. \tag{1.88}$$

Figure 1.15(a) shows an element with one side fixed. The force F places the element in pure shear and the work done is $U = F\delta/2$. The shear strain is $\gamma = \delta/l = \tau/G = F/AG$. Therefore, the stain energy due to shear is

$$U = \frac{F^2 l}{2AG}. \tag{1.89}$$

The expression for the strain energy due to bending can be developed by considering a section of a beam as shown in Fig. 1.15(b). The section PQ of the elastic curve has the length ds and the radius of curvature ρ. The strain energy is $dU = (M/2)d\theta$. Since $\rho d\theta = ds$, the strain energy becomes

$$dU = \frac{M ds}{2\rho}. \tag{1.90}$$

Considering Eq. (1.68), ρ can be eliminated in Eq. (1.90) and

$$dU = \frac{M^2 ds}{2EI}. \tag{1.91}$$

The strain energy due to bending for the whole beam can be obtained by integrating Eq. (1.91) and considering that $ds \approx dx$ for small deflections of the beam, that is

$$U = \int \frac{M^2 dx}{2EI}. \tag{1.92}$$

The strain energy stored in a unit volume u can be obtained by dividing Eqs. (1.87) - (1.89) by the total volume lA

$$u = \frac{\sigma^2}{2E} \quad \text{tension and compression,}$$
$$u = \frac{\tau^2}{2G} \quad \text{direct shear,}$$
$$u = \frac{\tau_{max}^2}{4G} \quad \text{torsion.}$$

Even if shear is present and the beam is not very short, Eq. (1.92) still gives good results. The expression of the strain energy due to shear loading of a beam can be approximated by considering Eq. (1.89) multiplied by a correction factor C. The values of C depend upon the shape of the cross section of the beam. Thus, the strain energy due to shear in bending is

$$U = \int \frac{CV^2 dx}{2AG}, \tag{1.93}$$

where V is the shear force. Table 1.6 lists the values of the correction factor C for various cross sections.

Castigliano's theorem provides an approach to deflection analysis.

Castigliano's theorem: *when forces act on a systems subject to small elastic displacements, the displacement corresponding to any force, collinear with the force, is equal to the partial derivative of the total strain energy with respect to that force.*

Castigliano's theorem can be written as

$$\delta_i = \frac{\partial U}{\partial F_i}, \tag{1.94}$$

where δ_i is the displacement of the point of application of the force F_i in the direction of F_i and U is the strain energy. For example, applying Castigliano's theorem for the cases of axial and torsional deflections and considering the expressions for the strain energy given by Eqs. (1.87) and (1.88), the following relations are obtained:

$$\delta = \frac{\partial}{\partial F}\left(\frac{F^2 l}{2AE}\right) = \frac{Fl}{AE}, \tag{1.95}$$

$$\theta = \frac{\partial}{\partial T}\left(\frac{T^2 l}{2GJ}\right) = \frac{Tl}{GJ}. \tag{1.96}$$

Even though no force or moment act at a point, Castigliano's theorem can be used to determine the deflection:

• consider a fictitious force or moment P_i at the point of interest and calculate the expression of the strain energy including the energy due to that dummy force or moment;

• find the expression for the deflection using Eq. (1.96) where the differentiation will be performed with respect to the fictitious force or moment P_i, that is,

$$\delta_i = \frac{\partial U}{\partial P_i}. \tag{1.97}$$

• Solve Eq. (1.97) and set $P_i = 0$, since P_i is a fictitious force or moment.

Compression

The analysis and design of *compression members* depend upon whether these members are loaded in tension or in torsion. The term *column* is applied to those members for which failure is not produced because of pure compression. Columns are classified according to their length and to whether the loading is central or eccentric. The problem of compression members is to find the *critical load* that produces the failure of the member. Next, the approach presented by Shigley and Mischke will be presented [20].

1. Long columns with central loading

Figure 1.16 shows long columns of length l having applied an axial load P and various *end conditions*. The load P is applied along the vertical symmetry axis of the column. The end conditions shown in Fig. 1.16 are

• rounded (or pivoted) - rounded ends [Fig. 1.16(a)]

• fixed - fixed ends [Fig. 1.16(b)]

• free - fixed ends [Fig. 1.16(c)]

• rounded - fixed ends [Fig. 1.16)(d)].

To develop the relationship between the critical load P_{cr} and the column material and geometry, the situation shown in Fig. 1.16(a) is considered. The figure shows that the bar is bent in the positive y direction and, thus, a negative moment is required:

$$M = -P\,y. \tag{1.98}$$

Equations (1.76) and (1.98) give

$$\frac{d^2y}{dx^2} = -\frac{P}{EI}y, \tag{1.99}$$

or

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0, \tag{1.100}$$

with the solution

$$y = A \sin\sqrt{\frac{P}{EI}}x + B \cos\sqrt{\frac{P}{EI}}x, \tag{1.101}$$

where A and B are constants of integration which can be determined by considering the boundary conditions $y = 0$ at $x = 0$ and $y = 0$ at $x = l$. Substituting the two boundary conditions in Eq. (1.101), results in $B = 0$ and

$$0 = A \sin\sqrt{\frac{P}{EI}}l. \tag{1.102}$$

If $A = 0$ is considered into the previous equation, the trivial solution of no *buckling* is obtained.

If $A \neq 0$, then

$$\sin\sqrt{\frac{P}{EI}}l = 0, \tag{1.103}$$

which is satisfied if $\left(\sqrt{P/EI}\right) l = n\pi$, where $n = 1, 2, 3, \ldots$.

The critical load associated with $n = 1$ is called the *first critical load* and is given by the following expression:

$$P_{cr} = \frac{\pi^2 EI}{l^2}. \tag{1.104}$$

Equation (1.104) is called *Euler column formula* and applied only for columns with rounded ends. Substituting Eq. (1.104) into Eq. (1.101), the equation of the deflection curve is obtained:

$$y = A \sin\frac{\pi x}{l}. \tag{1.105}$$

The minimum critical load occurs for $n = 1$.

Consider the relation $I = Ak^2$ for the second moment of area I, where A is the cross-section area and k the radius of gyration. Equation (1.104) can be rewritten as

$$\frac{P_{cr}}{A} = \frac{\pi^2 E}{(l/k)^2}, \tag{1.106}$$

where the ratio l/k is called the *slenderness ratio* and P_{cr}/A is the *critical unit load*. The critical unit load is the load per unit area which can place the column in *unstable equilibrium*. Equation (1.106) shows that the critical unit load depends only upon the modulus of elasticity and the slenderness ratio.

Figure 1.16(b) depicts a column with both ends fixed. The inflection points are at A and B located at a distance $l/4$ from the ends. The distance AB is the same curve as for a column with rounded ends. Substituting the length l by $l/2$ in Eq. (1.104), the expression for the first critical load is

$$P_{cr} = \frac{\pi^2 EI}{(l/2)^2} = \frac{4\pi^2 EI}{l^2}. \tag{1.107}$$

Fig. 1.16(c) shows a column with one end free and the other one fixed. The curve of the free-fixed ends column is equivalent to half of the curve for columns with rounded ends. Therefore, if $2l$ is substituted in Eq. (1.104) for l, then the critical load for this case is obtained:

$$P_{cr} = \frac{\pi^2 EI}{(2l)^2} = \frac{\pi^2 EI}{4l^2}. \tag{1.108}$$

Figure 1.16(d) shows a column with one end fixed and the other one rounded. The inflection point is the point A located at a distance of $0.707l$ from the rounded end. Therefore,

$$P_{cr} = \frac{\pi^2 EI}{(0.707l)^2} = \frac{2\pi^2 EI}{l^2}. \tag{1.109}$$

The above situations can be summarized by writing the Euler equation in the following forms

$$P_{cr} = \frac{C\pi^2 EI}{l^2} \qquad \frac{P_{cr}}{A} = \frac{C\pi^2 E}{(l/k)^2}, \tag{1.110}$$

where C is called the *end-condition constant*. It can have one of the values listed in Table 1.7.

Figure 1.17 shows the unit load P_{cr}/A as a function of the slenderness ratio l/k. The curve PQR is obtained. The quantity S_y corresponds to point Q and represents the yield strength of the material. From the graph it results that any compression member having the l/k value less than $(l/k)_Q$ should be treated as a pure compression member, while all others can be treated as Euler columns. In practice, this fact is not true. Several tests showed the failure of columns with the slenderness ratio below or very close to point Q. For this reason, neither simple compression methods nor the Euler column equation should be used when the slenderness ratio is near $(l/k)_Q$. The solution in this case is to consider a point T on the Euler curve of Fig. 1.17 such that, if the slenderness ratio corresponding to T is $(l/k)_1$, the Euler equation should be used only when the actual slenderness ratio of the column is greater than $(l/k)_1$. Point T can be selected such that $P_{cr}/A = S_y/2$. From Eq. (1.110), the slenderness ratio $(l/k)_1$ is obtained:

$$\left(\frac{l}{k}\right)_1 = \left(\frac{2\pi^2 CE}{S_y}\right)^{1/2}. \tag{1.111}$$

2. Intermediate-length columns with central loading

When the actual slenderness ratio l/k is less than $(l/k)_1$ (the region in Fig. 1.17 where Euler formula is not suitable), the *parabolic* or *J. B. Johnson formula* can be used

$$\frac{P_{cr}}{A} = a - b\left(\frac{l}{k}\right)^2, \tag{1.112}$$

where a and b are constants that can be obtained by fitting a parabola to the Euler curve in Fig. 1.17 (the dashed line ending at T). The constants are

$$a = S_y, \tag{1.113}$$

and

$$b = \left(\frac{S_y}{2\pi}\right)^2 \frac{1}{CE}. \tag{1.114}$$

Substituting Eqs. (1.113) and (1.114) into Eq. (1.112) yields

$$\frac{P_{cr}}{A} = S_y - \left(\frac{S_y}{2\pi}\frac{l}{k}\right)^2 \frac{1}{CE}, \tag{1.115}$$

which can be applied if $\dfrac{l}{k} \le \left(\dfrac{l}{k}\right)_1$.

3. Columns with eccentric loading

Figure 1.18(a) shows a column acted upon by a force P that is applied at a distance e, also called eccentricity, from the centroidal axis of the column. The free-body diagram in shown in Fig. 1.18(b). Equating the sum of moments about the origin O to zero gives

$$\sum M_O = M + Pe + Py = 0. \tag{1.116}$$

Substituting M from Eq. (1.116) into Eq. (1.76), a nonhomogeneous second order differential equation is obtained:

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = -\frac{Pe}{EI}. \tag{1.117}$$

Considering the following boundary conditions

$$x = 0, \quad y = 0,$$
$$x = \frac{l}{2}, \quad \frac{dy}{dx} = 0,$$

and substituting $x = l/2$ in the resulting solution, the maximum deflection δ and the maximum bending moment M_{max} are obtained:

$$\delta = e\left[\sec\left(\frac{1}{2}\sqrt{\frac{P}{EI}}\right) - 1\right], \tag{1.118}$$

$$M_{max} = -P(e+\delta) = -Pe\sec\left(\frac{1}{2}\sqrt{\frac{P}{EI}}\right). \tag{1.119}$$

At $x = l/2$, the compressive stress σ_c is maximum and can be calculated by adding the axial component produced by the load P and the bending component produced by the bending moment M_{max}, that is

$$\sigma_c = \frac{P}{A} - \frac{Mc}{I} = \frac{P}{A} - \frac{Mc}{Ak^2}. \tag{1.120}$$

Substituting Eq. (1.119) into the previous equation yields

$$\sigma_c = \frac{P}{A}\left[1 + \frac{ec}{k^2}\sec\left(\frac{1}{2k}\sqrt{\frac{P}{EA}}\right)\right]. \tag{1.121}$$

Considering the yield strength S_y of the column material as σ_c and manipulating Eq. (1.121) gives

$$\frac{P}{A} = \frac{S_{yc}}{1 + (ec/k^2)\sec[(l/2k)\sqrt{P/AE}]}. \tag{1.122}$$

The previous equation is called the *secant column formula* and the term ec/k^2 is the *eccentricity ratio*. Since Eq. (1.122) cannot be solved explicitly for the load P, root-finding techniques using numerical methods can be applied.

4. Short compression member

A short compression member is illustrated in Fig. 1.19. At point D of coordinate y, the compressive stress in the x-direction has two components, one due to the axial load P which is equal to P/A and the other due to the bending moment which is equal to My/I. The compressive stress is

$$\sigma_c = \frac{P}{A} + \frac{My}{I} = \frac{P}{A} + \frac{PeyA}{IA} = \frac{P}{A}\left(1 + \frac{ey}{k^2}\right), \tag{1.123}$$

where $k = (I/A)^{1/2}$ is the radius of gyration, y the coordinate of point D, and e the eccentricity of loading. Setting $\sigma_c = 0$ and solving, the y-coordinate of a line parallel to the x-axis along which the normal stress is zero is obtained:

$$y = -\frac{k^2}{e}. \tag{1.124}$$

If $y = c$ at point B in Fig. 1.19, the largest compressive stress is obtained. Hence, substituting $y = c$ in Eq. (1.123) gives

$$\sigma_c = \frac{P}{A}\left(1 + \frac{ec}{k^2}\right). \tag{1.125}$$

For design or analysis, the previous equation can be used only if the range of lengths for which the equation is valid is known. For a strut, it is desired that the effect of bending deflection be within a certain small percentage of eccentricity. If the limiting percentage is 1% of e, then the slenderness ratio is bounded by

$$\left(\frac{1}{k}\right)_2 = 0.282\left(\frac{AE}{P_{cr}}\right)^{1/2}. \tag{1.126}$$

Therefore, the limiting slenderness ratio for using Eq. (1.125) is given by Eq. (1.126).

1.3 Examples

Example 1.1. For a stress element having $\sigma_x = 100$ MPa and $\tau_{xy} = 60$ MPa (cw), find the principal stresses and directions on a stress element with respect to the xy system. Plot the maximum and minimum shear stresses τ_1 and τ_2, and find the corresponding normal stresses on another stress element. The stress components that are not given are taken as zero.

Solution.

First, the Mohr's circle diagram corresponding to the given data will be constructed. Then, using the diagram the stress components will be calculated. Finally, the stress components will be drawn.

The first step to construct Mohr's diagram is to draw the σ- and τ-axes [Fig. 1.20(a)] and locate the points A of $\sigma_x = 100$ MPa and C of $\sigma_y = 0$ MPa on the σ-axis.

Then, $\tau_{xy} = 60$ MPa is represented in the cw direction and $\tau_{yx} = 60$ MPa in the ccw direction. Hence, point B has the coordinates $\sigma_x = 100$ MPa, $\tau_{xy} = 60$ MPa and point D has the coordinates $\sigma_x = 0$ MPa, $\tau_{yx} = -60$ MPa. The line BD is the diameter and point $E(0, 50)$ the center of the Mohr's circle. The intersections of the circle with the σ-axis give the principal stresses σ_1 and σ_2 at points F and G, respectively.

The x-axis of the stress elements is line EB and the y axis line ED. The segments BA and AE have the length of 60 and 50 MPa, respectively. The length of segment BE is

$$BE = HE = \tau_1 = \sqrt{(60)^2 + (50)^2} = 78.1 \text{ MPa}.$$

Since the intersection E is 50 MPa from the origin, the principal stresses are

$$\sigma_1 = 50 + 78.1 = 128.1 \text{ MPa}, \qquad \sigma_2 = 50 - 78.1 = -28.1 \text{ MPa}.$$

The angle 2ϕ with respect to the x-axis cw to σ_1 is

$$2\phi = \tan^{-1}\frac{60}{50} = 50.2^\circ.$$

For the first stress element, the x- and y-axes are parallel to the original axes as shown in Fig. 1.20(b). The angle ϕ is in the same direction as the angle 2ϕ in the Mohr's circle diagram. Thus, measuring 25.1° (half of 50.2°) clockwise from x-axis, σ_1-axis is located. The σ_2-axis will be at 90° with respect with the σ_1-axis, as shown in Fig. 1.20(b).

For the second stress element, the two extreme shear stresses occur at the points H and I in Fig. 1.20(a). The two normal stresses corresponding to these shear stresses are each equal to 50 MPa. Point H is 39.8° ccw from point B in the Mohr's circle diagram. Therefore, the stress element is oriented 19.9° (half of 39.8°) ccw from x as shown in Fig. 1.20(c).

Example 1.2. Develop the expressions for the load, the shear force, and the bending moment for the beam in Fig. 1.21.

Solution.

The beam shown in Fig. 1.21 is loaded with the concentrated forces F_1 and F_2. The reactions R_1 and R_2 are also concentrated loads. Thus, using Table 1.3, the load intensity has the following expression:

$$q(x) = R_1\langle x\rangle^{-1} - F_1\langle x - l_1\rangle^{-1} - F_2\langle x - l_2\rangle^{-1} + R_2\langle x - l\rangle^{-1}.$$

The shear force is $V = 0$ at $x = -\infty$. Hence,

$$V(x) = \int_{-\infty}^{x} q(x)\,dx = R_1\langle x\rangle^{0} - F_1\langle x - l_1\rangle^{0} - F_2\langle x - l_2\rangle^{0} + R_2\langle x - l\rangle^{0}.$$

A second integration yields

$$M(x) = \int_{-\infty}^{x} V(x)\,dx = R_1\langle x\rangle^{1} - F_1\langle x - l_1\rangle^{1} - F_2\langle x - l_2\rangle^{1} + R_2\langle x - l\rangle^{1}.$$

To calculate the reactions R_1 and R_2, the functions $V(x)$ and $M(x)$ are evaluated at x slightly larger than l. At that point, both shear force and bending moment must be zero. Therefore, $V(x) = 0$ at x slightly larger than l, that is,

$$V = R_1 - F_1 - F_2 + R_2 = 0.$$

Similarly, the moment equation yields

$$M = R_1 l - F_1(l - l_1) - F_2(l - l_2) = 0.$$

The preceding two equations can be solved to obtain the reaction forces R_1 and R_2.

Example 1.3. A cantilever beam with a uniformly distributed load w is shown in Fig. 1.22. The load w acts on the portion $a \le x \le l$. Determine the expressions of the shear force and the bending moment.

Solution.

The moment M_1 and the force R_1 are the support reactions. Using Table 1.2, the load intensity function is

$$q(x) = -M_1\langle x\rangle^{-2} + R_1\langle x\rangle^{-1} - w\langle x-a\rangle^0.$$

Integrating successively two times gives

$$V(x) = \int_{-\infty}^{x} q(x)\,dx = -M_1\langle x\rangle^{-1} + R_1\langle x\rangle^{0} - w\langle x-a\rangle^1,$$
$$M(x) = \int_{-\infty}^{x} V(x)\,dx = -M_1\langle x\rangle^{0} + R_1\langle x\rangle^{1} - \frac{w}{2}\langle x-a\rangle^2.$$

The reactions can be calculated by evaluating $V(x)$ and $M(x)$ at x slightly larger than l and observing that both V and M are zero in this region. Shear force equation yields

$$-M_1\cdot 0 + R_1 - w(l-a) = 0,$$

which can be solved to obtain the reaction R_1. The moment equations gives

$$-M_1 + R_1 l - \frac{w}{2}(l-a) = 0,$$

which can be solved to obtain the moment M_1.

Example 1.4. Determine the diameter of a solid round shaft OC, represented in Fig. 1.23, such that the bending stress does not exceed 10 kpsi. The transversal loads are $F_A = 800$ lb and $F_B = 300$ lb. The length of the shaft is $l = 36$ in., $a = 12$ in., and $b = 16$ in.

Solution.

The moment equation for the shaft about C yields

$$\sum M_C = -lR_O + (l-a)F_A + (l-a-b)F_B = -36R_O + 24(800) + 8(300) = 0.$$

This equation gives $R_O = 600$ lb.

The force equation for the shaft with respect to the y-axis is

$$\sum F_y = R_O - F_A - F_B + R_C = R_O - 800 - 300 + R_C,$$

yielding $R_C = 500$ lb.

The shear force and the bending moment diagrams are shown in Figs. 1.23(b) and (c).

The maximum bending moment is

$$M = 600(12) = 7200 \ \text{lb} \cdot \text{in.}$$

The section modulus is

$$\frac{I}{c} = \frac{\pi d^3}{32} = 0.0982d^3.$$

Then, the bending stress is

$$\sigma = \frac{M}{I/c} = \frac{7200}{0.0982d^3}.$$

Considering $\sigma = 10\,000$ psi and solving for d, it results:

$$d = \sqrt[3]{\frac{7200}{0.0982(10000)}} = 1.94 \ \text{in.}$$

Example 1.5. Figure 1.24 [20] shows the link 1 with the length $l = 4$ in., the width $w = 1.25$ in., and the thickness $t = 0.25$ in. The link is loaded by the force $F = 1000$ lb. at the distance $a = 1$ in. This force causes the twisting and bending of a shaft 2 with the diameter $D = 0.75$ in. and the length $L = 5$ in. Find: a) the force, the moment, and the torque at the origin A; and b) the maximum torsional stress and the bending stress in the arm BC.

Solution.

The free-body diagrams of links 1 and 2 are shown in Fig. 1.25.

The force and torque at point C are

$$\mathbf{F} = -1000\boldsymbol{\jmath} \ \text{lb}, \qquad \mathbf{T} = -1000\mathbf{k} \ \text{lb} \cdot \text{in.}$$

The force, moment, and torque at the end B of the arm BC are

$$\mathbf{F} = 1000\boldsymbol{\jmath} \ \text{lb}, \quad \mathbf{M} = 4000\boldsymbol{\imath} \ \text{lb} \cdot \text{in.}, \quad \mathbf{T} = 1000\mathbf{k} \ \text{lb} \cdot \text{in.}$$

The force, moment, and torque at the end B of the shaft AB are

$$\mathbf{F} = -1000\boldsymbol{\jmath} \ \text{lb}, \quad \mathbf{T} = -4000\boldsymbol{\imath} \ \text{lb} \cdot \text{in}, \quad \mathbf{M} = -1000\mathbf{k} \ \text{lb} \cdot \text{in.}$$

The force, moment, and torque at the end A of the shaft AB are

$$\mathbf{F} = 1000\boldsymbol{\jmath}\ \text{lb}, \quad \mathbf{M} = 6000\mathbf{k}\ \text{lb}\cdot\text{in.}, \quad \mathbf{T} = 4000\boldsymbol{\imath}\ \text{lb}\cdot\text{in.}$$

For the arm BC, the bending stress will reach a maximum near the shaft at B. The bending stress for the rectangular cross section of the arm is

$$\sigma = \frac{M}{I/c} = \frac{6M}{bh^2} = \frac{6(4000)}{0.25(1.25)^2} = 61\,440 \text{ psi.}$$

The torsional stress is

$$\tau_{max} = \frac{T}{wt^2}\left(3 + 1.8\frac{t}{w}\right) = \frac{1000}{1.25(0.25)^2}\left(3 + 1.8\frac{0.25}{1.25}\right) = 43\,008 \text{ psi.}$$

Example 1.6. Figure 1.26 shows a beam AC loaded by the uniform distributed force w between B and C. Find the analytical expression for the deflection y as a function of x.

Solution.

The loading equation for x in the range $0 < x < l$ is

$$q = R_B\langle x - a\rangle^{-1} - w\langle x - a\rangle^0, \tag{1.127}$$

where R_B is the reaction at B. Integrating this equation four times according to Eqs. (1.74)-(1.78) yields

$$V = R_B\langle x - a\rangle^0 - w\langle x - a\rangle^1 + C_1, \tag{1.128}$$

$$M = R_B\langle x - a\rangle^1 - \frac{w}{2}\langle x - a\rangle^2 + C_1 x + C_2, \tag{1.129}$$

$$EI\theta = \frac{R_B}{2}\langle x - a\rangle^2 - \frac{w}{6}\langle x - a\rangle^3 + \frac{C_1}{2}x^2 + C_2 x + C_3, \tag{1.130}$$

$$EIy = \frac{R_B}{6}\langle x - a\rangle^3 - \frac{w}{24}\langle x - a\rangle^4 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4. \tag{1.131}$$

The integration constants C_1 to C_4 are found using the boundary conditions.

At $x = 0$ both $EI\theta = 0$ and $EIy = 0$. This gives $C_3 = 0$ and $C_4 = 0$.

At $x = 0$ the shear force is equal to the reaction at A. Therefore, Eq. (1.128) gives $V(0) = R_A = C_1$.

The deflection must be zero at $x = a$. Thus, Eq. (1.131) yields

$$\frac{C_1}{6}a^3 + \frac{C_2}{2}a^2 = 0 \quad \text{or} \quad C_1\frac{a}{3} + C_2 = 0. \tag{1.132}$$

At the free end, $x = l$, the moment must be zero. For this boundary condition Eq. (1.129) gives

$$R_B(l-a) - \frac{w}{2}(l-a)^2 + C_1 l + C_2 = 0,$$

and using the notation $l - a = b$, the equation resulted from the sum of the forces in the y direction, namely $R_B = R_A + wb = -C_1 + wb$,

$$C_1 a + C_2 = -\frac{wb^2}{2}. \tag{1.133}$$

Solving Eqs. (1.132) and (1.133) simultaneously for C_1 and C_2 gives

$$C_1 = R_A = \frac{3wb^2}{4a}, \qquad C_2 = \frac{wb^2}{4}.$$

Therefore, the reaction R_B is obtained:

$$R_B = -R_A + wb = \frac{wb}{4a}(4a + 3b).$$

Equation (1.129) for $x = 0$ gives

$$M(0) = M_A = C_2 = \frac{wb^2}{4}.$$

The analytical expression for the deflection curve is obtained by substituting the expressions for R_B and the constants C_1, C_2, C_3, and C_4 in Eq. (1.131), that is

$$EIy = \frac{wb}{24a}(4a + 3b)\langle x - a\rangle^3 - \frac{w}{24}\langle x - a\rangle^4 - \frac{wb^2x^3}{8a} + \frac{wb^2x^2}{8}.$$

Example 1.7. Consider a simply supported beam of length l and rectangular cross section as shown in Fig. 1.27. A uniformly distributed load w is applied to the beam. Find the strain energy due to shear.

Solution.

The shear force at an arbitrary distance x from the origin is

$$V = R_1 - wx = \frac{wl}{2} - wx.$$

The strain energy given by Eq. (1.93) with $C = 1.5$ (see Table 1.1) is

$$U = \frac{1.5}{2AG}\int_0^l \left(\frac{wl}{2} - wx\right)^2 dx = \frac{3w^2l^3}{48AG}.$$

Example 1.8. A concentrated load F is applied to the end of a cantilever beam (Fig. 1.28). Find the strain energy by neglecting the shear.

Solution.

The bending moment at any point x along the beam has the expression $M = -Fx$. Substituting M into Eq. (1.92), the strain energy is

$$U = \int_0^l \frac{F^2x^2}{2EI}\,dx = \frac{F^2l^3}{6EI}.$$

Example 1.9. A cantilever of length l is loaded by a transversal force F at a distance a as shown in Fig. 1.29. Find the maximum deflection of the cantilever if shear is neglected.

Solution.

The maximum deflection of the cantilever will be at its free end. To apply Castigliano's theorem, a fictitious force Q is considered at that point. The bending moments corresponding to the segments OA and AB are, respectively,

$$M_{OA} = F\,(x - a) + Q(x - l)$$
$$M_{AB} = Q(x - l).$$

The total strain energy is obtained

$$U = \int_0^a \frac{M_{OA}^2}{2EI}\,dx + \int_a^l \frac{M_{AB}^2}{2EI}\,dx.$$

Applying Castigliano's theorem, the deflection is

$$y = \frac{\partial U}{\partial Q} = \frac{1}{2EI}\left[\int_0^a 2M_{OA}\frac{\partial M_{OA}}{\partial Q}\,dx + \int_a^l 2M_{AB}\frac{\partial M_{AB}}{\partial Q}\,dx\right].$$

Since

$$\frac{\partial M_{OA}}{\partial Q} = \frac{\partial M_{AB}}{\partial Q} = x - l,$$

the expression for the deflection becomes

$$y = \frac{F}{EI}\left\{\int_0^a \left[F\,(x-a) + Q(x-l)\right](x-l)\,dx + \int_a^l [Q(x-l)](x-l)\,dx\right\}.$$

Since Q is a dummy force, setting $Q = 0$ in the previous equation gives

$$y = \frac{F}{EI}\int_0^a (x-a)\,(x-l)\,dx = \frac{a^2(3l-a)}{6EI}.$$

1.4 Problems

1.1 Find the total elongation of a straight bar of length L if a tensile load F is applied at the ends of the bar. The cross section of the bar is A and the modulus of elasticity is E.

1.2 The bar in Fig. 1.30 has a constant cross section and is held rigidly between the walls. An axial load F is applied to the bar at a distance a from the left end. The length of the bar is l. Find the reactions of the walls upon the bar.

1.3 Consider a straight bar of uniform cross section A loaded with the axial load F. Find a) the normal and shearing stress intensities on a plane inclined at an angle ϕ to the axis of the bar and b) the magnitude and direction of the maximum shearing stress.

1.4 A straight bar with the uniform cross section of 1.2 in.^{2} is acted upon by an axial force of 14 000 lb at each end. Determine a) the normal and shearing stress intensities on a plane inclined at an angle 45° to the axis of the bar and b) the maximum shearing stress.

1.5 A plane element in a body is subjected to a normal stress in the x-direction of $\sigma_x = 12\,000$ lb/in.^{2}, as well as shearing stress (cw) of $\tau_{xy} =$ 4000 lb/in.^{2} Determine a) the normal and shearing stress intensities on a plane inclined at an angle 30° to the normal stress and b) the maximum shearing stress on the inclined plane.

1.6 A plane element in a body is subjected to a normal compressive stress in the x-direction of $\sigma_x = -12\ 000$ lb/in^2, as well as shearing stress (ccw) of $\tau_{xy} = -4000$ lb/in.^{2} Determine a) the normal and shearing stress intensities on a plane inclined at an angle 30° to the normal stress and b) the maximum shearing stress on the inclined plane.

1.7 A plane element in a body is subjected to a normal stress in the x-direction of $\sigma_x = 12\ 000$ lb/in.^{2}, a normal stress in the y-direction of $\sigma_x = 15\,000$ lb/in.^{2}, as well as shearing stress (cw) of $\tau_{xy} = 8000$ lb/in.^{2} Determine a) the principal stresses and their directions and b) the maximum shearing stresses and the directions of the planes on which they occur.

1.8 A plane element in a body is subjected to a normal compressive stress in the x-direction of $\sigma_x = -12\,000$ lb/in^2, a normal stress in the y-direction of $\sigma_x = 15000$ lb/in.^{2}, as well as shearing stress (ccw) of $\tau_{xy} = -8000$ lb/in.^{2} Determine a) the principal stresses and their directions and b) the maximum shearing stresses and the directions of the planes on which they occur.

1.9 A bolted joint is shown in Fig. 1.31. The diameter of the bolt is 0.75 in. and the force is $F = 8000$ lb. Determine the average shearing stress across either of the planes $a - a$ or $b - b$.

1.10 Two plates are joined by a single rivet of 1 in. diameter as shown in Fig. 1.32. The load is $F = 9000$ lb and the rivet holes are 1/16 in. larger in diameter than the rivet. The rivet fills the hole completely. Find the average shearing stress developed in the rivet.

1.11 The fillet weld is a common type of weld used for joining two plates as shown in Fig. 1.33. The dimensions in Fig. 1.33 are $a = 8$ in., $b = 7$ in., and $h = 0.5$ in. The throat of the weld is t. The allowable working stress for shear loading is 11 000 lb/in.^{2} Determine the allowable tensile force F that is applied midway between the two welds. Only shearing stresses are considered in the weld.

1.12 The shafts and gears are usually fastened together by means of a key, as shown in Fig. 1.34. Consider the gear with the radius $R = 10$ in. subject to a force F of 1000 lb. The shaft has a radius $r = 1$ in. The dimensions of key are $t = b = 1/2$ in. and $L = 3$ in. Determine the shear stress on a horizontal plane through the key.

1.13 Consider the simply supported beam, shown in Fig. 1.35, subjected to a concentrated moment M. Find the equation of the deflection curve.

1.14 Consider a simply supported beam subjected to a uniform load distributed, w, over a portion of its length as indicated in Fig. 1.36. Determine the equation of the deflection curve.

1.15 Consider the cantilever beam of Fig. 1.37, subjected to a uniform load distributed, w, over a portion of its length. Find the equation of the deflection curve.

1.16 A sphere of weight W is falling freely through a height h above a cantilever beam as shown in Fig. 1.38. The beam is struck at its tip by the sphere. Determine the total deflection of the tip. Neglect the weight of the beam.

1.17 Consider the simply supported beam, shown in Fig. 1.39, loaded by a concentrated moment M at the left end. Find the equation of the deflection curve and the slope at the left end using Castigliano's theorem.

1.18 The overhanging beam of Fig. 1.40 is loaded by two equal forces F. Find the deflection at the left end using Castigliano's theorem.

1.19 Determine the slenderness ratio for a wood column 10×10 in. in cross-section and 30 ft. long.

1.20 A steel bar with the rectangular cross-section of 2×2 in. and pined at each end is subjected to axial compression. The critical unit load of the material is 33 000 lb/in.^{2} and $E = 30 \times 10^6$ lb/in.^{2} Find a) the minimum length for which Euler's equation may be used to determine the buckling load and b) the critical load if the bar is 75 in. long.

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Figure captions

Figure 1.1. Stress element: a) three-dimensional case; b) planar case.

Figure 1.2. Normal and shear stresses on a planar surface.

Figure 1.3. Mohr's circle.

Figure 1.4. Mohr's circle application: (a) Mohr's circle diagram; (b) principal stresses.

Figure 1.5. (a) Simply supported beam loaded by concentrated forces and (b) free-body diagram of the left-hand part of the beam.

Figure 1.6. Normal stress in flexure.

Figure 1.7. Bending stress in flexure.

Figure 1.8. Normal stress of asymmetrical section beam.

Figure 1.9. Shear stresses.

Figure 1.10. Stresses in beam with rectangular cross-section: (a) side view; (b) cross section view.

Figure 1.11. Bar in torsion.

Figure 1.12. Beam described as a (a) linear spring; (b) nonlinear stiffening spring.

Figure 1.13. Uniformly loaded beam.

Figure 1.14. Simply supported beam loaded by a concentrated force.

Figure 1.15. Strain energy due to: (a) direct shear; (b) bending.

Figure 1.16 Long columns: (a) both round ends column; (b) both fixed ends column; (c) one free end and one fixed end column; (d) one rounded end and one fix end column.

Figure 1.17. Euler's curve.

Figure 1.18. (a) Eccentric loaded column and (b) free-body diagram

Figure 1.19. Short compression member

Figure 1.20. Mohr's circle application for Example 1.1.

Figure 1.21. Free-body diagram for simply supported beam loaded by concentrated forces in Example 1.2.

Figure 1.22. Free-body diagram for cantilever beam with a uniformly distributed load in Example 1.3.

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Figure 1.24. Crank mechanism for Example 1.5.

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Figure 1.26. Cantilever beam loaded by a uniformly distributed force at the free end for Example 1.6.

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Figure 1.39. Simply supported beam for Problem 1.17.
Figure 1.40. Overhanging beam for Problem 1.18.

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