Impact of a compound pendulum with a spring

Figure 1 depicts a uniform rod of mass m and length L and a spring with elastic constant k. The rod is connected to the ground by a pin joint and is free to rotate in a vertical plane and the end point of the rod makes contact with the spring. The spring compresses under the weight of the rod. The compression phase ends when the velocity of the end point of the rod is zero. Next phase is the restitution phase, when the spring expands and the rod is moving upward. At the end of the restitution phase there is the separation of the end point of the rod from the spring.

The plane of motion will be designated the xy plane. The y-axis is vertical, with the positive sense directed vertically upward. The x-axis is horizontal and is contained in the plane of motion. The z-axis is also horizontal and is perpendicular to the plane of motion. These axes define an inertial reference frame. The unit vectors for the inertial reference frame are $\mathbf{\imath}$, $\mathbf{\jmath}$, and $\mathbf{k}$.

The mass center of the rod is at the point C. As the rod is uniform, its mass center is coincident with its geometric center. The mass center, C, is at a distance $L/2$ from the pivot point O fixed to the ground, and the position vector is

$$\mathbf{r}_{OC} = \mathbf{r}_C = x_C \mathbf{\imath} + y_C \mathbf{\jmath},$$

where x_C and y_C are the coordinates of C

$$x_C = \frac{L}{2}\cos\theta, \;\; y_C = \frac{L}{2}\sin\theta.$$

Hence

$$\mathbf{r}_C = \frac{L}{2}\cos\theta\,\mathbf{\imath} + \frac{L}{2}\sin\theta\,\mathbf{\jmath}. \tag{1}$$

The point at the end of the rod A, that will be in contact with the spring, is located at the distance L from the pivot point O. The position vector of the tip A is

$$\mathbf{r}_{OA} = \mathbf{r}_A = x_A\,\mathbf{\imath} + y_A\,\mathbf{\jmath},$$

where

$$x_A = L\cos\theta \quad \text{and} \quad y_A = L\sin\theta.$$

Hence

$$\mathbf{r}_A = L\cos\theta\,\imath + L\sin\theta\,\jmath. \tag{2}$$

The motion of the rod is planar, consisting of pure rotation about the pivot point. The directions of the angular velocity and angular acceleration vectors will be perpendicular to the plane of motion, in the z direction. The angular velocity of the rod can be expressed as

$$\boldsymbol{\omega} = \omega\mathbf{k} = \frac{d\theta}{dt}\mathbf{k} = \dot{\theta}\,\mathbf{k}.$$

This problem involves only a single moving rigid body and the angular velocity vector refers to that body.
The angular acceleration of the rod can be expressed as

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \alpha\mathbf{k} = \frac{d^2\theta}{dt^2}\,\mathbf{k} = \ddot{\theta}\,\mathbf{k}. \tag{3}$$

The positive sense is clockwise.

The mass moment of inertia of the rod about the fixed pivot point O can be evaluated from the mass moment of inertia about the mass center C using the transfer theorem. Thus

$$I_O = I_C + m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{12} + \frac{mL^2}{4} = \frac{mL^2}{3}. \tag{4}$$

Three cases will be considered: I. the motion of the rod before it makes contact with the spring; II. the motion of the rod when it compresses the spring under its weight; III. the restitution phase.

I. The force driving the motion of the rod is gravity. The weight of the rod is acting through its mass center will cause a moment about the pivot point. This moment will give the rod a tendency to rotate about the pivot point. This moment will be given by the cross product of the vector from the pivot point, O to the mass center, C, crossed into the weight force $\mathbf{G} = -mg\,\jmath$. The sum of the moments about this point will be equal to the mass moment of inertia about the pivot point multiplied by the angular acceleration of the rod. Thus

$$I_O\,\boldsymbol{\alpha} = \Sigma\mathbf{M}_O = \mathbf{r}_C \times \mathbf{G} \tag{5}$$

Substituting Eq. (1) and Eq. (3) into Eq. (5) the equation of motion for this case is

$$\frac{mL^2}{3}\ddot{\theta}\mathbf{k} = \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ \frac{L}{2}\cos\theta & \frac{L}{2}\sin\theta & 0 \\ 0 & -mg & 0 \end{vmatrix},$$

or

$$\ddot{\theta} = -\frac{3g}{2L}\cos\theta. \tag{6}$$

The initial conditions at t=0 are

$$\theta(t=0) = \theta(0) = \theta_0 \quad \text{and} \quad \dot{\theta}(t=0) = \omega_0.$$

This case ends at the moment $t = t_1$ when the end point A makes contact with the spring at the angle $\theta(t_1) = \theta_1$.

II. At the moment of contact between the rod and the spring, the forces that are acting in the rod are the weight force, $\mathbf{G} = -mg\,\jmath$ (at C), and the elastic force that is acting upward on y-axis with the point of application at A. The elastic force is

$$\mathbf{P} = \mathbf{F}_e = -k\,\delta\,\jmath, \tag{7}$$

where δ is the deformation of the spring

$$\delta = L\sin\theta - L\sin\theta_1. \tag{8}$$

The moment equation of motion with respect to the fixed point O is

$$I_O\,\boldsymbol{\alpha} = \Sigma\mathbf{M}_O = \mathbf{r}_C \times \mathbf{G} + \mathbf{r}_A \times \mathbf{P}, \tag{9}$$

or

$$\frac{mL^2}{3}\ddot{\theta}\mathbf{k} = \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ \frac{L}{2}\cos\theta & \frac{L}{2}\sin\theta & 0 \\ 0 & -mg & 0 \end{vmatrix} + \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ L\cos\theta & L\sin\theta & 0 \\ 0 & -k(L\sin\theta - L\sin\theta_1) & 0 \end{vmatrix},$$

or

$$\ddot{\theta} = -\frac{3g}{2L}\cos\theta - \frac{3k}{2m}\sin 2\theta + \frac{3}{m}\cos\theta\sin\theta_1. \tag{10}$$

The initial conditions at $t = t_1$ for this phase are

$$\theta(t = t_1) = \theta(t_1) = \theta_1 \quad \text{and} \quad \dot{\theta}(t = t_1) = \omega_1.$$

The maximum compression will end when the velocity of the tip A is zero $[\mathbf{v}_A(t = t_m) = \mathbf{0}$ or $\dot{\theta}(t_m) = \omega_m = 0]$ and δ reaches its maximum deflection.

III. The restitution phase, when the rod is pushed by the spring upward. The equation of motion is the same as in the previous case. This phase will end at $t = t_2$, when the rod will separate from the rod. Because there are no losses, the angular velocity at the end of the restitution, ω_2 is equal to ω_1, the angular velocity at the instant when the rod makes contact with the rod. The initial conditions are

$$\theta(t_m) = \theta_m \quad \text{or} \quad \dot{\theta}(t_m) = \omega_m = 0.$$

The restitution phase will end when the rod separates from the spring $\delta = 0$.

Numerical Example

The mass of the rod is m=2 kg, the length of the rod L=1 m, the elastic constant of the spring is $k = 2 \times 10^3$ N/m, and the mass of the spring is negligible.

The rod is dropped at the initial angle $\theta(0) = \theta_0 = \pi/3$, and the rod gets makes contact with the spring at $\theta(t_1) = \theta_1 = \pi/6$ ($t_1 = 0.354702$ s). The angular velocity at this time is $\omega = -3.28209$ rad/s. The spring compresses under the weight of the rod, until it reaches its maximum compression $\delta_m = 0.0650279$ m, at the time $t_m = 0.388673$ s, and the angle $\theta_m = 25.7835°$. At this moment, when $\omega_m = 0$ and the restitution phase starts. The maximum elastic force is P=130.056 N. The spring expands pushing the rod, until the rod separates from the spring at the angular velocity $\omega_2 = 3.28209$ rad/s, at the time $t_2 = 0.422644$ s. After that, the rod continues its rotational motion until $\omega_3 = 0$ rad/s. The total time of the motion is $t_3 = 0.777346$ s.

In Appendix 1 is given a *Mathematica* program for this numerical example.