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6 Analytical Dynamics

6.1 Kane’s Dynamical Equations

A two-link kinematic chain is considered in Fig. 5.7. The bars 1 and 2 are homogenuos and have the lengths \( L_1 = L_2 = L \). The masses of the rigid links are \( m_1 = m_2 = m \) and the gravitational acceleration is \( g \).

The plane of motion is \( xy \) plane with the \( y \)-axis vertical, with the positive sense directed downward. The origin of the reference frame is at \( A \). The system has two degrees of freedom. To characterize the instantaneous configuration of the system, two generalized coordinates \( q_1(t) \) and \( q_2(t) \) are employed. The generalized coordinates \( q_1 \) and \( q_2 \) denote the radian measure of the angles between the link 1 and 2 and the horizontal \( x \)-axis.

There are two generalized speeds defined as

\[
\begin{align*}
    u_1 &= \dot{q}_1 \\
    u_2 &= \dot{q}_2.
\end{align*}
\]

(6.1)

The mass centers of the links are designated by \( C_1(x_{C_1}, y_{C_1}, 0) \) and \( C_2(x_{C_2}, y_{C_2}, 0) \).

Kinematics

The position vector of the center of the mass \( C_1 \) of the link 1 is

\[
\mathbf{r}_{C_1} = x_{C_1} \mathbf{i} + y_{C_1} \mathbf{j},
\]

where \( x_{C_1} \) and \( y_{C_1} \) are the coordinates of \( C_1 \)

\[
\begin{align*}
    x_{C_1} &= \frac{L_1}{2} \cos q_1, \\
    y_{C_1} &= \frac{L_1}{2} \sin q_1.
\end{align*}
\]

The velocity vector of \( C_1 \) is the derivative with respect to time of the position vector of \( C_1 \)

\[
\mathbf{v}_{C_1} = \dot{\mathbf{r}}_{C_1} = \dot{x}_{C_1} \mathbf{i} + \dot{y}_{C_1} \mathbf{j},
\]

where

\[
\begin{align*}
    \dot{x}_{C_1} &= -\frac{L_1}{2} \dot{q}_1 \sin q_1 \\
    \dot{y}_{C_1} &= \frac{L_1}{2} \dot{q}_1 \cos q_1,
\end{align*}
\]

or

\[
\mathbf{v}_{C_1} = -\frac{L_1}{2} u_1 \sin q_1 \mathbf{i} + \frac{L_1}{2} u_1 \cos q_1 \mathbf{j}.
\]

The acceleration vector of \( C_1 \) is the double derivative with respect to time of the position vector of \( C_1 \)

\[
\mathbf{a}_{C_1} = \ddot{\mathbf{r}}_{C_1} = \ddot{x}_{C_1} \mathbf{i} + \ddot{y}_{C_1} \mathbf{j},
\]
where
\[ \ddot{x}_{C_1} = -\frac{L_1}{2} \dot{q}_1 \sin q_1 - \frac{L_1}{2} \dot{q}_1^2 \cos q_1, \]
\[ \ddot{y}_{C_1} = \frac{L_1}{2} \dot{q}_1 \cos q_1 - \frac{L_1}{2} \dot{q}_1^2 \sin q_1, \]
or
\[ \mathbf{a}_{C_1} = \left( -\frac{L_1}{2} \dot{u}_1 \sin q_1 - \frac{L_1}{2} u_1^2 \cos q_1 \right) \mathbf{i} + \left( \frac{L_1}{2} \dot{u}_1 \cos q_1 - \frac{L_1}{2} u_1^2 \sin q_1 \right) \mathbf{j}. \]

The position vector of the center of the mass \( C_2 \) of the link 2 is
\[ \mathbf{r}_{C_2} = x_{C_2} \mathbf{i} + y_{C_2} \mathbf{j}, \]
where \( x_{C_2} \) and \( y_{C_2} \) are the coordinates of \( C_2 \)
\[ x_{C_2} = L_1 \cos q_1 + \frac{L_2}{2} \cos q_2 \quad \text{and} \quad y_{C_2} = L_1 \sin q_1 + \frac{L_2}{2} \sin q_2. \]
The velocity vector of \( C_2 \) is the derivative with respect to time of the position vector of \( C_2 \)
\[ \mathbf{v}_{C_2} = \dot{\mathbf{r}}_{C_2} = \dot{x}_{C_2} \mathbf{i} + \dot{y}_{C_2} \mathbf{j}, \]
where
\[ \dot{x}_{C_2} = -L_1 \dot{q}_1 \sin q_1 - \frac{L_2}{2} \dot{q}_2 \sin q_2, \]
\[ \dot{y}_{C_2} = L_1 \dot{q}_1 \cos q_1 + \frac{L_2}{2} \dot{q}_2 \cos q_2, \]
or
\[ \mathbf{v}_{C_2} = \left( -L_1 u_1 \sin q_1 - \frac{L_2}{2} u_2 \sin q_2 \right) \mathbf{i} + \left( L_1 u_1 \cos q_1 + \frac{L_2}{2} u_2 \cos q_2 \right) \mathbf{j}. \]
The acceleration vector of \( C_2 \) is the double derivative with respect to time of the position vector of \( C_2 \)
\[ \mathbf{a}_{C_2} = \ddot{\mathbf{r}}_{C_2} = \ddot{x}_{C_2} \mathbf{i} + \ddot{y}_{C_2} \mathbf{j}, \]
where
\[ \ddot{x}_{C_2} = -L_1 \ddot{q}_1 \sin q_1 - \frac{L_1}{2} \ddot{q}_1^2 \cos q_1 - \frac{L_2}{2} \ddot{q}_2 \sin q_2 - \frac{L_2}{2} \ddot{q}_2^2 \cos q_2, \]
\[ \ddot{y}_{C_2} = L_1 \ddot{q}_1 \cos q_1 - L_1 \ddot{q}_1^2 \sin q_1 + \frac{L_2}{2} \ddot{q}_2 \cos q_2 - \frac{L_2}{2} \ddot{q}_2^2 \sin q_2, \]
or
\[ a_{C_2} = (-L_1 \dot{u}_1 \sin q_1 - L_1 u_1^2 \cos q_1 - \frac{L_2}{2} \ddot{u}_2 \sin q_2 - \frac{L_2}{2} u_2^2 \cos q_2) \mathbf{i} + \\
(L_1 \dot{u}_1 \cos q_1 - L_1 u_1^2 \sin q_1 + \frac{L_2}{2} \ddot{u}_2 \cos q_2 - \frac{L_2}{2} u_2^2 \sin q_2) \mathbf{j}. \]

The position vector of the end point \( D \) is
\[ \mathbf{r}_D = x_D \mathbf{i} + y_D \mathbf{j}, \]
where
\[ x_D = L_1 \cos q_1 + L_2 \cos q_2 \quad \text{and} \quad y_D = L_1 \sin q_1 + L_2 \sin q_2. \]
The velocity of the end point \( D \) is
\[ \mathbf{v}_D = \dot{\mathbf{r}}_D = \dot{x}_D \mathbf{i} + \dot{y}_D \mathbf{j}, \]
where
\[ \dot{x}_D = -L_1 \dot{q}_1 \sin q_1 - L_2 \dot{q}_2 \sin q_2, \]
\[ \dot{y}_D = L_1 \dot{q}_1 \cos q_1 + L_2 \dot{q}_2 \cos q_2, \]
or
\[ \mathbf{v}_D = (-L_1 u_1 \sin q_1 - L_2 u_2 \sin q_2) \mathbf{i} + (L_1 u_1 \cos q_1 + L_2 u_2 \cos q_2) \mathbf{j}. \]
The acceleration of the end point \( D \) is
\[ \mathbf{a}_D = \ddot{\mathbf{r}}_D = \ddot{x}_D \mathbf{i} + \ddot{y}_D \mathbf{j}, \]
where
\[ \ddot{x}_D = -L_1 \ddot{q}_1 \sin q_1 - L_1 \dot{q}_1^2 \cos q_1 - L_2 \ddot{q}_2 \sin q_2 - L_2 \dot{q}_2^2 \cos q_2, \]
\[ \ddot{y}_D = L_1 \ddot{q}_1 \cos q_1 - L_1 \dot{q}_1^2 \sin q_1 + L_2 \ddot{q}_2 \cos q_2 - L_2 \dot{q}_2^2 \sin q_2, \]
or
\[ \mathbf{a}_D = (-L_1 \ddot{u}_1 \sin q_1 - L_1 u_1^2 \cos q_1 - L_2 \ddot{u}_2 \sin q_2 - L_2 u_2^2 \cos q_2) \mathbf{i} + \\
(L_1 \ddot{u}_1 \cos q_1 - L_1 u_1^2 \sin q_1 + L_2 \ddot{u}_2 \cos q_2 - L_2 u_2^2 \sin q_2) \mathbf{j}. \]
The angular velocity vectors of the links 1 and 2 are
\[ \omega_1 = \dot{q}_1 k = u_1 k \quad \text{and} \quad \omega_2 = \dot{q}_2 k = u_2 k. \]

The angular acceleration vectors of the links 1 and 2 are
\[ \alpha_1 = \ddot{q}_1 k = \dot{u}_1 k \quad \text{and} \quad \alpha_2 = \ddot{q}_2 k = \dot{u}_2 k. \]

The mass moment of inertia of the link 1 with respect to the center of mass \( C_1 \) is
\[ I_{C_1} = \frac{m_1 L_1^2}{12}. \]

The mass moment of inertia of the link 1 with respect to the fixed point of rotation \( A \) is
\[ I_A = I_{C_1} + m_1 \left( \frac{L_1}{2} \right)^2 = \frac{m_1 L_1^2}{3}. \]

The mass moment of inertia of the link 2 with respect to the center of mass \( C_2 \) is
\[ I_{C_2} = \frac{m_2 L_2^2}{12}. \]

**Generalized inertia forces**

The generalized inertia forces for a rigid body \( RB \) are
\[ K_{in} = \frac{\partial v_{CG}}{\partial u_r} \cdot F_{in} + \frac{\partial \omega}{\partial u_r} \cdot T_{in}, \quad (6.2) \]

where \( v_{CG} \) is the velocity of the mass center \( RB \), and \( \omega = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \) is the angular velocity of \( RB \).

The inertia force for the rigid body \( RB \) is
\[ F_{in} = -M a_{CG}, \quad (6.3) \]

where \( M \) is the mass of \( RB \), and \( a_{CG} \) is the acceleration of the mass center of \( RB \).

The inertia torque \( T_{in} \) for \( RB \) is
\[ T_{in} = -\alpha \cdot \vec{I} - \vec{\omega} \times (\vec{I} \cdot \vec{\omega}), \quad (6.4) \]

where \( \alpha = \dot{\omega} = \alpha_x \hat{i} + \alpha_y \hat{j} + \alpha_z \hat{k} \) is the angular acceleration of \( RB \), and \( \vec{I} = (I_x) \hat{i} + (I_y) \hat{j} + (I_z) \hat{k} \) is the central inertia dyadic of \( RB \). The central principal axes of \( RB \) are parallel to \( \hat{i}, \hat{j}, \hat{k} \) and the associated moments of inertia have the values \( I_x, I_y, I_z \), respectively.
Kane’s Dynamical Equations

• Link 1:

\[ \mathbf{\dot{F}}_{i1} = -m_{1} \mathbf{\dot{a}}_{C1} = \]
\[ -m_{1} \frac{L_1}{2} \left[ (-\dot{u}_{1} \sin q_1 - u_1^2 \cos q_1) \mathbf{i} + (\dot{u}_{1} \cos q_1 - u_1^2 \sin q_1) \mathbf{j} \right], \]
\[ \mathbf{T}_{i1} = -\mathbf{\alpha}_{10} \cdot \mathbf{\ddot{I}}_1 = -\mathbf{\alpha}_1 I_{C1} = -\frac{m_{1} L_1^2}{12} \dot{u}_{1} \mathbf{k}. \] (6.5)

• Link 2:

\[ \mathbf{\dot{F}}_{i2} = -m_{2} \mathbf{\dot{a}}_{C2} = \]
\[ -m_{2} \left( -L_1 \dot{u}_{1} \sin q_1 - L_1 u_1^2 \cos q_1 - \frac{L_2}{2} \dot{u}_{2} \sin q_2 - \frac{L_2}{2} u_2^2 \cos q_2 \right) \mathbf{i} \]
\[ -m_{2} \left( L_1 \dot{u}_{1} \cos q_1 - L_1 u_1^2 \sin q_1 + \frac{L_2}{2} \dot{u}_{2} \cos q_2 - \frac{L_2}{2} u_2^2 \sin q_2 \right) \mathbf{j} \]
\[ \mathbf{T}_{i2} = -\mathbf{\alpha}_2 \cdot \mathbf{\ddot{I}}_2 = -\mathbf{\alpha}_2 I_{C2} = -\frac{m_{2} L_2^2}{12} \dot{u}_{2} \mathbf{k}. \] (6.6)

The generalized inertia forces associated to \( q_1 \) and \( q_2 \) are

\[ K_{i1} = \frac{\partial \mathbf{v}_{C1}}{\partial u_1} \cdot \mathbf{F}_{i1} + \frac{\partial \mathbf{\omega}_1}{\partial u_1} \cdot \mathbf{T}_{i1} + \frac{\partial \mathbf{v}_{C2}}{\partial u_1} \cdot \mathbf{F}_{i2} + \frac{\partial \mathbf{\omega}_1}{\partial u_1} \cdot \mathbf{T}_{i2}, \]
\[ K_{i2} = \frac{\partial \mathbf{v}_{C1}}{\partial u_2} \cdot \mathbf{F}_{i1} + \frac{\partial \mathbf{\omega}_1}{\partial u_2} \cdot \mathbf{T}_{i1} + \frac{\partial \mathbf{v}_{C2}}{\partial u_2} \cdot \mathbf{F}_{i2} + \frac{\partial \mathbf{\omega}_1}{\partial u_2} \cdot \mathbf{T}_{i2}. \] (6.7)
Generalized active forces
The weight forces on the links 1 and 2 are
\[ \mathbf{G}_1 = m_1 g \mathbf{j}, \text{ acts at } C_1, \]
\[ \mathbf{G}_2 = m_2 g \mathbf{j}, \text{ acts at } C_2. \]

The impact force act at the end point \( D \)
\[ \mathbf{P} = -\text{sign}(\mathbf{v}_D \cdot \mathbf{1}) \mu F_1 + F \mathbf{j}, \]
where \( F \) is the normal impulsive force and \( \mu \) is the coefficient of friction.

The generalized active forces associated to \( q_1 \) and \( q_2 \) are
\[ Q_1 = \frac{\partial \mathbf{v}_{C_1}}{\partial u_1} \cdot \mathbf{G}_1 + \frac{\partial \mathbf{v}_{C_2}}{\partial u_1} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{v}_D}{\partial u_1} \cdot \mathbf{P}, \]
\[ Q_2 = \frac{\partial \mathbf{v}_{C_1}}{\partial u_2} \cdot \mathbf{G}_1 + \frac{\partial \mathbf{v}_{C_2}}{\partial u_2} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{v}_D}{\partial u_2} \cdot \mathbf{P}. \]

The Kane’s dynamical equations are
\[ K_{ir} + Q_r = 0, \quad r = 1, 2. \quad (6.8) \]

The solution of the system is obtained from Kane’s dynamical relations Eq. (6.8) and from kinematical relations Eq. (6.1) with the initial conditions \( q_{10} = q_1(0), q_{20} = q_2(0), u_{10} = u_1(0), \) and \( u_{20} = u_2(0). \)
6.2 Lagrange’s Equations of Motion

Kinetic energy
The kinetic energy of the link 1 which is in rotational motion is

\[ T_1 = \frac{1}{2} I_A \omega_1 \cdot \omega_1 = \frac{1}{2} I_A \dot{q}_1^2 = \frac{1}{2} mL^2 \frac{\dot{q}_1}{3} \cdot \ddot{q}_1 = \frac{mL^2}{6} \ddot{q}_1, \]

where \( I_A \) is the mass moment of inertia about the center of rotation \( A \), \( I_A = mL^2/3 \).

The kinetic energy of the bar 2 is due to the translation and rotation and can be expressed as

\[ T_2 = \frac{1}{2} I_{C_2} \omega_1 \cdot \omega_1 + \frac{1}{2} m_2 v_{C_2} \cdot v_{C_2} = \frac{1}{2} I_{C_2} \dot{q}_2^2 + \frac{1}{2} m_2 v_{C_2} \cdot v_{C_2}, \]

where \( I_{C_2} \) is the mass moment of inertia about the center of mass \( C_2 \) and

\[ v_{C_2} \cdot v_{C_2} = v_{C_2}^2 = L^2 \dot{q}_1^2 + \frac{1}{4} L^2 \dot{q}_2^2 + L^2 \dot{q}_1 \dot{q}_2 \cos(q_2 - q_1). \]

Equation (6.9) becomes

\[ T_2 = \frac{1}{2} mL^2 \left[ \dot{q}_2 + \frac{1}{2} \dot{q}_2 + \dot{q}_1 \dot{q}_2 \cos(q_2 - q_1) \right]. \]

The total kinetic energy of the system is

\[ T = T_1 + T_2 = \frac{mL^2}{6} \left[ 4\dot{q}_1^2 + 3\dot{q}_2 \cos(q_2 - q_1) + \dot{q}_2^2 \right]. \]

The left hand sides of Lagrange’s equations \( \partial T / \partial \dot{q}_i, \ i = 1, 2 \) are

\[ \frac{\partial T}{\partial \dot{q}_1} = \frac{mL^2}{6} \left[ 8\dot{q}_1 + 3\dot{q}_2 \cos(q_2 - q_1) \right], \]

\[ \frac{\partial T}{\partial \dot{q}_2} = \frac{mL^2}{6} \left[ 3\dot{q}_1 \cos(q_2 - q_1) + 2\dot{q}_2 \right]. \]

The two Lagrange’s equations are

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} = Q_1, \]

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} = Q_2, \tag{6.9} \]
where the generalized active forces associated to $q_1$ and $q_2$ are

$$Q_1 = \frac{\partial r_{c_1}}{\partial q_1} \cdot G_1 + \frac{\partial r_{c_2}}{\partial q_1} \cdot G_2 + \frac{\partial r_d}{\partial q_1} \cdot P,$$

$$Q_2 = \frac{\partial r_{c_1}}{\partial q_2} \cdot G_1 + \frac{\partial r_{c_2}}{\partial q_2} \cdot G_2 + \frac{\partial r_d}{\partial q_2} \cdot P.$$