2 Lagrange’s Equations

Example 2.1. The planar mechanical system considered is shown in Fig. 2.1 has a slider 1 of mass \( m \) and a mathematical pendulum 2 with the mass \( M \) concentrated at the point \( B \). The length of \( AB \) is \( L \) and the elastic constant of the spring \( R \) is \( k \). The spring deflects only horizontally. Find the equations of motion using Lagrange’s method.

Solution.

To characterize the instantaneous configuration of the system the generalized coordinate are employed. The generalized coordinates are quantities associated with the position of the system. The first generalized coordinate \( q_1(t) \) denotes the linear displacement of the slider. The second generalized coordinate \( q_2(t) \) is the radian measure of the angle between the vertical axis and the line \( AB \).

Kinematics

A cartesian reference frame \( xOyz \) with the unit vectors \( [\mathbf{i}, \mathbf{j}, \mathbf{k}] \) is selected, Fig. 2.1.

The position vector of mass 1 is

\[
\mathbf{r}_1 = \mathbf{r}_A = q_1(t) \mathbf{i}.
\]  

(2.1)

The position vector of mass 2 is

\[
\mathbf{r}_2 = \mathbf{r}_B = [q_1(t) + L \sin q_2(t)] \mathbf{i} + L \cos q_2(t) \mathbf{j}.
\]  

(2.2)

The velocity of the slider 1 is

\[
\mathbf{v}_A = \frac{d\mathbf{r}_A}{dt} = \dot{\mathbf{r}}_A = \dot{q}_1 \mathbf{i},
\]  

(2.3)

and the velocity of the particle at \( B \) is

\[
\mathbf{v}_B = \frac{d\mathbf{r}_B}{dt} = \dot{\mathbf{r}}_B = (\dot{q}_1 + L \dot{q}_2 \cos q_2) \mathbf{i} - L \dot{q}_2 \sin q_2 \mathbf{j}.
\]  

(2.4)

Kinetic energy

The kinetic energy of the slider 1 is

\[
T_1 = \frac{1}{2} m \mathbf{v}_A \cdot \mathbf{v}_A = \frac{1}{2} m \dot{q}_1^2.
\]  

(2.5)
and the kinetic energy of the mass 2 is
\[ T_2 = \frac{1}{2} M v_B \cdot v_B = \frac{1}{2} M \left( \dot{q}_1^2 + 2L \dot{q}_1 \dot{q}_2 \cos q_2 + L^2 \dot{q}_2^2 \right). \quad (2.6) \]
The total kinetic energy is
\[ T = T_1 + T_2. \quad (2.7) \]

**Generalized forces**
The forces that act on 1 at A are the spring force and the gravity force
\[ F_A = -kq_1 \mathbf{i} + mg \mathbf{j}, \quad (2.8) \]
where \( g = 9.81 \text{ m/s}^2 \) is the gravity acceleration. The gravity force acts on mass 2 at B
\[ F_B = Mg \mathbf{j}. \quad (2.9) \]

There are two generalized forces. The generalized force associated to \( q_1 \) is
\[ Q_1 = F_A \cdot \frac{\partial \mathbf{r}_A}{\partial q_1} + F_B \cdot \frac{\partial \mathbf{r}_B}{\partial q_1} = (-kq_1 \mathbf{i} + mg \mathbf{j}) \cdot \mathbf{i} + Mg \mathbf{j} \cdot \mathbf{i} = -kq_1. \quad (2.10) \]
The generalized force associated to \( q_2 \) is
\[ Q_2 = F_A \cdot \frac{\partial \mathbf{r}_A}{\partial q_2} + F_B \cdot \frac{\partial \mathbf{r}_B}{\partial q_2} = (-kq_1 \mathbf{i} + mg \mathbf{j}) \cdot \mathbf{0} + Mg \mathbf{j} \cdot (L \cos q_2 \mathbf{i} - L \sin q_2 \mathbf{j}) = -MgL \sin q_2. \quad (2.11) \]

**Lagrange’s equations**
The two Lagrange’s equations are
\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} = Q_1, \]
\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} = Q_2. \quad (2.12) \]
Lagrange’s equations - examples

One can calculate for \( q_1 \)

\[
\frac{\partial T}{\partial \dot{q}_1} = (m + M)\dot{q}_1 + LM\dot{q}_2 \cos q_2,
\]

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) = (m + M)\ddot{q}_1 + LM\ddot{q}_2 \cos q_2 - LM\dot{q}_2^2 \sin q_2,
\]

\[
\frac{\partial T}{\partial \dot{q}_1} = 0.
\]  \( 2.13 \)

For the generalized coordinate \( q_2 \) the left hand side of Lagrange’s equation is

\[
\frac{\partial T}{\partial \dot{q}_2} = LM \left( \dot{q}_1 \cos q_2 + L \dot{q}_2 \right),
\]

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_2} \right) = LM \left( \ddot{q}_1 \cos q_2 - \ddot{q}_2 \dot{q}_2 \sin q_2 + \dot{q}_1 \dot{q}_2 \right),
\]

\[
\frac{\partial T}{\partial q_2} = -LM \dot{q}_1 \dot{q}_2 \sin q_2.
\]  \( 2.14 \)

The equations of motion are

\[
(m + M)\ddot{q}_1 + LM\ddot{q}_2 \cos q_2 - LM\dot{q}_2^2 \sin q_2 = -kq_1,
\]

\[
LM \left( \ddot{q}_1 \cos q_2 - \ddot{q}_2 \dot{q}_2 \sin q_2 + \ddot{q}_1 \dot{q}_2 \right) + LM\dot{q}_1 \dot{q}_2 \sin q_2 = -MgL \sin q_2
\]  \( 2.15 \)

or

\[
(m + M)\ddot{q}_1 + LM\dot{q}_2 \cos q_2 - LM\dot{q}_2^2 \sin q_2 + kq_1 = 0,
\]

\[
LM \ddot{q}_1 \cos q_2 + ML^2 \dot{q}_2 + MgL \sin q_2 = 0.
\]  \( 2.16 \)

For small oscillations of the pendulum \( \sin q_2 \approx q_2 \) and \( \cos q_2 \approx 1 \), the equations of motion are

\[
(m + M)\ddot{q}_1 + LM\ddot{q}_2 - LM\dot{q}_2^2 \sin q_2 + kq_1 = 0,
\]

\[
LM\ddot{q}_1 + ML^2 \dot{q}_2 + MgLq_2 = 0.
\]  \( 2.17 \)

The \textit{Mathematica}™ program with the equations of motion are given in Program 2.1. The equations of motion are numerically solved for \( m = M = 1 \) kg, \( L = 1 \) m, \( k = 1 \) N/m, and \( g = 10 \) m/s\(^2\), and the following initial conditions

\[
q_1(0) = q_2(0) = 0.1, \ \dot{q}_1(0) = \dot{q}_2(0) = 0.
\]
**Example 2.2.** A double compound pendulum is considered in Fig. 2.2. The bars 1 and 2 are homogennous and have the lengths $OA = AB = L$ and the masses $m_1 = m_2 = m$. At $O$ and $A$ there are pin joints. The mass centers of links 1 and 2 are at $C_1$ and at $C_2$. Find the Lagrange’s equations of motion.

**Solution.**

To characterize the instantaneous configuration of the system, two generalized coordinates $q_1(t)$ and $q_2(t)$ are employed. The generalized coordinates $q_1$ and $q_2$ denote the radian measure of the angles between the link 1 and 2 and the vertical $y$ axis.

**Kinematics**

The position vector of the mass center of link 1 is

$$r_{C_1} = 0.5L \sin q_1 \mathbf{i} + 0.5L \cos q_1 \mathbf{j},$$

and the position vector of the mass center of link 2 is

$$r_{C_2} = (L \sin q_1 + 0.5L \sin q_2) \mathbf{i} + (L \cos q_1 + 0.5L \cos q_2) \mathbf{j}.$$  

(2.18)

(2.19)

The velocity of $C_1$ is

$$v_{C_1} = \frac{dr_{C_1}}{dt} = \dot{r}_{C_1} = 0.5L \dot{q}_1 \cos q_1 \mathbf{i} - 0.5L \dot{q}_1 \sin q_1 \mathbf{j},$$

(2.20)

and the velocity of $C_2$ is

$$v_{C_2} = \frac{dr_{C_2}}{dt} = \dot{r}_{C_2} =$$

$$(L \dot{q}_1 \cos q_1 + 0.5L \dot{q}_2 \cos q_2) \mathbf{i} - (L \dot{q}_1 \sin q_1 + 0.5L \dot{q}_2 \sin q_2) \mathbf{j}.$$  

(2.21)

**Kinetic energy**

The kinetic energy of the link 1 which is in rotational motion is

$$T_1 = \frac{1}{2} I_0 \dot{q}_1^2 = \frac{1}{2} \frac{mL^2}{3} \dot{q}_1^2 = \frac{ML^2}{6} \dot{q}_1^2,$$

(2.22)

where $I_0$ is the mass moment of inertia about the center of rotation $O$, $I_O = mL^2/3$.

The kinetic energy of the bar 2 is due to the translation and rotation and can be expressed as

$$T_2 = \frac{1}{2} I_{C_2} \dot{q}_2^2 + \frac{1}{2} m_2 v_{C_2}^2,$$

(2.23)
where $I_{C_2}$ is the mass moment of inertia about the center of mass $C_2$, $I_{C_2} = mL^2/12$, and
\[
v_{C_2}^2 = v_{C_2} \cdot v_{C_2} = L^2 \dot{q}_2^2 + \frac{1}{4} L^2 \dot{q}_2^2 + L^2 \dot{q}_1 \dot{q}_2 \cos(q_2 - q_1) \tag{2.24}
\]

Equation (2.23) becomes
\[
T_2 = \frac{1}{2} \frac{mL^2}{12} \ddot{q}_2^2 + \frac{1}{2} mL^2 \left[ \dot{q}_1^2 + \frac{1}{4} \dot{q}_2^2 + \dot{q}_1 \dot{q}_2 \cos(q_2 - q_1) \right]. \tag{2.25}
\]

The total kinetic energy of the system is
\[
T = T_1 + T_2 = \frac{mL^2}{6} \left[ 4\dot{q}_1^2 + 3 \dot{q}_1 \dot{q}_2 \cos(q_2 - q_1) + \dot{q}_2^2 \right]. \tag{2.26}
\]

The left hand sides of Lagrange’s equations are
\[
\frac{\partial T}{\partial \dot{q}_1} = \frac{mL^2}{6} \left[ 8\dot{q}_1 + 3\dot{q}_2 \cos(q_2 - q_1) \right],
\]
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) = \frac{mL^2}{6} \left[ 8\ddot{q}_1 + 3\ddot{q}_2 \cos(q_2 - q_1) - 3\dot{q}_2 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1) \right],
\]
\[
\frac{\partial T}{\partial q_1} = \frac{mL^2}{6} \dot{q}_1 \dot{q}_2 \sin(q_2 - q_1) = \frac{mL^2}{6} \dot{q}_1 \dot{q}_2 \sin(q_2 - q_1);
\]
\[
\frac{\partial T}{\partial \dot{q}_2} = \frac{mL^2}{6} \left[ 3\dot{q}_1 \cos(q_2 - q_1) + 2\dot{q}_2 \right],
\]
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_2} \right) = \frac{mL^2}{6} \left[ 3\ddot{q}_1 \cos(q_2 - q_1) - 3\dot{q}_1 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1) + 2\ddot{q}_2 \right],
\]
\[
\frac{\partial T}{\partial q_2} = -\frac{mL^2}{6} \dot{q}_1 \dot{q}_2 \sin(q_2 - q_1) = -\frac{mL^2}{6} \dot{q}_1 \dot{q}_2 \sin(q_2 - q_1). \tag{2.27}
\]

**Generalized forces**

The gravity forces on links 1 and 2 at the mass centers $C_1$ and $C_2$
\[
\mathbf{F}_{C_1} = \mathbf{G}_1 = m_1 g \mathbf{j} = mg \mathbf{j} \quad \text{and} \quad \mathbf{F}_{C_2} = \mathbf{G}_2 = mg \mathbf{j}. \tag{2.28}
\]

There are two generalized forces. The generalized force associated to $q_1$ is
\[
Q_1 = \mathbf{F}_{C_1} \cdot \frac{\partial \mathbf{r}_{C_1}}{\partial q_1} + \mathbf{F}_{C_2} \cdot \frac{\partial \mathbf{r}_{C_2}}{\partial q_1} = mg \mathbf{j} \cdot (0.5 L \cos q_1 \mathbf{j} - 0.5 L \sin q_1 \mathbf{j}) + mg \mathbf{j} \cdot (L \cos q_1 \mathbf{j} - L \sin q_1 \mathbf{j})
\]
\[
= -1.5mgL \sin q_1. \tag{2.29}
\]
The generalized force associated to \( q_2 \) is

\[
Q_2 = \mathbf{F}_{C_1} \cdot \frac{\partial \mathbf{r}_{C_1}}{\partial q_2} + \mathbf{F}_{C_2} \cdot \frac{\partial \mathbf{r}_{C_2}}{\partial q_2} = \\
mg \mathbf{j} \cdot \mathbf{0} + mg \mathbf{j} \cdot (0.5L \cos q_2 + 0.5L \sin q_2) \\
= -0.5mgL \sin q_2.
\]

The two Lagrange’s equations are

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} = Q_1, \\
1.333mL^2 \ddot{q}_1 + 0.5mL^2 \ddot{q}_2 \cos(q_2 - q_1) - 0.5mL^2 \dot{q}_2^2 \sin(q_2 - q_1) \\
+ 1.5mgL \sin q_1 = 0;
\]

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} = Q_2, \\
0.5mL^2 \ddot{q}_1 \cos(q_2 - q_1) + 0.333mL^2 \ddot{q}_2 + 0.5mL^2 \dot{q}_1^2 \sin(q_2 - q_1) \\
+ 0.5mgL \sin q_2 = 0.
\]

The Mathematica™ program with the equations of motion are given in Program 2.2. The equations of motion are numerically solved for \( m_1 = m_2 = m = 1 \text{ kg}, L_1 = L_2 = 1 \text{ m}, \) and \( g = 10 \text{ m/s}^2, \) and the following initial conditions

\[
q_1(0) = q_2(0) = 0.1, \quad \dot{q}_1(0) = \dot{q}_2(0) = 0.
\]
Example 2.3. The mechanism shown in Fig. 2.3(a) is considered. The lengths of the links are \( L_1 = 0.001 \) m, \( L_2 = 0.470 \) m, and \( L_3 = 0.047 \) m. The links 1 and 2 are rectangular prisms with the depth \( d = 0.001 \) m and height \( h = 0.01 \) m. Link 3 has the height \( h_3 = 0.02 \) m, and the depth \( d_3 = 0.05 \) m. The mass density of the links is \( \rho = 7850 \) Kg/m\(^3\). The angle of the driver link with the horizontal axis is \( \theta(t) = \angle BAC \) and the angular velocity of the driver link is \( \omega(t) = \dot{\theta} \). A motor moment acts on link 1 and is given by \( M_m = M k \) [Fig. 2.39(b)]. For a D.C. electric motor, \( M = M_0 \left( 1 - \frac{\omega}{\omega_0} \right) \), where \( M_0 \) and \( \omega_0 \) are given in catalogues. For the considered mechanism \( M_0 = 1 \) N m and \( \omega_0 = 4 \) rad/s. The initial conditions \( \theta(0) = \pi/6 \) rad and \( \omega(0) = \dot{\theta}(0) = 0 \) rad/s are given. Find the Lagrange’s equation of motion of the mechanism.

Solution.

A fixed reference frame \( xyz \) is chosen with the origin at \( A \). The center of mass locations of the links \( i = 1, 2, 3 \) are designated by \( C_i(x_{Ci}, y_{Ci}, 0) \). The mechanism has one degree of freedom and the angle \( \theta(t) \) is selected as the generalized coordinate.

Kinematics

The position vector of the center of the mass \( C_1 \) of the link 1 is

\[
\mathbf{r}_{C_1} = x_{C_1}\mathbf{i} + y_{C_1}\mathbf{j},
\]

where \( x_{C_1} \) and \( y_{C_1} \) are the coordinates of \( C_1 \)

\[
x_{C_1} = \frac{L_1}{2} \cos \theta, \quad y_{C_1} = \frac{L_1}{2} \sin \theta.
\]

The position vector of the center of the mass \( C_2 \) of the link 2 is

\[
\mathbf{r}_{C_2} = x_{C_2}\mathbf{i} + y_{C_2}\mathbf{j},
\]

where \( x_{C_2} \) and \( y_{C_2} \) are the coordinates of \( C_2 \)

\[
x_{C_2} = L_1 \cos \theta + \frac{L_2}{2} \cos \theta_2, \quad y_{C_2} = L_1 \sin \theta + \frac{L_2}{2} \sin \theta_2,
\]

where \( \theta_2 = \arctan \frac{L_1 \sin \theta}{L_1 \cos \theta - AC} \).
The position vector of the center of the mass $C_3$ of the link 3 is

$$\mathbf{r}_{C_3} = AC_1. \quad (2.36)$$

The velocity vector of $C_1$ is the derivative with respect to time of the position vector of $C_1$

$$\mathbf{v}_{C_1} = \dot{\mathbf{r}}_{C_1} = \dot{x}_{C_1}\mathbf{1} + \dot{y}_{C_1}\mathbf{J}, \quad (2.37)$$

where

$$\dot{x}_{C_1} = -\frac{L_1}{2} \dot{\theta} \sin \theta, \quad \dot{y}_{C_1} = \frac{L_1}{2} \dot{\theta} \cos \theta. \quad (2.38)$$

The velocity vector of $C_2$ is the derivative with respect to time of the position vector of $C_2$

$$\mathbf{v}_{C_2} = \dot{\mathbf{r}}_{C_2} = \dot{x}_{C_2}\mathbf{1} + \dot{y}_{C_2}\mathbf{J}, \quad (2.39)$$

where

$$\dot{x}_{C_2} = -L_1 \dot{\theta} \sin \theta - \frac{L_2}{2} \dot{\theta}_2 \sin \theta_2, \quad (2.40)$$

$$\dot{y}_{C_2} = L_1 \dot{\theta} \cos \theta + \frac{L_2}{2} \dot{\theta}_2 \cos \theta_2.$$

The velocity vector of $C_3$ is zero

$$\mathbf{v}_{C_3} = \mathbf{0}. \quad (2.41)$$

The acceleration vector of $C_1$ is the double derivative with respect to time of the position vector of $C_1$

$$\mathbf{a}_{C_1} = \ddot{\mathbf{r}}_{C_1} = \ddot{x}_{C_1}\mathbf{1} + \ddot{y}_{C_1}\mathbf{J}, \quad (2.42)$$

where

$$\ddot{x}_{C_1} = -\frac{L_1}{2} \ddot{\theta} \sin \theta - \frac{L_1}{2} \dot{\theta}_2^2 \cos \theta, \quad (2.43)$$

$$\ddot{y}_{C_1} = \frac{L_1}{2} \ddot{\theta} \cos \theta - \frac{L_1}{2} \dot{\theta}_2^2 \sin \theta.$$

The acceleration vector of $C_2$ is the double derivative with respect to time of the position vector of $C_2$

$$\mathbf{a}_{C_2} = \ddot{\mathbf{r}}_{C_2} = \ddot{x}_{C_2}\mathbf{1} + \ddot{y}_{C_2}\mathbf{J}, \quad (2.44)$$
where
\[\ddot{x}_{C_2} = -L_1 \dot{\theta} \sin \theta - L_1 \dot{\theta}^2 \cos \theta - \frac{L_2}{2} \dot{\theta}_2 \sin \theta_2 - \frac{L_2}{2} \dot{\theta}_2^2 \cos \theta_2,\]
\[\ddot{y}_{C_2} = L_1 \dot{\theta} \cos \theta - L_1 \dot{\theta}^2 \sin \theta + \frac{L_2}{2} \dot{\theta}_2 \cos \theta_2 - \frac{L_2}{2} \dot{\theta}_2^2 \sin \theta_2.\]

The acceleration vector of \(C_3\) is zero
\[a_{C_3} = 0.\]

The angular velocity vectors of the links 1, 2, and 3 are
\[\omega = \dot{\theta} \mathbf{k},\]
\[\omega_2 = \omega_3 = \dot{\theta}_2 \mathbf{k}.\]

The angular acceleration vectors of the links 1, 2, and 3 are
\[\alpha = \ddot{\theta} \mathbf{k},\]
\[\alpha_2 = \alpha_3 = \ddot{\theta}_2 \mathbf{k}.\]

**Kinetic energy**

The masses of the links 1, 2 and 3 are
\[m_1 = \rho L_1 h d,\]
\[m_2 = \rho L_2 h d,\]
\[m_3 = m_{3a} - m_{3b},\]

where \(m_{3a} = \rho L_3 h_3 d_3\) and \(m_{3b} = \rho L_3 h d.\)

The mass moment of inertia of the link 1 with respect to the center of mass \(C_1\) is
\[I_{C_1} = \frac{m_1}{12} \left(L_1^2 + h^2\right).\]

The mass moment of inertia of the link 2 with respect to the center of mass \(C_2\) is
\[I_{C_2} = \frac{m_2}{12} \left(L_2^2 + h^2\right).\]

The mass moment of inertia of the link 3 with respect to the center of mass \(C_3\) is
\[I_{C_3} = \frac{m_{3a}}{12} \left(L_3^2 + h_3^2\right) - \frac{m_{3b}}{12} \left(L_3^2 + h^2\right).\]

The kinetic energy \(T_1\) for the link 1 is
\[T_1 = \frac{1}{2} m_1 \mathbf{v}_{C_1} \cdot \mathbf{v}_{C_1} + \frac{1}{2} I_{C_1} \mathbf{\omega} \cdot \mathbf{\omega}.\]
The kinetic energy $T_2$ for the link 2 is
\[
T_2 = \frac{1}{2} m_2 v_{C_2} \cdot v_{C_2} + \frac{1}{2} I_{C_2} \omega_2 \cdot \omega_2.
\] (2.51)

The kinetic energy $T_3$ for the link 3 is
\[
T_3 = \frac{1}{2} I_{C_3} \omega_3 \cdot \omega_3.
\] (2.52)

The total kinetic energy is
\[
T = \sum_{i=1}^{3} T_i = T_1 + T_2 + T_3.
\] (2.53)

**Generalized force**

The gravitational forces on links 1, 2, and 3 are
\[
G_1 = -m_1 g \mathbf{j}, \quad G_2 = -m_2 g \mathbf{j}, \quad G_3 = -m_3 g \mathbf{j}.
\] (2.54)

The generalized force $Q_i$ associated with the gravitational force $G_i$ is
\[
Q_i = \frac{\partial r_{C_i}}{\partial \theta} \cdot G_i.
\] (2.55)

The generalized force $Q_m$ associated to the motor moment is
\[
Q_m = \frac{\partial \omega}{\partial \theta} \cdot M_m = M_0 \left( 1 - \frac{\dot{\theta}}{\omega_0} \right).
\] (2.56)

The total generalized force $Q$ for the mechanism is
\[
Q = \sum_{i=1}^{3} Q_i + Q_m = \sum_{i=1}^{3} \frac{\partial r_{C_i}}{\partial \theta} \cdot G_i + \frac{\partial \omega}{\partial \theta} \cdot M_m.
\] (2.57)

**Lagrange’s equation**

The Lagrange’s differential equation for the mechanism with one degree of freedom is
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} = Q,
\] (2.58)

where $T$ is the total kinetic energy of the system, and $Q$ is the generalized force.
Lagrange’s equations - examples

For the link 1 some calculations are given

\[ T_1 = \frac{1}{2}(I_{C1} + \frac{1}{4}L_1^2 m_1)\dot{\theta}^2, \]
\[ \frac{\partial T_1}{\partial \dot{\theta}} = (I_{C1} + \frac{1}{4}L_1^2 m_1)\dot{\theta}, \]
\[ \frac{\partial r_{C1}}{\partial \dot{\theta}} = -\frac{1}{2}L_1(\sin \theta \hat{i} + \cos \theta \hat{j}), \]
\[ Q_1 = \frac{\partial r_{C1}}{\partial \theta} \cdot (-m_1 g \hat{j}) = -\frac{1}{2}m_1 g L_1 \cos \theta. \]

The Mathematica™ program with the equations of motion are given in Program 2.3.
Remark: Lagrange’s method does not require the calculation of the joint forces.
Example 2.4. Figure 2.4(a) is a schematic representation of an open kinematic chain (robot arm) consisting of three links 1, 2, 3, and a rigid body $RB$. Link 1 can be rotated at $A$ in a “fixed” cartesian reference frame $(0)$ of unit vectors $[ı_0, ĵ_0, k_0]$ about a vertical axis $ı_0$. The unit vector $ı_0$ is fixed in link 1. Link 1 is connected to link 2 through pin joints $B$ and $B'$. The link 2 rotates relative to 1 about an axis fixed in both 1 and 2, passing through $B$, and $B'$. The link 3 is connected to 2 by means of a slider joint 2’. The slider joint is rigidly attached to link 2. The last link 3 holds rigidly the rigid body $RB$. The mass centers of links 1, 2, 2’ and 3 are $C_1$, $C_2 = C_2'$, and $C_3$, respectively. The mass center of $RB$ is at $C_R$. The length of link 2 is $l$ and the length of link 3 is $L$. The mass of the link 1 is $m_1$, the masses of the bars 2 and 3 are $m_2$ and $m_3$, the mass of the slider 2’ is $m_2'$, and the mass of $RB$ is $m_R$. Find the equations of motion for the robotic system.

Solution

A reference frame $(1)$ of unit vectors $[ı_1, ĵ_1, k_1]$ is attached to body 1, with $ı_1 = ı_0$.

A reference frame $(2)$ of unit vectors $[ı_2, ĵ_2, k_2]$ is attached to link 2, as it is shown in Fig. 2.4. The unit vector $ĵ_2$ is parallel to the axis of link 2, $BB'$, and $ĵ_2 = ĵ_1$. The unit vector $k_2$ is parallel to the axis of link 3, $C_2C_R$.

To characterize the instantaneous position of the arm, the generalized coordinates $q_1(t)$, $q_2(t)$, $q_3(t)$ are employed. The first generalized coordinate $q_1$ denotes the radian measure of the angle between the axes of $(1)$ and $(0)$ [Fig. 2.4(b)]. The second generalized coordinate $q_2$ designates the radian measure of rotation of the angle between $(1)$ and $(2)$ [Fig. 2.4(c)]. The last generalized coordinate $q_3$ is the distance from $C_2$ to $C_3$.

Angular velocities

Next the angular velocities of the links and the rigid body will be expressed in the fixed reference frame $(0)$. The angular velocity of link 1 in $(0)$ is

$$\omega_{10} = \dot{q}_1 ı_1 = \dot{q}_1 ı_0. \quad (2.59)$$

The angular velocity of link 2 with respect to $(1)$ is

$$\omega_{21} = \dot{q}_2 ĵ_2 = \dot{q}_2 ĵ_1, \quad (2.60)$$

and the angular velocity of link 2 with respect to the fixed reference frame $(0)$ is

$$\omega_{20} = \omega_{10} + \omega_{21} = \dot{q}_1 ı_1 + \dot{q}_2 ĵ_2. \quad (2.61)$$
The unit vectors $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$ can be expressed as functions of $\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$ [Fig. 2.4(b)]

\[
\begin{align*}
\mathbf{i}_1 &= \mathbf{i}_0, \\
\mathbf{j}_1 &= c_1 \mathbf{j}_0 + s_1 \mathbf{k}_0, \\
\mathbf{k}_1 &= -s_1 \mathbf{j}_0 + c_1 \mathbf{k}_0,
\end{align*}
\]

(2.62)

where $s_1 = \sin q_1$ and $c_1 = \cos q_1$.

The unit vectors $\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2$ can be expressed as [Fig. 2.4(c)]

\[
\begin{align*}
\mathbf{i}_2 &= c_2 \mathbf{i}_1 - s_2 \mathbf{k}_1 \\
&= c_2 \mathbf{i}_0 + s_1 s_2 \mathbf{j}_0 - c_1 s_2 \mathbf{k}_0, \\
\mathbf{j}_2 &= \mathbf{j}_1, \\
&= c_1 \mathbf{j}_0 + s_1 \mathbf{k}_0, \\
\mathbf{k}_2 &= s_2 \mathbf{i}_1 + c_2 \mathbf{k}_1 \\
&= s_2 \mathbf{i}_0 - c_2 s_1 \mathbf{j}_0 + c_1 c_2 \mathbf{k}_0,
\end{align*}
\]

(2.63)

where $s_2 = \sin q_2$ and $c_2 = \cos q_2$.

The angular velocity of link 2 in (0) can be written in terms of the unit vectors of the reference frame (2) as

\[
\omega_{20} = \dot{q}_1 c_2 \mathbf{i}_2 + \dot{q}_2 \mathbf{j}_2 + \dot{q}_1 s_2 \mathbf{k}_2,
\]

(2.64)

and in terms of the unit vectors of the reference frame (0) as

\[
\omega_{20} = \dot{q}_1 \mathbf{i}_0 + \dot{q}_2 c_1 \mathbf{j}_0 + \dot{q}_2 s_1 \mathbf{k}_0.
\]

(2.65)

The link 3 and the rigid body RB have the same rotation motion as link 2, i.e.,

\[
\omega_{30} = \omega_{R0} = \omega_{20},
\]

where $\omega_{30}$ is the angular velocity of link 3 in (0) and $\omega_{R0}$ is the angular velocity of RB in (0).

**Linear velocities**

The position vector of $C_1$, the mass center of link 1 is

\[
r_{C_1} = L_{111} = L_{110},
\]

(2.66)
and the velocity of $C_1$ in (0) is

$$v_{C_1} = \frac{d}{dt} r_{C_1} = \dot{r}_{C_1} = 0. \quad (2.67)$$

The position vector of $C_2$, the mass center of link 2, is

$$r_{C_2} = L_2 \hat{1}_1 = L_2 \hat{1}_0,$$

or written in terms of the unit vectors of the reference frame (2)

$$r_{C_2} = L_2 c_2 \hat{1}_2 + L_2 s_2 \hat{k}_2.$$

The velocity of $C_2$ in (0) is

$$v_{C_2} = \frac{d}{dt} r_{C_2} = \frac{d}{dt} (L_2 \hat{1}_0) = 0.$$

The position vector of $C_3$ with respect to reference frame (0) is

$$r_{C_3} = r_{C_2} + q_3 \hat{k}_2 = L_2 \hat{1}_0 + q_3 \hat{k}_2, \quad (2.68)$$

or expressing $\hat{k}_2$ in terms of reference (0) unit vectors yields

$$r_{C_3} = (L_2 + q_3 s_2) \hat{1}_0 - q_3 c_2 s_1 \hat{j}_0 + q_3 c_2 c_1 \hat{k}_0.$$

The position vector of $C_3$ with respect to reference frame (0) written in terms of the unit vectors of the reference frame (2) is

$$r_{C_3} = L_2 c_2 \hat{1}_2 + (q_3 + L_2 s_2) \hat{k}_2.$$

The velocity of the mass center $C_3$ in (0), written in terms of the unit vectors of the reference frame (0), can be calculated taking the derivative with respect to time of Eq. (2.69)

$$v_{C_3} = \frac{d}{dt} r_{C_3} = (c_2 q_2 \dot{q}_2 + s_2 \dot{q}_3) \hat{1}_0 +$$

$$(s_1 s_2 \dot{q}_2 q_3 - c_1 c_2 q_2 \dot{q}_1 - s_1 c_2 \dot{q}_3) \hat{j}_0 +$$

$$(c_1 c_2 \dot{q}_3 - s_1 c_2 q_3 \dot{q}_1 - c_1 s_2 q_3 \dot{q}_2) \hat{k}_0. \quad (2.69)$$
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The velocity of $C_3$ in (0) can be computed using the derivation formula for the moving vector $\mathbf{r}_{C_3}$

$$\mathbf{v}_{C_3} = \frac{d}{dt} \mathbf{r}_{C_3} = \frac{(2)}{dt} \mathbf{r}_{C_3} + \mathbf{\omega}_{20} \times \mathbf{r}_{C_3},$$

(2.70)

where $\frac{(2)}{dt}$ represents the partial derivative with respect to time in reference frame (2), $[\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2]$,

$$\frac{(2)}{dt} \mathbf{r}_{C_3} = \frac{(2)}{dt} [L_2 c_2 \mathbf{i}_2 + (q_3 + L_2 s_2) \mathbf{k}_2] = -\dot{q}_2 L_2 s_2 \mathbf{i}_2 + (\dot{q}_3 + \dot{q}_2 L_2 c_2) \mathbf{k}_2.$$ (2.71)

Using Eqs. (2.65)(2.69)(2.70)(2.71) the velocity of $C_3$ in (0), written in terms of the unit vectors of the reference frame (2) is

$$\mathbf{v}_{C_3} = -\dot{q}_2 L_2 s_2 \mathbf{i}_2 + (\dot{q}_3 + \dot{q}_2 L_2 c_2) \mathbf{k}_2 + \begin{vmatrix} \mathbf{i}_2 & \mathbf{j}_2 & \mathbf{k}_2 \\ \dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\ L_2 c_2 & 0 & q_3 + L_2 s_2 \end{vmatrix}$$

$$= \dot{q}_2 q_3 \mathbf{i}_2 - \dot{q}_1 q_3 c_2 \mathbf{j}_2 + \dot{q}_3 \mathbf{k}_2.$$ (2.72)

The position vector of the mass center $C_R$ of the rigid body $RB$ is

$$\mathbf{r}_{C_R} = \mathbf{r}_{C_3} + \mathbf{r}_{C_3 C_R}$$

$$= \mathbf{r}_{C_3} + \frac{L}{2} \mathbf{k}_2,$$ (2.73)

or expressed in terms of the reference frame (0) is

$$\mathbf{r}_{C_R} = \left[ L_2 + \left( q_3 + \frac{L}{2} \right) s_2 \right] \mathbf{i}_0 - \left( q_3 + \frac{L}{2} \right) \mathbf{j}_0 + \left( q_3 + \frac{L}{2} \right) \mathbf{k}_0.$$

The velocity of $C_R$ in (0) is

$$\mathbf{v}_{C_R} = \frac{d}{dt} \mathbf{r}_{C_R} = \left[ \left( q_3 + \frac{L}{2} \right) c_2 \dot{q}_2 + s_2 \dot{q}_3 \right] \mathbf{i}_0 +$$

$$\left[ s_1 s_2 \dot{q}_2 \left( q_3 + \frac{L}{2} \right) - s_1 c_2 \dot{q}_3 - c_1 c_2 \dot{q}_1 \left( q_3 + \frac{L}{2} \right) \right] \mathbf{j}_0 +$$

$$\left[ -c_2 s_1 \dot{q}_1 \left( q_3 + \frac{L}{2} \right) - c_1 s_2 \dot{q}_2 \left( q_3 + \frac{L}{2} \right) + c_1 c_2 \dot{q}_3 \right] \mathbf{k}_0.$$ (2.74)
The velocity of \( C_R \) in (0) can be computed much easier using mobile reference frame (2)

\[
v_{C_R} = \frac{d}{dt}r_{C_R} = \frac{(2)\,d}{dt}r_{C_R} + \omega_{20} \times r_{C_R},
\]

where

\[
r_{C_R} = L_2 c_2 \hat{1}_2 + (q_3 + L_2 s_2 + L/2) \hat{k}_2.
\]

The velocity of \( C_R \) is

\[
v_{C_R} = -\dot{q}_2 L_2 s_2 \hat{1}_2 + (\dot{q}_3 + \dot{q}_2 L_2 c_2) \hat{k}_2 + \begin{vmatrix}
\hat{1}_2 & \hat{j}_2 & \hat{k}_2 \\
\dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\
L_2 c_2 & 0 & q_3 + L_2 s_2 + L/2
\end{vmatrix}
= (L/2 + q_3)\dot{q}_2 \hat{1}_2 - c_2 \dot{q}_1 (q_3 + L/2) \hat{j}_2 + \dot{q}_3 \hat{k}_2.
\]

Remark: The angular velocity \( \omega_{10} \) was expressed in terms of unit vectors \([\hat{1}_1, \hat{j}_1, \hat{k}_1]\) and \( \omega_{20} \) expressed in terms of unit vectors \([\hat{1}_2, \hat{j}_2, \hat{k}_2]\). This will facilitate later work, where it will be assumed that the central principal axes of inertia of link 1 are parallel to \([\hat{1}_1, \hat{j}_1, \hat{k}_1]\) and the central principal axis of inertia of links 2 and 3 are parallel to \([\hat{1}_2, \hat{j}_2, \hat{k}_2]\). When it comes to dealing with the velocities of \( C_1, C_2, C_3 \), and \( C_R \) it is best to use whatever vector basis permits one to write the simplest expression.

**Kinetic energy**

The kinetic energy of a rigid body is

\[
T = \frac{1}{2} m v_C \cdot v_C + \frac{1}{2} \omega \cdot (\bar{I} \cdot \omega),
\]

where \( m \) is the mass, \( v_C \) is the velocity of the mass center, \( \omega = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \) is the angular velocity of the rigid body in (0), and \( \bar{I} = (I_x \hat{i} + (I_y \hat{j} + (I_z \hat{k})) \) is the central inertia dyadic of the rigid body. The central principal axes of the rigid body are parallel to \( \hat{i}, \hat{j}, \hat{k} \) and the associated moments of inertia have the values \( I_x, I_y, I_z \), respectively. The inertia matrix associated to \( \bar{I} \) is

\[
\bar{I} \rightarrow I = \begin{bmatrix}
I_x & 0 & 0 \\
0 & I_y & 0 \\
0 & 0 & I_z
\end{bmatrix}.
\]
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The dot product of the vector $\mathbf{\omega}$ with the central inertia dyadic $I$ is

$$\mathbf{\omega} \cdot I = I \cdot \mathbf{\omega} = \omega_x I_x \mathbf{i} + \omega_y I_y \mathbf{j} + \omega_z I_z \mathbf{k}.$$  (2.79)

The total kinetic energy of the robot arm is

$$T = T_1 + T_2 + T'_2 + T_3 + T_R,$$  (2.80)

where $T_1$ is the kinetic energy of link 1, $T_2$ is the kinetic energy of bar 2, $T'_2$ is the kinetic energy of slider 2', $T_3$ is the kinetic energy of bar 3, and $T_R$ is the kinetic energy of $RB$.

The kinetic energy of link 1 is

$$T_1 = \frac{1}{2} m_1 \mathbf{v}_{C1} \cdot \mathbf{v}_{C1} + \frac{1}{2} \omega_{10} \cdot (\bar{I}_1 \cdot \omega_{10}) = \frac{1}{2} \omega_{10} \cdot (\bar{I}_1 \cdot \omega_{10}),$$  (2.81)

where $m_1$ is the mass of the link, $\bar{I}_1 = (I_{1x} \mathbf{i}_1) + (I_{1y} \mathbf{j}_1) + (I_{1z} \mathbf{k}_1)$ is the central inertia dyadic of link 1, and $\omega_{10} = \dot{q}_1 \mathbf{i}_1$. Using the above relation the kinetic energy of link 1 is

$$T_1 = \frac{1}{2} I_{1x} \dot{q}_1^2.$$  (2.82)

The kinetic energy of bar 2 is

$$T_2 = \frac{1}{2} m_2 \mathbf{v}_{C2} \cdot \mathbf{v}_{C2} + \frac{1}{2} \omega_{20} \cdot (\bar{I}_2 \cdot \omega_{20}) = \frac{1}{2} \omega_{20} \cdot (\bar{I}_2 \cdot \omega_{20})$$  (2.83)

where $m_2$ is the mass of the bar and

$$\bar{I}_2 = (I_{2x} \mathbf{i}_2) + (I_{2y} \mathbf{j}_2) + (I_{2z} \mathbf{k}_2) = \left(\frac{m_2 l^2}{12} \mathbf{i}_2\right) + \left(\frac{m_2 l^2}{12} \mathbf{k}_2\right),$$

is the central inertia dyadic of bar 2 with the length $l$. The kinetic energy is

$$T_2 = \frac{m_2 l^2}{24} \dot{q}_1^2.$$  (2.84)

The kinetic energy of slider 2' is

$$T'_2 = \frac{1}{2} m'_2 \mathbf{v}_{C2'} \cdot \mathbf{v}_{C2'} + \frac{1}{2} \omega_{20} \cdot (\bar{I}'_2 \cdot \omega_{20}) = \frac{1}{2} \omega_{20} \cdot (\bar{I}'_2 \cdot \omega_{20})$$  (2.85)

where $m'_2$ is the mass of the slider and

$$\bar{I}'_2 = (I_{2'x} \mathbf{i}_2) + (I_{2'y} \mathbf{j}_2) + (I_{2'z} \mathbf{k}_2),$$
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is the central inertia dyadic of the slider. The kinetic energy is

$$T_2' = \frac{1}{2} \left[ (I_{2x} c_2^2 + I_{2z} s_2^2) \dot{q}_1^2 + I_{2y} \dot{q}_2^2 \right].$$  \hspace{1cm} (2.86)

The kinetic energy of bar 3 is

$$T_3 = \frac{1}{2} m_3 \mathbf{v}_{C3} \cdot \mathbf{v}_{C3} + \frac{1}{2} \mathbf{\omega}_{20} \cdot (\bar{I}_3 \cdot \mathbf{\omega}_{20}),$$  \hspace{1cm} (2.87)

where \(m_3\) is the mass of the bar and

$$\bar{I}_3 = (I_{3x} \mathbf{1}_2) \mathbf{1}_2 + (I_{3y} \mathbf{k}_2) \mathbf{k}_2 + (I_{3z} \mathbf{j}_2) \mathbf{j}_2 = (\frac{m_3 L^2}{12}) \mathbf{1}_2 + (\frac{m_3 L^2}{12}) \mathbf{k}_2,$$

is the central inertia dyadic of bar 3.

The rigid body \(RB\) is considered as a particle with the mass \(m_R\) concentrated at \(C_R\). The kinetic energy of \(RB\) is

$$T_R = \frac{1}{2} m_R \mathbf{v}_{CR} \cdot \mathbf{v}_{CR} = \frac{m_R}{2} \left[ (L/2 + q_3)^2 \dot{q}_2^2 + c_2^2(q_3 + L/2)^2 \dot{q}_1^2 + \dot{q}_3^2 \right].$$  \hspace{1cm} (2.88)

**Generalized forces**

In the case of the robot arm, there are two kinds of forces that contribute to the generalized forces \(Q_1, Q_2, Q_3\) namely, contact forces applied in order to drive 1, 2, 3 and \(RB\), and gravitational forces exerted on 1, 2, 3, and \(RB\) by the Earth. The contact forces are neglected for this example. The gravitational forces exerted on 1, 2, 3, and \(RB\) by the Earth, are denoted by \(G_1, G_2, G_3, G_R\), respectively, and can be expressed as

$$G_1 = -m_1 g \mathbf{1}_0,$$
$$G_2 = -(m_2 + m_2') g \mathbf{1}_0,$$
$$G_3 = -m_3 g \mathbf{1}_0,$$
$$G_R = -m_R g \mathbf{1}_0.$$  \hspace{1cm} (2.89)

One can express the contribution to the generalized force of all forces and torques acting on the system, as

$$Q_r = \frac{\partial \mathbf{r}_{C1}}{\partial q_r} \cdot \mathbf{G}_1 + \frac{\partial \mathbf{r}_{C2}}{\partial q_r} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{r}_{C3}}{\partial q_r} \cdot \mathbf{G}_3 + \frac{\partial \mathbf{r}_{CB}}{\partial q_r} \cdot \mathbf{G}_R, \hspace{0.5cm} r = 1, 2, 3.$$  \hspace{1cm} (2.90)
The vectors in Eq. (2.90) must be expressed in terms of the fixed reference frame (0). The generalized forces are

\[
Q_1 = 0,
Q_2 = -gc_2(m_r L/2 + m_r q_3 + m_3 q_3),
Q_3 = -g(m_R + m_3)s_2.
\]

The same results can be obtained using the relations

\[
Q_r = \frac{\partial \mathbf{v}_{C_1}}{\partial \dot{q}_r} \cdot \mathbf{G}_1 + \frac{\partial \mathbf{v}_{C_2}}{\partial \dot{q}_r} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r} \cdot \mathbf{G}_3 + \frac{\partial \mathbf{v}_{CB}}{\partial \dot{q}_r} \cdot \mathbf{G}_R, \quad r = 1, 2, 3. \tag{2.91}
\]

In Eq. (2.91) the vectors \( \mathbf{v}_{C_1} \), \( \mathbf{G}_1 \) are expressed in terms of the mobile reference frame (1), and the vectors \( \mathbf{v}_{C_2} \), \( \mathbf{G}_2 \), \( \mathbf{v}_{C_3} \), \( \mathbf{G}_3 \), \( \mathbf{v}_{C_R} \), \( \mathbf{G}_R \) are expressed in terms of the mobile reference frame (2).

The Lagrange’s equations of motion are

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = Q_r,
\]

where \( r = 1, 2, 3 \).

The Mathematica™ program with the equations of motion are given in Program 2.4.
Example 2.5. Figure 2.5(a) is a schematic representation of an open kinematic chain (robot arm) consisting of three links 1, 2, and 3. Let \( m_1, m_2, m_3 \) be the masses of 1, 2, 3, respectively. Link 1 can be rotated at \( A \) in a “fixed” reference frame \((0)\) of unit vectors \([\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0]\) about a vertical axis \( \mathbf{i}_0 \). The unit vector \( \mathbf{i}_0 \) is fixed in 1. The link 1 is connected to link 2 at the pin joint \( B \). The element 2 rotates relative to 1 about an axis fixed in both 1 and 2, passing through \( B \), and perpendicular to the axis of 1. The last link 3 is connected to 2 by means of a slider joint. The mass centers of links 1, 2, and 3 are \( C_1, C_2, \) and \( C_3 \), respectively. The distances \( L_1 = AC_1, L_2 = BC_2, \) and \( L_B = AB \) are indicated in Fig. 2.5. The reference frame \((1)\) of the unit vectors \([\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1]\) is attached to link 1, and the reference frame \((2)\) of the unit vectors \([\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2]\) is attached to link 2, as shown in Fig. 2.5(b). Find and solve the Lagrange’s equations of motion.

Solution

The generalized coordinates (quantities associated with the instantaneous position of the system) are \( q_1(t), q_2(t), q_3(t) \).

The first generalized coordinate \( q_1 \) denotes the radian measure of the angle between the axes of \((1)\) and \((0)\). The unit vectors \( \mathbf{i}_1, \mathbf{j}_1, \) and \( \mathbf{k}_1 \) can be expressed as functions of \( \mathbf{i}_0, \mathbf{j}_0, \) and \( \mathbf{k}_0 \)

\[
\begin{align*}
\mathbf{i}_1 &= \mathbf{i}_0, \\
\mathbf{j}_1 &= c_1 \mathbf{j}_0 + s_1 \mathbf{k}_0, \\
\mathbf{k}_1 &= -s_1 \mathbf{j}_0 + c_1 \mathbf{k}_0,
\end{align*}
\]

or

\[
\begin{bmatrix}
\mathbf{i}_1 \\
\mathbf{j}_1 \\
\mathbf{k}_1
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & c_1 & s_1 \\
0 & -s_1 & c_1
\end{bmatrix}
\begin{bmatrix}
\mathbf{i}_0 \\
\mathbf{j}_0 \\
\mathbf{k}_0
\end{bmatrix},
\]

where \( s_1 = \sin q_1 \) and \( c_1 = \cos q_1 \). The transformation matrix from \((1)\) to \((0)\) is

\[
R_{10} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & c_1 & s_1 \\
0 & -s_1 & c_1
\end{bmatrix}.
\]

The second generalized coordinate designates also a radian measure of the rotation angle between \((1)\) and \((2)\). The unit vectors \( \mathbf{i}_2, \mathbf{j}_2 \) and \( \mathbf{k}_2 \) can be
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expressed as

\[ \mathbf{1}_2 = c_2 \mathbf{1}_1 - s_2 \mathbf{k}_1 \]
\[ = c_2 \mathbf{1}_0 + s_1 s_2 \mathbf{J}_0 - c_1 s_2 \mathbf{k}_0, \]
\[ \mathbf{J}_2 = \mathbf{J}_1, \]
\[ = c_1 \mathbf{J}_0 + s_1 \mathbf{k}_0, \]
\[ \mathbf{k}_2 = s_2 \mathbf{1}_1 + c_2 \mathbf{k}_1 \]
\[ = s_2 \mathbf{1}_0 - c_2 s_1 \mathbf{J}_0 + c_1 c_2 \mathbf{k}_0, \]

where \( s_2 = \sin q_2 \) and \( c_2 = \cos q_2 \). The transformation matrix from (2) to (1) is

\[ R_{21} = \begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix}. \]

The last generalized coordinate \( q_3 \) is the distance from \( C_2 \) to \( C_3 \).

Angular velocities

Next the angular velocity of the links 1, 2, and 3 will be expressed in the fixed reference frame (0). The angular velocity of 1 in (0) is

\[ \mathbf{\omega}_{10} = \dot{q}_1 \mathbf{1}_1. \]

The angular velocity of the link 2 with respect to (1) is

\[ \mathbf{\omega}_{21} = \dot{q}_2 \mathbf{J}_2. \]

The angular velocity of the link 2 with respect to the fixed reference frame (0) is

\[ \mathbf{\omega}_{20} = \mathbf{\omega}_{10} + \mathbf{\omega}_{21} = \dot{q}_1 \mathbf{1}_1 + \dot{q}_2 \mathbf{J}_2. \]

With \( \mathbf{1}_0 = \mathbf{1}_1 = c_1 \mathbf{1}_2 + s_1 \mathbf{k}_2 \) the angular velocity of the link 2 in the reference frame (0) written in terms of the reference frame (2) is

\[ \mathbf{\omega}_{20} = \dot{q}_1 (c_2 \mathbf{1}_2 + s_2 \mathbf{k}_2) + \dot{q}_2 \mathbf{J}_2 = \dot{q}_1 c_2 \mathbf{1}_2 + \dot{q}_2 \mathbf{J}_2 + \dot{q}_1 s_2 \mathbf{k}_2. \]

The link 3 has the same rotational motion as link 2, i.e., \( \mathbf{\omega}_{30} = \mathbf{\omega}_{20} \).

The angular acceleration of the link 1 in the reference frame (0) can be expressed as

\[ \mathbf{\alpha}_{10} = \ddot{q}_1 \mathbf{1}_1. \]
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The angular acceleration of the link 2 with respect to the reference frame (0) is

\[ \alpha_{20} = \frac{d}{dt} \omega_{20} = \frac{(2)}{dt} \omega_{20}. \]

where \( \frac{(2)}{dt} \) represents the derivative with respect to time in reference frame (2), \([i_2, j_2, k_2]\). The link 3 has the same angular acceleration as link 2, i.e., \( \alpha_{30} = \alpha_{20} \).

**Linear velocities**

The position vector of \( C_1 \), the mass center of link 1, is

\[ \mathbf{r}_{C_1} = L_1 \mathbf{k}_1, \]

and the velocity of \( C_1 \) in (0) is

\[ \mathbf{v}_{C_1} = \frac{d}{dt} \mathbf{r}_{C_1} = \frac{(1)}{dt} \mathbf{r}_{C_1} + \omega_{10} \times \mathbf{r}_{C_1} = \begin{vmatrix} 1_1 & 0 & 0 \\ \dot{q}_1 & 0 & 0 \\ 0 & 0 & L_1 \end{vmatrix} = -L_1 \dot{q}_1 \mathbf{j}_1. \]

The position vector of \( C_2 \), the mass center of link 2, is

\[ \mathbf{r}_{C_2} = L_B \mathbf{k}_1 + L_2 \mathbf{k}_2 = L_B(-s_2 i_2 + c_2 k_2) + L_2 k_2 = -L_B s_2 i_2 + (L_B c_2 + L_2) k_2. \]

The velocity of \( C_2 \) in (0) is

\[ \mathbf{v}_{C_2} = \frac{d}{dt} \mathbf{r}_{C_2} = \frac{(2)}{dt} \mathbf{r}_{C_2} + \omega_{20} \times \mathbf{r}_{C_2} = -L_B c_1 \dot{q}_2 i_2 - L_B c_2 \dot{q}_2 k_2 + \begin{vmatrix} i_2 & \mathbf{j}_2 & \mathbf{k}_2 \\ \dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\ -L_B s_2 & 0 & L_B c_2 + L_2 \end{vmatrix} = -L_2 \dot{q}_2 i_2 + (L_B + L_2 c_2) \dot{q}_1 j_2. \]

The position vector of \( C_3 \) with respect to reference frame (0) is

\[ \mathbf{r}_{C_3} = \mathbf{r}_{C_2} + q_3 \mathbf{k}_2 = -L_B s_2 i_2 + (L_B c_2 + L_2 + q_3) \mathbf{k}_2, \]
and the velocity of this mass center in (0) is

$$v_{C_3} = \frac{d}{dt} r_{C_3} = \frac{(2)d}{dt} r_{C_3} + \omega_{20} \times r_{C_3}$$

$$= -L_B c_2 \dot{q}_2 \mathbf{i}_2 -(L_B c_2 \dot{q}_2 + \dot{q}_3)k_2 + \begin{vmatrix} \mathbf{i}_2 & \mathbf{j}_2 & k_2 \\ \dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\ -L_B s_2 & 0 & L_B c_2 + L_2 + q_2 \end{vmatrix}$$

$$= (L_2 + q_3) \dot{q}_2 \mathbf{i}_2 - (L_B + L_2 c_2 + c_2 q_2) \dot{q}_1 \mathbf{j}_2 + \dot{q}_3 k_2.$$ 

### Kinetic energy

The total kinetic energy of the robot arm in the reference frame (0) is

$$T = \sum_{i=1}^{3} T_i.$$ 

The kinetic energy of the link $i$, $i = 1, 2, 3$, is

$$T_i = \frac{1}{2} m_i v_{C_i} \cdot v_{C_i} + \frac{1}{2} \omega_{i0} \cdot (\bar{I}_i \cdot \omega_{i0}).$$

**Remark:** The kinetic energy for a rigid body is

$$T_{\text{rigid body}} = \frac{1}{2} m v_C \cdot v_C + \frac{1}{2} \omega \cdot (\bar{I}_C \cdot \omega),$$

where $m$ is the mass of the rigid body, $v_C$ is the velocity of the mass center of the rigid body in (0), $\omega = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$ is the angular velocity of the rigid body in (0), and $\bar{I} = (I_x \mathbf{i} + (I_y \mathbf{j}) + (I_z \mathbf{k}) \mathbf{k}$ is the central inertia dyadic of the rigid body. The central principal axes of the rigid body are parallel to $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and the associated moments of inertia have the values $I_x, I_y, I_z$, respectively. The inertia matrix associated with $\bar{I}$ is

$$\bar{I} \rightarrow \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}.$$ 

The dot product of the vector $\omega$ with the dyadic $\bar{I}$ is

$$\omega \cdot \bar{I} = \bar{I} \cdot \omega = \omega_x I_x \mathbf{i} + \omega_y I_y \mathbf{j} + \omega_z I_z \mathbf{k}.$$
Lagrange’s equations - examples

The links 1, 2, and 3 have the following mass distribution properties. The central principal axes of 1 are parallel to \( \mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1 \), Fig. 2.5, and the associated moments of inertia have the values \( I_{1x}, I_{1y}, I_{1z} \), respectively. The central inertia dyadic of 1 is

\[
\vec{I}_1 = (I_{1x}\mathbf{i}_1)\mathbf{i}_1 + (I_{1y}\mathbf{j}_1)\mathbf{j}_1 + (I_{1z}\mathbf{k}_1)\mathbf{k}_1.
\]

The central principal axes of 2 and 3 are parallel to \( \mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2 \) and the associated moments of inertia have values \( I_{2x}, I_{2y}, I_{2z} \), and \( I_{3x}, I_{3y}, I_{3z} \) respectively. The central inertia dyadic of 2 is

\[
\vec{I}_2 = (I_{2x}\mathbf{i}_2)\mathbf{i}_2 + (I_{2y}\mathbf{j}_2)\mathbf{j}_2 + (I_{2z}\mathbf{k}_2)\mathbf{k}_2,
\]

and the central inertia dyadic of 3 is

\[
\vec{I}_3 = (I_{3x}\mathbf{i}_2)\mathbf{i}_2 + (I_{3y}\mathbf{j}_2)\mathbf{j}_2 + (I_{3z}\mathbf{k}_2)\mathbf{k}_2.
\]

The central inertia dyadics of links 1 and 2 are calculated using Fig. 2.5(c). The kinetic energy is given in Program 2.5.

The left hand side of Lagrange’s equations is

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r}, \quad r = 1, 2, 3.
\]

**Generalized forces**

**Remark:** If a set of contact and/or body forces acting on a rigid body is equivalent to a couple of torque \( \mathbf{T} \) together with force \( \mathbf{R} \) applied at a point \( P \) of the rigid body, then the contribution of this set of forces to the generalized force, \( Q_r \), is given by

\[
Q_r = \frac{\partial \mathbf{\omega}}{\partial \dot{q}_r} \cdot \mathbf{T} + \frac{\partial \mathbf{v}_P}{\partial \dot{q}_r} \cdot \mathbf{R}, \quad r = 1, 2, ..., \]

where \( \mathbf{\omega} \) is the angular velocity of the rigid body in \( (0) \), \( \mathbf{v}_P \) is the velocity of \( P \) in \( (0) \), and \( r \) represents the generalized coordinates.

In the case of the robotic arm, there are two kinds of forces that contribute to the generalized forces \( Q_1, Q_2, \) and \( Q_3 \) namely, contact forces applied in order to drive the links 1, 2, and 3, and gravitational forces exerted on 1, 2, and 3 by the Earth.
The set of contact forces transmitted from 0 to 1 can be replaced with a couple of torque $T_{01}$ applied to 1 at $A$.

Similarly, the set of contact forces transmitted from 1 to 2 can be replaced with a couple of torque $T_{12}$ applied to 2 at $B$. The law of action and reaction then guarantees that the set of contact forces transmitted from 1 to 2 is equivalent to a couple of torque $-T_{12}$ to 1 at $B$.

Next, the set of contact forces exerted by link 2 on link 3 can be replaced with a force $F_{23}$ applied to 3 at $C$. The law of action and reaction guarantees that the set of contact forces transmitted from 3 to 2 is equivalent to a force $-F_{23}$ applied to 2 at $C$.

The point $C_{32}$ ($C_{32} \in \text{link2}$) instantaneously coincides with $C_3$, ($C_3 \in \text{link3}$).

The expressions $T_{01}$, $T_{12}$, and $F_{23}$ are

$$T_{01} = T_{01x} \mathbf{i}_1 + T_{01y} \mathbf{j}_1 + T_{01z} \mathbf{k}_1, \quad T_{12} = T_{12x} \mathbf{i}_2 + T_{12y} \mathbf{j}_2 + T_{12z} \mathbf{k}_2,$$

$$F_{23} = F_{23x} \mathbf{i}_2 + F_{23y} \mathbf{j}_2 + F_{23z} \mathbf{k}_2.$$

The external gravitational forces exerted on the links 1, 2, and 3 by the Earth, can be denoted by $G_1$, $G_2$, and $G_3$ respectively, and can be expressed as

$$G_1 = -m_1 g \mathbf{i}_1,$$

$$G_2 = -m_2 g \mathbf{i}_1 = -m_2 g (c_2 \mathbf{i}_2 + s_2 \mathbf{k}_2),$$

$$G_3 = -m_3 g \mathbf{i}_1 = -m_3 g (c_2 \mathbf{i}_2 + s_2 \mathbf{k}_2).$$

The reason for replacing $\mathbf{i}_1$ with $c_2 \mathbf{i}_2 + s_2 \mathbf{k}_2$ in connection with the forces $G_2$ and $G_3$ is that they are soon to be dot-multiplied with $\frac{\partial \mathbf{v}_{C_2}}{\partial \dot{q}_r}$ and $\frac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r}$ which have been expressed in terms of $\mathbf{i}_2, \mathbf{j}_2$, and $\mathbf{k}_2$.

One can express $(Q_r)_1$, the contribution to the generalized active force $Q_r$ of all the forces and torques acting on the particles of the link 1, as

$$(Q_r)_1 = \frac{\partial \mathbf{\omega}_{10}}{\partial \dot{q}_r} \cdot (T_{01} - T_{12}) + \frac{\partial \mathbf{v}_{C_1}}{\partial \dot{q}_r} \cdot G_1, \quad r = 1, 2, 3.$$

The contribution $(Q_r)_2$ to the generalized active force of all the forces and torques acting on the link 2 is

$$\begin{align*}
(Q_r)_2 &= \frac{\partial \mathbf{\omega}_{20}}{\partial \dot{q}_r} \cdot T_{12} + \frac{\partial \mathbf{v}_{C_2}}{\partial \dot{q}_r} \cdot G_2 + \frac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r} \cdot (F_{23}) \cdot (-F_{23}) \cdot \mathbf{k}_2, \\
&= \frac{\partial \mathbf{\omega}_{20}}{\partial \dot{q}_r} \cdot T_{12} + \frac{\partial \mathbf{v}_{C_2}}{\partial \dot{q}_r} \cdot G_2 + \frac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r} \cdot (-F_{23}), \quad r = 1, 2, 3.
\end{align*}$$
where
\[ v_{C32} = v_{C2} + \omega_{20} \times r_{C2C3} = v_{C2} + \omega_{20} \times q_3k_2. \]

The contribution \((Q_r)_3\), to the generalized active force of all the forces and torques acting on the link 3 is
\[ (Q_r)_3 = \frac{\partial \omega_{20}}{\partial \dot{q}_r} \cdot T_{23} + \frac{\partial v_{C3}}{\partial \dot{q}_r} \cdot G_3 + \frac{\partial v_{C3}}{\partial \dot{q}_r} \cdot F_{23}, \ r = 1, 2, 3. \]

The generalized active force \(Q_r\) of all the forces and torques acting on the links 1, 2, and 3 are
\[ Q_r = (Q_r)_1 + (Q_r)_2 + (Q_r)_3, \ r = 1, 2, 3, \]
or
\[ Q_1 = T_{01x}, \]
\[ Q_2 = T_{12y} - g m_2 L_2 c_2 - g m_3 c_2 (L_2 + q_3), \]
\[ Q_3 = F_{23z} - g m_3 s_2. \]

To arrive at the dynamical equations governing the robot arm, all that remains to be done is to substitute into Lagrange’s equations, namely,\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = Q_r, \ r = 1, 2, 3. \]

The Lagrange's equations are calculated in Program 2.5.

**Numerical simulation**

The robot arm is characterized by the following geometry: \(L_1 = 0.4\) m, \(L_2 = 0.7\) m, \(L_B = 0.8\) m, \(I_{3x} = 5\) kg\cdot m², \(I_{3y} = 4\) kg\cdot m², \(I_{3z} = 1\) kg\cdot m². The masses of the rigid bodies are \(m_1 = 90\) kg, \(m_2 = 60\) kg, \(m_3 = 40\) kg, and the gravitational acceleration is \(g = 9.81\) m/s².

The initial conditions, at \(t = 0\) s, are \(q_1(0) = \pi/18\) rad, \(q_2(0) = \pi/6\) rad, \(q_3(0) = 0.1\) m, and \(\dot{q}_1(0) = \dot{q}_2(0) = \dot{q}_3(0) = 0\).

The robot arm can be brought from an initial state of rest in reference frame \((0)\) to a final state of rest in \((0)\) in such a way that \(q_1, q_2,\) and \(q_3\) have specified
values $q_{1f}$, $q_{2f}$, and $q_{3f}$, respectively, by using the following feedback control laws

\[
T_{01x} = -\beta_{01}\dot{q}_1 - \gamma_{01}(q_1 - q_{1f}),
\]
\[
T_{12y} = -\beta_{12}\dot{q}_2 - \gamma_{12}(q_2 - q_{2f}) + g m_2 L_2 c_2 + g m_3 c_2 (L_2 + q_3),
\]
\[
F_{23z} = -\beta_{23}\dot{q}_3 - \gamma_{23}(q_3 - q_{3f}) + g m_3 s_2.
\]

The constant gains are: $\beta_{01} = 450 \text{ N} \cdot \text{m} \cdot \text{s/rad}$, $\gamma_{01} = 300 \text{ N} \cdot \text{m/rad}$, $\beta_{12} = 200 \text{ N} \cdot \text{m} \cdot \text{s/rad}$, $\gamma_{12} = 300 \text{ N} \cdot \text{m/rad}$, $\beta_{23} = 150 \text{ N} \cdot \text{s/m}$, $\gamma_{23} = 50 \text{ N/m}$. The values specified for the generalized coordinates are $q_{1f} = \pi/3 \text{ rad}$, $q_{2f} = \pi/3 \text{ rad}$, and $q_{3f} = 0.25 \text{ m}$.

The Mathematica\textsuperscript{TM} calculations that were used to compute and solve Lagrange’s equations are given in Program 2.5.
Figure 1
Figure 2
Figure 3
Figure 4
Figure 5
(* Lagrange's equations - example 1*)

Off[General::spell1];
Off[General::spell];

rA = {q1[t], 0, 0};
rB = {q1[t] + L*Sin[q2[t]],
    L*Cos[q2[t]], 0};

vA = D[rA, t];
vB = D[rB, t];

T1 = m vA.vA/2;
T2 = M vB.vB/2;

T = T1 + T2;

(* External forces*)
FA = {-k q1[t], m g, 0};
FB = {0, M g, 0};

(* Generalized forces*)
Q1 = FA.D[rA, q1[t]] + FB.D[rB, q1[t]]; 
Q2 = FA.D[rA, q2[t]] + FB.D[rB, q2[t]];

(* Lagrange's Equations*)
eq1 = D[D[T, q1'[t], t]] - D[T, q1[t]] - Q1;
eq2 = D[D[T, q2'[t], t]] - D[T, q2[t]] - Q2;

(* Small oscillations*)
rule = {Sin[q2[t]] -> q2[t],
    Cos[q2[t]] -> 1};

(* input data*)
rule1 = {m -> 1., M -> 1., L -> 1.,
    k -> 1., g -> 10.};
equation1 = eq1 /. rule /. rule1;
equation2 = eq2 /. rule /. rule1;
sol = NDSolve[
    {equation1 == 0, equation2 == 0,
    q1[0] == .1, q2[0] == .1,
    q1'[0] == 0., q2'[0] == 0.},
    {q1, q2}, {t, 0., 1.}];

Plot[Evaluate[q1[t]] /. sol,
    {t, 0., 1.}, PlotRange -> All,
    AxesLabel -> {"t[s]", "q1[m]"}];
Plot[Evaluate[q2[t]] /. sol,
    {t, 0., 1.}, PlotRange -> All,
    AxesLabel -> {"t[s]", "q2[0]"}]

0.12 0.14 0.16 0.18
q1 [m]

0.2 0.4 0.6 0.8 1

0.12 0.14 0.16 0.18
q1 [m]

0.2 0.4 0.6 0.8 1

0.12 0.14 0.16 0.18
q1 [m]
- Graphics -
(*Lagrange's equations - example 2 *)
Apply [Clear, Names["Global\"*\"]];

Off[General::spell1];
Off[General::spell];

(*position analysis*)
m1=m2=m;
L1=L2=L;
rA={L1*Sin[q1[t]], L1*Cos[q1[t]], 0};
rC1={L1*Sin[q1[t]]/2., L1*Cos[q1[t]]/2., 0};
rB=rA+{L2*Sin[q2[t]], L2*Cos[q2[t]], 0};
rC2=rA+{L2*Sin[q2[t]]/2., L2*Cos[q2[t]]/2., 0};

(*velocity analysis*)
vA=D[rA,t];
vC1=D[rC1,t];
vB=D[rB,t];
vC2=D[rC2,t];
w1={0, 0, -q1'[t]};
w2={0, 0, -q2'[t]};
JC1=m1*L1^2/12.;
JC2=m2*L2^2/12.;
T1=m1*vC1.vC1/2.+JC1*w1.w1/2.;
T2=m2*vC2.vC2/2.+JC2*w2.w2/2.;

(*T1 can be calculated as T1=J0 w1.w1/2. where J0=m1*L1^2/3 *)
Simplify[T1];
Simplify[T2];
T=T1+T2;
Simplify[T];

(*external forces*)
FC1={0, m1*g, 0};
FC2={0, m2*g, 0};

(*generalized forces*)
Q1=FC1.D[rC1,q1[t]]+FC2.D[rC2,q1[t]];
Q2=FC1.D[rC1,q2[t]]+FC2.D[rC2,q2[t]];

(*Lagrange's equations*)
eq1=D[D[T,q1'[t],t]]-D[T,q1[t]]-Q1;
eq2=D[D[T,q2'[t],t]]-D[T,q2[t]]-Q2;
(*input data*)

\[\text{inp} = \{m1 -> 1., L1 -> 1., m2 -> 1., L2 -> 1., g -> 10.\};\]

\[\text{equation1} = \text{eq1} /. \text{inp};\]

\[\text{equation2} = \text{eq2} /. \text{inp};\]

\[\text{sol} = \text{NDSolve}\{\text{equation1} == 0, \text{equation2} == 0, \]

\[q1[0] == 1, q2[0] == 1, q1'[0] == 0, q2'[0] == 0, \}

\[\{q1, q2\}, \{t, 0, 40.\}, \text{MaxSteps} -> 2000\};\]

\[\text{Plot}\left[\text{Evaluate}\left[q1[t]\right]/.\text{sol}, \{t, 0, 40.\}\right]\]

\[\text{Plot}\left[\text{Evaluate}\left[q2[t]\right]/.\text{sol}, \{t, 0, 40.\}\right]\]
(*Lagrange's equations of motion - example 3*)
Apply[Clear, Names["Global`*"]];
Off[General::spell];

(*Input data*)
data = {L1 -> .100, L2 -> .470, L3 -> .047, AC -> .280, h -> .01,
h3 -> .025, d -> .005, d3 -> .008, ro -> 7850, g -> 9.807, M0 -> 1., w0 -> 4.};

m1 = ro*L1*h*d;
m2 = ro*L2*h*d;
m3a = ro*L3*h3*d3;
m3b = ro*L3*h*d;
m3 = m3a - m3b;
IC1 = m1*D[t, (L1^2 + h^2)];
IC2 = m2*D[t, (L2^2 + h^2)];
IC3 = m3a*D[t, (L3^2 + h3^2) - m3b*D[t, (L3^2 + h^2)]];

(*Position, velocity and acceleration vectors*)
xB = L1*Cos[theta[t]]; 
yB = L1*Sin[theta[t]]; 
rB = {xB, yB, 0}; 
rC1 = rB/2.; 
vC1 = D[rC1, t]; 
xC = AC; 
yC = 0; 
rC = {xC, yC, 0}; 
theta2 = ArcTan[(yB - yC)/(xB - xC)]; 
rC2 = {xB + L2*Cos[theta2]/2., yB + L2*Sin[theta2]/2., 0}; 
vC2 = D[rC2, t]; 
rC3 = rC; 
vC3 = {0, 0, 0};

(*Angular velocities*)
omega = {0, 0, theta'[t]}; 
omega2 = Simplify[{0, 0, D[theta2, t]}];
omega3 = omega2;

(*Kinetic energy*)
T1 = m1*vC1.vC1/2. + IC1*omega.omega/2.;
T2 = m2*vC2.vC2/2. + IC2*omega2.omega2/2.;
T3 = m3*vC3.vC3/2. + IC3*omega3.omega3/2.;
T = Expand[T1 + T2 + T3];

(*Left hand side of the Lagrange equation*)
LHS = D[D[T, theta'[t]], t] - D[T, theta[t]]; 

(*Right hand side of the Lagrange equation (generalized forces]*)
G1 = {0, -m1*g, 0}; 
G2 = {0, -m2*g, 0}; 
G3 = {0, -m3*g, 0};
\[ Mm = \{ 0, 0, M0 (1 - \theta'[t]/w0) \}; \]
\[ \text{RHS} = \]
\[ D[rCl, \theta[t]].G1 + D[rC2, \theta[t]].G2 + D[rC3, \theta[t]].G3 + D[\omega, \theta'[t]].Mm; \]

(*Solution of the Lagrange equation*)
\[ \text{eqnLHS} = \text{LHS} /. \text{data} /. \{ \theta'[t] \to w[t], \theta''[t] \to w'[t] \}; \]
\[ \text{eqnRHS} = \text{RHS} /. \text{data} /. \{ \theta'[t] \to w[t], \theta''[t] \to w'[t] \}; \]
\[ \text{solution} = \text{NDSolve}[\{ \text{eqnLHS} = \text{eqnRHS}, \theta'[t] = w[t], \theta[0] = \frac{\Pi}{6}, w[0] = 0 \}, \]
\[ \{ \theta[t], w[t] \}, \{ t, 0, 5 \}]; \]

\[ \text{Plot}[	ext{Evaluate}[\theta[t] /. \text{solution}], \{ t, 0, 5 \}, \text{AxesLabel} \to \{ "t[s]", "\theta[rad]" \}]; \]
\[ \text{Plot}[	ext{Evaluate}[w[t] /. \text{solution}]*30/\Pi, \{ t, 0, 5 \}, \]
\[ \text{AxesLabel} \to \{ "t[s]", "n[rpm]" \}, \text{PlotRange} \to \{ \text{All}, \{ 20, 50 \} \}]; \]
\[ \text{Plot}[	ext{Evaluate}[w[t] /. \text{solution}]*30/\Pi, \{ t, 0, .05 \}, \]
\[ \text{AxesLabel} \to \{ "t[s]", "n[rpm]" \}, \text{PlotRange} \to \{ \text{All}, \{ 0, 50 \} \}]; \]
"Lagrange's equations of motion - example 4"
Apply [Clear, Names["Global`*"] ];
Off[General::spell]
Off[General::spell1]

"kinematics"
"transformation matrix from RF1 to RF0"
R10 = {{1,0,0},
      {0,Cos[q1[t]],Sin[q1[t]]},
      {0,-Sin[q1[t]],Cos[q1[t]]}};
MatrixForm[R10]
"transformation matrix from RF2 to RF1"
R21 = {{Cos[q2[t]],0,-Sin[q2[t]]},
      {0,1,0},
      {Sin[q2[t]],0,Cos[q2[t]]}};
MatrixForm[R21]
"angular velocity of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}"
w01 = {D[q1[t],t],0,0}

"angular velocity of link 2 in RF0 expressed in terms of RF1 {i1,j1,k1}"
w201 = {D[q1[t],t],D[q2[t],t],0}

"angular velocity of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}"
w202 = w201.Transpose[R21]

"angular velocity of link 2 in RF0 expressed in terms of RF0 {i0,j0,k0}"
w200 = w201.R10

"position vector of mass center C1 of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}"
r1C={L1,0,0}

"linear velocity of mass center C1 of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}"
v1C = D[r1C,t]+Cross[w10,r1C]

"position vector of mass center C2 of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}"
r2C={L2,0,0}.Transpose[R21]

"position vector of mass center C2 of link 2 in RF0 expressed in terms of RF0 {i0,j0,k0}"
r2C0={L2,0,0}.R10

"linear velocity of mass center C2 of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}"
v2C = D[r2C,t]+Cross[w202,r2C]

"position vector of mass center C3 of link 3 in RF0 expressed in terms of RF2 {i2,j2,k2}"
r3C=r2C+{0,0,q3[t]}

"linear velocity of mass center C3 of link 3 in RF0 expressed in terms of RF2 {i2,j2,k2}"
v3C = Expand[D[r3C,t]+Cross[w202,r3C]]; Simplify[v3C]

"position vector of mass center C3 of link 3 in RF0 expressed in terms of RF0 {i0,j0,k0}"
r3C0=({r2C+{0,0,q3[t]}}.R21).R10
\[ v_{C30} = D[r_{C30}, t] \]
\[ \text{Expand}[v_{C3}.v_{C3}, \text{Trig} \to \text{True}] = \text{Simplify}[v_{C30}.v_{C3}] \]

"position vector of mass center of rigid body RB in RF0 expressed in terms of RF2 \{i_2, j_2, k_2\}"
\[ r_{CR} = r_{C3} + \{0, 0, L/2\} \]

"linear velocity of mass center of rigid body RB in RF0 expressed in terms of RF2 \{i_2, j_2, k_2\}"
\[ v_{CR} = \text{Expand}[D[r_{CR}, t] + \text{Cross}[\text{w}202, r_{CR}]] \]
\[ \text{Simplify}[v_{CR}] \]
\[ \text{Simplify}[v_{C3}] \]

"position vector of mass center CR of rigid body RB in RF0 expressed in terms of RF0 \{i_0, j_0, k_0\}"
\[ r_{CR0} = (r_{CR}.R21).R10 \]
\[ v_{CR0} = D[r_{CR0}, t] \]
\[ \text{Expand}[v_{CR}.v_{C3}, \text{Trig} \to \text{True}] = \text{Simplify}[v_{CR0}.v_{C3}] \]

"inertia matrix associated to link 1 expressed in terms of RF1 \{i_1, j_1, k_1\}"
\[ I_1 = \{\{I_{1x}, 0, 0\}, \{0, I_{1y}, 0\}, \{0, 0, I_{1z}\}\} \]
\[ \text{MatrixForm}[I_1] \]

"inertia matrix associated to bar 2 expressed in terms of RF2 \{i_2, j_2, k_2\}"
\[ I_2 = \{\{I_{2x}, 0, 0\}, \{0, I_{2y}, 0\}, \{0, 0, I_{2z}\}\} \]
\[ \text{MatrixForm}[I_2] \]

"inertia matrix associated to slider 2' expressed in terms of RF2 \{i_2, j_2, k_2\}"
\[ I_2S = \{\{I_{2Sx}, 0, 0\}, \{0, I_{2Sy}, 0\}, \{0, 0, I_{2Sz}\}\} \]
\[ \text{MatrixForm}[I_2S] \]

"inertia matrix associated to bar 3 expressed in terms of RF2 \{i_2, j_2, k_2\}"
\[ I_3 = \{\{I_{3x}, 0, 0\}, \{0, I_{3y}, 0\}, \{0, 0, I_{3z}\}\} \]
\[ \text{MatrixForm}[I_3] \]

"kinetic energy"

"kinetic energy of link 1"
\[ T_1 = m v_{C1}.v_{C1}/2 + w_{10}.I_{11}.w_{10}/2 \]

"kinetic energy of link 2"
\[ T_2 = m v_{C2}.v_{C2}/2 + w_{202}.I_{22}.w_{202}/2; \text{Simplify}[T_2] \]

"kinetic energy of slider 2'"
\[ T_{2S} = m v_{C2}.v_{C2}/2 + w_{202}.I_{2S}.w_{202}/2 \]

"kinetic energy of link 3"
\[ T_3 = m v_{C3}.v_{C3}/2 + w_{202}.I_{33}.w_{202}/2; \text{Simplify}[T_3] \]

"kinetic energy of RB"
\[ TR = \text{Simplify}[mR v_{CR}.v_{CR}/2, \text{Trig} \to \text{True}] \]
"total kinetic energy"
Simplify[T1+T2+T2S+T3+TR] 
T=Expand[T1+T2+T2S+T3+TR];

"gravitational force that acts on link 1 at C1 
inRF0 expressed in terms of RF0 {i0,j0,k0} or in 
in terms of RF1 {i1,j1,k1}"
G1={ -m1 g, 0, 0 } 

"gravitational force that acts on bar 2 and 
slider 2' at C2 in RF0 expressed in terms of 
RF0 {i0,j0,k0} or RF1 {i1,j1,k1}"
G2={ -(m2+m2S) g, 0, 0 }

(*gravitational force that acts on bar 2 and 
slider 2' at C2 in RF0 expressed in terms of 
RF2 {i2,j2,k2}*)
G22={ -(m2+m2S) g, 0, 0 }.Transpose[R21];

"gravitational force that acts on link 3 at C3 
inRF0 expressed in terms of RF0 {i0,j0,k0}"
G3={ -m3 g, 0, 0 } 

(*gravitational force that acts on link 3 at C3 
inRF0 expressed in terms of RF2 {i2,j2,k2}*)
G32={ -m3 g, 0, 0 }.Transpose[R21];

"gravitational force that acts on link 3 at CR 
inRF0 expressed in terms of RF0 {i0,j0,k0}"
GR={ -mR g, 0, 0 }

(*gravitational force that acts on link 3 at C3 
inRF0 expressed in terms of RF2 {i2,j2,k2}*)
GR2={ -mR g, 0, 0 }.Transpose[R21];

"generalized active force Qj=∑Fj.∂(rj)/∂(qi) : Q1, Q2, Q3"
Q1=D[rC1,q1[t]].G1+D[rC20,q1[t]].G2+ 
D[rC30,q1[t]].G3+D[rCR0,q1[t]].GR;
Q2=D[rC1,q2[t]].G1+D[rC20,q2[t]].G2+ 
D[rC30,q2[t]].G3+D[rCR0,q2[t]].GR;
Q3=D[rC1,q3[t]].G1+D[rC20,q3[t]].G2+ 
D[rC30,q3[t]].G3+D[rCR0,q3[t]].GR;
Simplify[Q1] 
Simplify[Q2] 
Simplify[Q3] 

(*generalized active force*)

"generalized active force Qj=∑Fj.∂(vj)/∂(qi') : Q1, Q2, Q3"
F1=D[vC1,q1'[t]].G1+ 
D[vC2,q1'[t]].G22+ 
D[vC3,q1'[t]].G32+ 
D[vCR,q1'[t]].GR2;
F2=D[vC1,q2'[t]].G1+ 
D[vC2,q2'[t]].G22+ 
D[vC3,q2'[t]].G32+ 
D[vCR,q2'[t]].GR2;
D[vC3,q2'[t]].G32+
D[vCR,q2'[t]].GR2;

F3=D[vC1,q3'[t]].G1 +
D[vC2,q3'[t]].G22+
D[vC3,q3'[t]].G32+
D[vCR,q3'[t]].GR2;

Expand[Q1]==Expand[F1];
Expand[Q2]==Expand[F2];
Expand[Q3]==Expand[F3];

Simplify[F1]
Simplify[F2]
Simplify[F3]

" Lagrange's eom "

Leq1=D[D[T,q1'[t]],t]-D[T,q1[t]]-Q1==0;
Leq2=D[D[T,q2'[t]],t]-D[T,q2[t]]-Q2==0;
Leq3=D[D[T,q3'[t]],t]-D[T,q3[t]]-Q2==0;

Simplify[Leq1]
Simplify[Leq2]
Simplify[Leq3]

Lagrange's equations of motion – example 4

kinematics

transformation matrix from RF1 to RF0

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & \text{Cos}[q1[t]] & \text{Sin}[q1[t]] \\
0 & -\text{Sin}[q1[t]] & \text{Cos}[q1[t]]
\end{bmatrix}
\]

transformation matrix from RF2 to RF1

\[
\begin{bmatrix}
\text{Cos}[q2[t]] & 0 & -\text{Sin}[q2[t]] \\
0 & 1 & 0 \\
\text{Sin}[q2[t]] & 0 & \text{Cos}[q2[t]]
\end{bmatrix}
\]

angular velocity of link 1 in RF0 expressed in terms of RF1 \{i1,j1,k1\}

\{q1'[t], 0, 0\}

angular velocity of link 2 in RF0 expressed in terms of RF1 \{i1,j1,k1\}

\{q1'[t], q2'[t], 0\}

angular velocity of link 2 in RF0 expressed in terms of RF2 \{i2,j2,k2\}

\{\text{Cos}[q2[t]] q1'[t], q2'[t], \text{Sin}[q2[t]] q1'[t]\}

angular velocity of link 2 in RF0 expressed in terms of RF0 \{i0,j0,k0\}

\{q1'[t], \text{Cos}[q1[t]] q2'[t], \text{Sin}[q1[t]] q2'[t]\}
position vector of mass center C1
  of link 1 in RF0 expressed in terms of RF1 \{i1,j1,k1\}

\{L1, 0, 0\}

linear velocity of mass center C1
  of link 1 in RF0 expressed in terms of RF1 \{i1,j1,k1\}

\{0, 0, 0\}

position vector of mass center C2
  of link 2 in RF0 expressed in terms of RF2 \{i2,j2,k2\}

\{L2 \cos[q2[t]], 0, L2 \sin[q2[t]]\}

position vector of mass center C2
  of link 2 in RF0 expressed in terms of RF0 \{i0,j0,k0\}

\{L2, 0, 0\}

linear velocity of mass center C2 of link 2 in RF0 expressed in terms of RF2 \{i2,j2,k2\}

\{0, 0, 0\}

position vector of mass center C3 of link 3 in RF0 expressed in terms of RF2 \{i2,j2,k2\}

\{L2 \cos[q2[t]], 0, q3[t] + L2 \sin[q2[t]]\}

linear velocity of mass center C3 of link 3 in RF0 expressed in terms of RF2 \{i2,j2,k2\}

\{q3[t], q2'[t], -\cos[q2[t]] q3[t] q1'[t], q3'[t]\}

position vector of mass center C3 of link 3 in RF0 expressed in terms of RF0 \{i0,j0,k0\}

\{L2 \cos[q2[t]]^2 + \sin[q2[t]] (q3[t] + L2 \sin[q2[t]]),
-\sin[q1[t]] (-L2 \cos[q2[t]] \sin[q2[t]] + \cos[q2[t]] (q3[t] + L2 \sin[q2[t]])),
\cos[q1[t]] (-L2 \cos[q2[t]] \sin[q2[t]] + \cos[q2[t]] (q3[t] + L2 \sin[q2[t]]))\}

position vector of mass center of
  rigid body RB in RF0 expressed in terms of RF2 \{i2,j2,k2\}

\{L2 \cos[q2[t]], 0, \frac{L}{2} + q3[t] + L2 \sin[q2[t]]\}

linear velocity of mass center of
  rigid body RB in RF0 expressed in terms of RF2 \{i2,j2,k2\}

\{\frac{1}{2} (L + 2 q3[t]) q2'[t], -\frac{1}{2} \cos[q2[t]] (L + 2 q3[t]) q1'[t], q3'[t]\}

position vector of mass center CR of
  rigid body RB in RF0 expressed in terms of RF0 \{i0,j0,k0\}

\{L2 \cos[q2[t]]^2 + \sin[q2[t]] (\frac{L}{2} + q3[t] + L2 \sin[q2[t]]),
-\sin[q1[t]] (-L2 \cos[q2[t]] \sin[q2[t]] + \cos[q2[t]] (\frac{L}{2} + q3[t] + L2 \sin[q2[t]])),
\cos[q1[t]] (-L2 \cos[q2[t]] \sin[q2[t]] + \cos[q2[t]] (\frac{L}{2} + q3[t] + L2 \sin[q2[t]]))\}
inertia matrix associated to link 1 expressed in terms of RF1 \( \{i_1,j_1,k_1\} \)

\[
\begin{pmatrix}
I_{1x} & 0 & 0 \\
0 & I_{1y} & 0 \\
0 & 0 & I_{1z}
\end{pmatrix}
\]

inertia matrix associated to bar 2 expressed in terms of RF2 \( \{i_2,j_2,k_2\} \)

\[
\begin{pmatrix}
\frac{l^2 m_2}{12} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{l^2 m_2}{12}
\end{pmatrix}
\]

inertia matrix associated to slider 2' expressed in terms of RF2 \( \{i_2,j_2,k_2\} \)

\[
\begin{pmatrix}
I_{2Sx} & 0 & 0 \\
0 & I_{2Sy} & 0 \\
0 & 0 & I_{2Sz}
\end{pmatrix}
\]

inertia matrix associated to bar 3 expressed in terms of RF2 \( \{i_2,j_2,k_2\} \)

\[
\begin{pmatrix}
\frac{l^2 m_1}{12} & 0 & 0 \\
0 & \frac{l^2 m_2}{12} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

kinetic energy

kinetic energy of link 1

\[
\frac{1}{2} I_{1x} q_1'[t]^2
\]

kinetic energy of link 2

\[
\frac{1}{24} l^2 m_2 q_1'[t]^2
\]

kinetic energy of slider 2'

\[
\frac{1}{2} \left[ I_{2Sx} \cos[q_2[t]]^2 q_1'[t]^2 + I_{2Sx} \sin[q_2[t]]^2 q_1'[t]^2 + I_{2Sy} q_2'[t]^2 \right]
\]

kinetic energy of link 3

\[
\frac{1}{24} m_3 \left( \cos[q_2[t]]^2 (L^2 + 12 q_3[t]^2) q_1'[t]^2 + (L^2 + 12 q_3[t]^2) q_2'[t]^2 + 12 q_3'[t]^2 \right)
\]

kinetic energy of RB

\[
\frac{1}{6} m_R \left( \cos[q_2[t]]^2 (L + 2 q_3[t])^2 q_1'[t]^2 + (L + 2 q_3[t])^2 q_2'[t]^2 + 4 q_3'[t]^2 \right)
\]

total kinetic energy

\[
\frac{1}{24} \left( (12 I_{1x} + 12 I_{2Sx} + l^2 m_2 + L^2 (m_3 + 3 m_R)) \cos[q_2[t]]^2 + 12 L m_R \cos[q_2[t]]^2 q_3[t] + 12 (m_3 + m_R) \cos[q_2[t]]^2 q_3[t]^2 + 12 I_{2Sx} \sin[q_2[t]]^2 + l^2 m_2 \sin[q_2[t]]^2 \right) q_1'[t]^2 + \\
(12 I_{2Sy} + L^2 (m_3 + 3 m_R) + 12 L m_R q_3[t] + 12 (m_3 + m_R) q_3[t]^2) q_2'[t]^2 + 12 (m_3 + m_R) q_3'[t]^2
\]
gravitational force that acts on link 1 at C1 in RF0
expressed in terms of RF0 \{i0,j0,k0\} or in terms of RF1 \{i1,j1,k1\}
\{-g m1, 0, 0\}

gravitational force that acts on bar 2 and slider 2’ at
C2 in RF0 expressed in terms of RF0 \{i0,j0,k0\} or RF1 \{i1,j1,k1\}
\{-g (m2 + m2S), 0, 0\}

gravitational force that acts on
link 3 at C3 in RF0 expressed in terms of RF0 \{i0,j0,k0\}
\{-g m3, 0, 0\}

gravitational force that acts on
link 3 at CR in RF0 expressed in terms of RF0 \{i0,j0,k0\}
\{-g mR, 0, 0\}

generalized active force \( Q_i = \sum F_j \partial (r_j) / \partial (q_i) \): Q1, Q2, Q3
0
\[-\frac{1}{2} g \cos(q2[t]) (LmR + 2 (m3 + mR) q3[t]) \]
\[-g (m3 + mR) \sin(q2[t]) \]

generalized active force \( Q_i = \sum F_j \partial (v_j) / \partial (q_i') \): Q1, Q2, Q3
0
\[-\frac{1}{2} g \cos(q2[t]) (LmR + 2 (m3 + mR) q3[t]) \]
\[-g (m3 + mR) \sin(q2[t]) \]

LaGrange’s eom

\[ (12 I1x + (12 I2Sx + L^2 (m3 + 3 mR)) \cos(q2[t])^2 + 12 LmR \cos(q2[t]) q3[t] + 12 (m3 + mR) \cos(q2[t]) q3[t]^2 + 12 I2Sx \sin(q2[t])^2 + L^2 m2 \sin(q2[t])^2 \] \[ q1''[t] = 2 \cos(q2[t]) q1'[t] ((12 I2Sx - 12 I2Sx + L^2 m3 + 3 L^2 mR + 12 LmR q3[t] + 12 (m3 + mR) q3[t]^2) \]
\[ \sin(q2[t]) q2'[t] - 6 \cos(q2[t]) (LmR + 2 (m3 + mR) q3[t]) q3'[t] q3''[t] ] \]

\[ (12 I2Sx - 12 I2Sx + L^2 (m3 + 3 mR)) \sin(q2[t]) q1'[t]^2 \]
\[ 24 (m3 + mR) q3[t] q2'[t] (\cos(q2[t]) \sin(q2[t]) q1'[t]^2 + q2'[t]) + \]
\[ 24 q3[t] (g m3 \cos(q2[t]) + g mR \cos(q2[t]) + LmR \cos(q2[t]) \sin(q2[t]) q1'[t]^2 + 2 (m3 + mR) q2'[t] q3'[t] + LmR q2''[t]) + \]
\[ 2 (6 g LmR \cos(q2[t]) + 12 LmR q2'[t] q3'[t] + (12 I2Sy + L^2 (m3 + 3 mR)) q2''[t]) = 0 \]

\[ g LmR \cos(q2[t]) + 2 (m3 + mR) q3''[t] = LmR (\cos(q2[t])^2 q1'[t]^2 + q2'[t]) + \]
\[ 2 (m3 + mR) q3[t] (-g \cos(q2[t]) + \cos(q2[t]) q1'[t]^2 + q2'[t])^2 \]
(* Example 5*)
"LAGRANGE's equations of motion - 3 DOF Robot"

Apply [Clear,Names["Global`*"]];
Off[General::spell];
Off[General::spell1];

"transformation matrix from RF1 to RF0: R10=
R10 = {{1,0,0},
       {0,Cos[q1[t]],Sin[q1[t]]},
       {0,-Sin[q1[t]],Cos[q1[t]]}};
MatrixForm[R10]

"transformation matrix from RF2 to RF1: R21=
R21=({Cos[q2[t]],0,-Sin[q2[t]]},
     {0,1,0},
     {Sin[q2[t]],0,Cos[q2[t]]});
MatrixForm[R21]

"angular velocity of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}: w10="
w10 = {D[q1[t],t],0,0}

"angular velocity of link 2 in RF0 expressed in terms of RF1 {i1,j1,k1}: w201="
w201 = {D[q1[t],t],D[q2[t],t],0}

"angular velocity of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}: w20="
w20=w201.Transpose[R21]

"angular acceleration of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}: a10="
a10=D[w10,t]

"angular acceleration of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}: a20="
a20=D[w20,t]

"position vector of mass center C1 of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}: rC1="
rC1={0,0,L1}

"linear velocity of mass center C1 of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}: vC1="
vC1=D[rC1,t]+Cross[w10,rC1]

"linear velocity of joint B in RF0 expressed in terms of RF1 {i1,j1,k1}: vB="
vB =D[{0,0, 2 L1},t]+Cross[w10,{0,0,2 L1}]

"position vector of mass center C2 of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}: rC2="
rC2={0,0,2 L1}.Transpose[R21]+{0,0,L2}

"linear velocity of mass center C2 of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}: vC2="
vC2 =D[rC2,t]+Cross[w20,rC2]

"position vector of mass center C3 of link 3 in RF0 expressed in terms of RF2 {i2,j2,k2}: rC3="
rC3=rC2+{0,0,q3[t]}

"linear velocity of C32 of link 2 expressed in terms of RF2 {i2,j2,k2}
C32 of link 2 is superposed with C3 of link 3: vC32="
vC32 = vC2 + Cross[w20, {0, 0, q3[t]}]

"linear velocity of mass center C3 of link 3 in RF0 expressed in terms of RF2 {i2, j2, k2}: vC3 = 

vC3 = D[vC3, t] + Cross[w20, vC3]

(* another way of computing vC3 is: *)

vC3 = vC3 + D[({0, 0, q3[t]})], t];

("*vC3 - vC3' = (0, 0, 0); ** 

"linear acceleration of mass center C1 of link 1 in RF0 expressed in terms of RF1 {i1, j1, k1}: aC1 = 

aC1 = D[vC1, t] + Cross[w10, vC1]

"linear acceleration of mass center C2 of link 2 in RF0 expressed in terms of RF2 {i2, j2, k2}: aC2 = 

aC2 = D[vC2, t] + Cross[w20, vC2]

"linear acceleration of mass center C3 of link 3 in RF0 expressed in terms of RF2 {i2, j2, k2}: aC3 = 

aC3 = D[vC3, t] + Cross[w20, vC3]

"gravitational force that acts on link 1 at C1 in RF0 expressed in terms of RF1 {i1, j1, k1}: G1 = 

G1 = {-m1 g, 0, 0}

"gravitational force that acts on link 2 at C2 in RF0 expressed in terms of RF2 {i2, j2, k2}: G2 = 

G2 = {-m2 g, 0, 0}. Transpose[R21]

"gravitational force that acts on link 3 at C3 in RF0 expressed in terms of RF2 {i2, j2, k2}: G3 = 

G3 = {-m3 g, 0, 0}. Transpose[R21]

"contact torque of 0 that acts on link 1 in RF0 expressed in terms of RF1 {i1, j1, k1}: T01 = 

T01 = {T01x, T01y, T01z}

"contact torque of link 1 that acts on link 2 in RF0 expressed in \ terms of RF2 {i2, j2, k2}: T12 = 

T12 = {T12x, T12y, T12z}

"contact force of link 2 that acts on link 3 in RF0 expressed \ in terms of RF2 {i2, j2, k2}: F23 = 

F23 = {F23x, F23y, F23z}

"generalized active force Q1 = 

Q1 = D[w10, q1'[t]]. T01 +

D[vC1, q1'[t]]. G1 +

D[w10, q1'[t]]. Transpose[R21]. (-T12) +

D[w20, q1'[t]]. T12 +

D[vC2, q1'[t]]. G2 +

D[vC32, q1'[t]]. (-F23) +

D[vC3, q1'[t]]. G3 +

D[vC3, q1'[t]]. F23

"generalized active force Q2 = 

Q2 = D[w10, q2'[t]]. T01 +

D[vC1, q2'[t]]. G1 +

D[w10, q2'[t]]. Transpose[R21]. (-T12) +

D[w20, q2'[t]]. T12 +

D[vC2, q2'[t]]. G2 +

D[vC32, q2'[t]]. (-F23) +

D[vC3, q2'[t]]. G3 +

D[vC3, q2'[t]]. F23

"generalized active force Q3 = 

Q3 = D[w20, q3'[t]]. T01 +

D[vC1, q3'[t]]. G1 +

D[w10, q3'[t]]. Transpose[R21]. (-T12) +

D[w20, q3'[t]]. T12 +

D[vC2, q3'[t]]. G2 +

D[vC32, q3'[t]]. (-F23) +

D[vC3, q3'[t]]. G3 +

D[vC3, q3'[t]]. F23
Q3=D[w10,q3'[t]].T01+
   D[vc1,q3'[t]].G1+
   D[w10,q3'[t]].Transpose[R21].(-T12)+
   D[w20,q3'[t]].T12+
   D[vc2,q3'[t]].G2+
   D[vc32,q3'[t]].(-F23)+
   D[vc3,q3'[t]].G3+
   D[vc3,q3'[t]].F23

(* inertia dyadics *)

"central inertia dyadic for link 1 expressed in terms of RF1 {i1,j1,k1}: I1="
I1={{m1(2 L1)^2/12,0,0},{0,m1(2 L1)^2/12,0},{0,0,0}}

"central inertia dyadic for link 2 expressed in terms of RF2 {i2,j2,k2}: I2="
I2={{m2(2 L2)^2/12,0,0},{0,m2(2 L2)^2/12,0},{0,0,0}}

"central inertia torque for link 3 expressed in terms of RF2 {i2,j2,k2}: I3="
I3={{{I3x,0,0},{0,I3y,0},{0,0,I3z}}}
q1ref→N[Pi/3], q2ref→N[Pi/3], q3ref→0.25

"control"
control={(T01x→b01 q1'[t]→g01(q1[t]−q1ref),
T12y→b12 q2'[t]→g12(q2[t]−q2ref)+g (m2 L2+m3 (L2+q3[t])) Cos[q2[t]],
F23z→b23 q3'[t]→g23(q3[t]−q3ref)+g m3 Sin[q2[t]] )/.indata

(*Lagrange's equations of motion*)
lageq={
(Lagr1/.indata)=0,(Lagr2/.indata)=0,(Lagr3/.indata)=0,
q1'[0]=0,q2'[0]=0,q3'[0]=0,
q1[0]=N[Pi/18],q2[0]=N[Pi/6],q3[0]=0.1);

(*numerical simulation of Lagrange's eom*)
lagrange=NDSolve[lageq/.control,{q1,q2,q3},{t,0,15}]
Plot[Evaluate[q1[t]/.lagrange],{t,0,15},PlotRange→{All,All},
AxesLabel→"t[\text{s}]",q1[\text{rad}]"
]
Plot[Evaluate[q2[t]/.lagrange],{t,0,15},PlotRange→{All,All},
AxesLabel→"t[\text{s}]",q2[\text{rad}]"
]
Plot[Evaluate[q3[t]/.lagrange],{t,0,15},PlotRange→{All,All},
AxesLabel→"t[\text{s}]",q3[\text{m}]"
]

LAGRANGE's equations of motion - 3 DOF Robot

transformation matrix from RF1 to RF0: R10=
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos[q1[t]] & \sin[q1[t]] \\
0 & -\sin[q1[t]] & \cos[q1[t]]
\end{pmatrix}
\]

transformation matrix from RF2 to RF1: R21=
\[
\begin{pmatrix}
\cos[q2[t]] & 0 & -\sin[q2[t]] \\
0 & 1 & 0 \\
\sin[q2[t]] & 0 & \cos[q2[t]]
\end{pmatrix}
\]

angular velocity of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}: w10=
\{q1'[t], 0, 0\}

angular velocity of link 2 in RF0 expressed in terms of RF1 {i1,j1,k1}: w20=
\{q1'[t], q2'[t], 0\}

angular acceleration of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}: a10=
\{q1''[t], 0, 0\}

angular acceleration of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}: a20=
\{(\cos[q2[t]] q1'[t], q2'[t], \sin[q2[t]] q1'[t])

Example 5*)

aC3=D[vC3,t]+Cross[w20,vC3]

...
\[-\sin(q_2(t)) q_1'(t) q_2''(t) + \cos(q_2(t)) q_1''(t),
q_2''(t), \cos(q_2(t)) q_1'(t) q_2'(t) + \sin(q_2(t)) q_1'(t)\]

position vector of mass center C1 of
link 1 in RF0 expressed in terms of RF1 \(\{i_1,j_1,k_1\}\): \(r_{C1} = (0, 0, L_1)\)

linear velocity of mass center C1 of
link 1 in RF0 expressed in terms of RF1 \(\{i_1,j_1,k_1\}\): \(v_{C1} = (0, -L_1 q_1'(t), 0)\)

linear velocity of joint B in RF0 expressed in terms of RF1 \(\{i_1,j_1,k_1\}\): \(v_B = (0, -2 L_1 q_1'(t), 0)\)

position vector of mass center C2 of
link 2 in RF0 expressed in terms of RF2 \(\{i_2,j_2,k_2\}\): \(r_{C2} = (-2 L_1 \sin(q_2(t)), 0, L_2 + 2 L_1 \cos(q_2(t)))\)

linear velocity of mass center C2 of
link 2 in RF0 expressed in terms of RF2 \(\{i_2,j_2,k_2\}\): \(v_{C2} = (L_2 q_2'(t), -L_2 \cos(q_2(t)) q_1'(t) - 2 L_1 \cos(q_2(t))^2 q_1'(t) - 2 L_1 \sin(q_2(t))^2 q_1'(t), 0)\)

position vector of mass center C3 of
link 3 in RF0 expressed in terms of RF2 \(\{i_2,j_2,k_2\}\): \(r_{C3} = (-2 L_1 \sin(q_2(t)), 0, L_2 + 2 L_1 \cos(q_2(t)) + q_3(t))\)

linear velocity of C32 of link 2 expressed in terms of RF2 \(\{i_2,j_2,k_2\}\): \(v_{C32} = (L_2 q_2'(t) + q_3(t) q_2'(t), -L_2 \cos(q_2(t)) q_1'(t) - 2 L_1 \cos(q_2(t))^2 q_1'(t) - \cos(q_2(t)) q_3[t] q_1'(t) - 2 L_1 \sin(q_2(t))^2 q_1'(t), 0)\)

linear velocity of mass center C3 of
link 3 in RF0 expressed in terms of RF2 \(\{i_2,j_2,k_2\}\): \(v_{C3} = (L_2 q_2'(t) + q_3[t] q_2'(t), -L_2 \cos(q_2(t)) q_1'(t) - 2 L_1 \cos(q_2(t))^2 q_1'(t) - \cos(q_2(t)) q_3[t] q_1'(t) - 2 L_1 \sin(q_2(t))^2 q_1'(t), q_3'(t))\)

linear acceleration of mass center C1 of
link 1 in RF0 expressed in terms of RF1 \(\{i_1,j_1,k_1\}\): \(a_{C1} = (0, -L_1 q_1''(t), -L_1 q_1'(t)^2)\)

linear acceleration of mass center C2 of
link 2 in RF0 expressed in terms of RF2 \(\{i_2,j_2,k_2\}\): \(a_{C2} = (L_2 \cos(q_2(t)) \sin(q_2(t)) q_1'(t)^2 - 2 L_1 \cos(q_2(t))^2 \sin(q_2(t)) q_1'(t)^2 + 2 L_1 \sin(q_2(t))^3 q_1'(t)^2 + L_2 q_2''(t), 2 L_2 \sin(q_2(t)) q_1'(t) q_2'(t) - L_2 \cos(q_2(t)) q_1''(t) - 2 L_1 \cos(q_2(t))^2 q_1''(t) - 2 L_1 \sin(q_2(t))^2 q_1''(t), -L_2 \cos(q_2(t))^2 q_1''(t)^2 + q_1''(t)^2 - 2 L_1 \cos(q_2(t))^3 q_1''(t)^2 - 2 L_1 \cos(q_2(t)) \sin(q_2(t)) q_1'(t)^2 q_1''(t)^2 - L_2 q_2''(t)^2)\)
linear acceleration of mass center C3 of
link 3 in RF0 expressed in terms of RF2 \{i2,j2,k2\}: aC3 =

\[
\begin{align*}
&[L_2 \cos(q_2(t)) \sin(q_2(t))] \ q_1'(t)^2 + 2 L_1 \cos(q_2(t)) ^2 \sin(q_2(t))] \ q_1'(t)^2 + \\
&\cos(q_2(t)) \ q_3(t) \sin(q_2(t))] \ q_1'(t)^2 + 2 L_1 \sin(q_2(t))] q_1'(t)^2 + \\
&2 \ q_2'(t) \ q_3'(t) + L_2 \ q_2''(t) + q_3(t) \ q_2''(t) + 2 L_2 \sin(q_2(t))] q_1'(t) \ q_2'(t) + \\
&2 \ q_3(t) \ sin(q_2(t))] q_1'(t) \ q_2'(t) - 2 \cos(q_2(t))] q_1'(t) \ q_3'(t) - L_2 \cos(q_2(t))] q_1''(t) - \\
&2 L_1 \cos(q_2(t))] ^2 q_1''(t) - \cos(q_2(t))] q_3(t) \ q_1''(t) - 2 L_1 \sin(q_2(t))] q_1''(t), \\
&-L_2 \cos(q_2(t))] ^2 q_1'(t)^2 - 2 L_1 \cos(q_2(t))] q_1'(t)^2 - \cos(q_2(t))] ^2 q_3(t) q_1'(t)^2 - \\
&2 L_1 \cos(q_2(t))] \sin(q_2(t))] q_1'(t)^2 - L_2 q_2'(t)^2 - q_3(t) q_2'(t)^2 + q_3''(t)]
\end{align*}
\]

gravitational force that acts on link
1 at C1 in RF0 expressed in terms of RF1 \{i1,j1,k1\}: G1 =
\(-g m1, 0, 0\)

gravitational force that acts on link
2 at C2 in RF0 expressed in terms of RF2 \{i2,j2,k2\}: G2 =
\(-g m2 \cos(q_2(t)), 0, -g m2 \sin(q_2(t))\)}

gravitational force that acts on link
3 at C3 in RF0 expressed in terms of RF2 \{i2,j2,k2\}: G3 =
\(-g m3 \cos(q_2(t)), 0, -g m3 \sin(q_2(t))\)}

contact torque of 0 that acts on link
1 in RF0 expressed in terms of RF1 \{i1,j1,k1\}: T01 =
\(T01x, T01y, T01z\)

contact torque of link 1 that acts on link 2 in RF0 expressed in terms of RF2 \{i2,j2,k2\}: T12 =
\(T12x, T12y, T12z\)

contact force of link 2 that acts on link 3 at C3 in RF0 expressed in terms of RF2 \{i2,j2,k2\}: F23 =
\(F23x, F23y, F23z\)

generalized active force Q1 = T01x

generalized active force Q2 = T12y - \ g L2 \ m2 \ \cos(q_2(t)) \ - \ g m3 \ \cos(q_2(t)) \ (L2 - q_3(t))

generalized active force Q3 = F23z - \ g m3 \ \sin(q_2(t))

central inertia dyadic for link 1 expressed in terms of RF1 \{i1,j1,k1\}: I1 =
\(\left\{ \frac{L_1^2 m_1}{3}, 0, 0 \right\}, \left\{ 0, \frac{L_1^2 m_1}{3}, 0 \right\}, \left\{ 0, 0, 0 \right\} \)
central inertia dyadic for link 2 expressed in terms of RF2 \((i_2,j_2,k_2)\): 
\[
I_2 = 
\left\{ \left[ \frac{L_2^2 m_2}{3}, 0, 0 \right], \left[ 0, \frac{L_2^2 m_2}{3}, 0 \right], \left[ 0, 0, 0 \right] \right\}
\]

central inertia torque for link 3 expressed in terms of RF2 \((i_2,j_2,k_2)\): 
\[
I_3 = 
\left\{ \left[ I_{3x}, 0, 0 \right], \left[ 0, I_{3y}, 0 \right], \left[ 0, 0, I_{3z} \right] \right\}
\]

kinetic energy of link 1: 
\[
T_1 = \frac{2}{3} L_1^2 m_1 q_1'[t]^2
\]

kinetic energy of link 2: 
\[
T_2 = \frac{1}{3} m_2 \left( (6 L_1^2 + L_2^2 + 6 L_1 L_2 \cos[q_2[t]] + L_2^2 \cos[2 q_2[t]]) q_1'[t]^2 + 2 L_2^2 q_2'[t]^2 \right)
\]

kinetic energy of link 3: 
\[
T_3 = \frac{1}{3} m_2 \left( (6 L_1^2 + L_2^2 + 6 L_1 L_2 \cos[q_2[t]] + L_2^2 \cos[2 q_2[t]]) q_1'[t]^2 + 2 L_2^2 q_2'[t]^2 \right)
\]

total kinetic energy: 
\[
T = \frac{1}{12} \left( 3 I_{3x} + 3 I_{3z} + 8 L_1^2 m_1 + 24 L_1^2 m_2 + 4 L_2^2 m_2 + 24 L_1^2 m_3 + 3 L_2^2 m_3 + 24 L_1 L_2 \cos[q_2[t]] \right)  \]

\[
+ 3 I_{3x} \cos[2 q_2[t]] - 3 I_{3z} \cos[2 q_2[t]] + 4 L_2^2 \cos[2 q_2[t]] + 3 L_2^2 \cos[2 q_2[t]] + 12 m_3 \cos[q_2[t]] q_3[t] + 6 m_3 \cos[q_2[t]] q_3[t]^2 q_2'[t]^2 + q_1'[t]^2 + 2 \left( 3 I_{3y} + L_2^2 \left( 4 m_2 + 3 m_3 \right) + 6 L_2 m_3 q_3[t] + 3 m_3 q_3[t]^2 \right) q_2'[t]^2 + 6 m_3 q_3'[t]^2
\]

First Lagrange’s equation of motion
\[
D[D[T,q_1'[t]],t] - D[T,q_1[t]] = Q_1
\]
\[
\frac{1}{6} \left( -6 T_{01x} - 2 q_1'[t] \left( 2 \left( 6 L_1 L_2 \cos[q_2[t]] + 3 I_{3x} \cos[2 q_2[t]] \right) \left( 3 m_3 \cos[q_2[t]] q_3[t] + 6 m_3 \cos[q_2[t]] q_3[t]^2 \right) \sin[q_2[t]] q_2'[t] - 6 m_3 \cos[q_2[t]] q_3[t] \right) \cos[q_2[t]] \right) + 6 m_3 L_1 \cos[q_2[t]] q_3[t] \left( 6 m_3 \cos[q_2[t]] q_3[t] \right) \cos[q_2[t]] q_3'[t]
\]
\[
\left( 3 I_{3x} + 8 L_1^2 m_1 + 24 L_1^2 m_2 + 4 L_2^2 m_2 + 24 L_1^2 m_3 + 3 L_2^2 m_3 + 12 m_3 \cos[q_2[t]] q_3[t] + 3 I_{3x} \cos[2 q_2[t]] \right) - 3 I_{3z} \cos[2 q_2[t]] + 4 L_2^2 \cos[2 q_2[t]] + 3 L_2^2 \cos[2 q_2[t]] + 12 m_3 \cos[q_2[t]] q_3[t] \left( 2 L_1 L_2 \cos[q_2[t]] q_3[t] + 6 m_3 \cos[q_2[t]] q_3[t]^2 \right) q_2'[t]
\]

Second Lagrange’s equation of motion
\[
D[D[T,q_2'[t]],t] - D[T,q_2[t]] = Q_2
\]
\[
- T_{12y} + g L_2 m_2 \cos[q_2[t]] + g L_2 m_3 \cos[q_2[t]] + \frac{1}{3} \left( 6 L_1 L_2 \cos[q_2[t]] \sin[q_2[t]] q_1'[t]^2 + q_2''[t] \right) + 2 L_2 m_3 q_2'[t] q_3'[t] + 3 m_3 q_3'[t] \left( \cos[q_2[t]] \sin[q_2[t]] q_1'[t]^2 + q_2''[t] + q_3'[t] \right)
\]
\[
\left( 3 m_3 q_3[t]^2 + 2 \left( 6 L_1 + L_2 \cos[q_2[t]] \sin[q_2[t]] q_1'[t]^2 + 2 q_2'[t] q_3'[t] + 2 L_2 q_2''[t] \right) \right)
\]

Third Lagrange’s equation of motion
\[ D[D[T, q3'[t]], t] - D[T, q3[t]] = Q3 \]

\[-F23z + g m3 \sin[q2[t]] - m3 \cos[q2[t]] (2 L1 + L2 \cos[q2[t]] + \cos[q2[t]] q3[t]) q1'[t]^2 - m3 (L2 + q3[t]) q2'[t]^2 + m3 q3''[t] \]

**Numerical data**

\[
\begin{align*}
L1 & \rightarrow 0.4, L2 \rightarrow 0.7, LB \rightarrow 0.8, I3x \rightarrow 5, I3y \rightarrow 4, I3z \rightarrow 1, \quad m1 \rightarrow 90, \\
m2 & \rightarrow 60, m3 \rightarrow 40, g \rightarrow 9.81, b01 \rightarrow 450, \quad \quad \quad g01 \rightarrow 300, \ b12 \rightarrow 200, \quad g12 \rightarrow 300, \\
b23 & \rightarrow 150, q23 \rightarrow 50, \quad q1ref \rightarrow 1.0472, \quad q2ref \rightarrow 1.0472, \quad q3ref \rightarrow 0.25
\end{align*}
\]

**Control**

\[
\begin{align*}
T01x & \rightarrow -300 (-1.0472 + q1[t]) - 450 q1'[t], \\
T12y & \rightarrow -300 (-1.0472 + q2[t]) + 9.81 \cos[q2[t]] (42. + 40 (0.7 + q3[t])) - 200 q2'[t], \\
F23z & \rightarrow -50 (-0.25 + q3[t]) + 392.4 \sin[q2[t]] - 150 q3''[t]
\end{align*}
\]

\[
\{(q1 \rightarrow \text{InterpolatingFunction}[\{(0., 15.), \}, <>], \\
q2 \rightarrow \text{InterpolatingFunction}[\{(0., 15.), \}, <>], \\
q3 \rightarrow \text{InterpolatingFunction}[\{(0., 15.), \}, <>])\}
\]

**Graphics**

- **q1[rad]**

- **q2[rad]**
- Graphics -