Ordinary Differential Equations

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1 Introduction

An *ordinary differential equation* is a relation involving one or several derivatives of a function $y(x)$ with respect to x. The relation may also be composed of constants, given functions of x, or y itself.

The equation

$$y^{'}(x) = e^{x}, \tag{1}$$

where $y^{'} = dy/dx$, is of a first order ordinary differential equation, the equation

$$y^{''}(x) + 2y(x) = 0, \tag{2}$$

where $y^{''} = d^2y/dx^2$ is of a second order ordinary differential equation, and the equation

$$2x^2\, y^{'''}(x)\, y^{'}(x) + 3e^{-x}\, y^{''}(x) = (x^2 + 1)y^2(x), \tag{3}$$

where $y^{'''} = d^3y/dx^3$ is a third order ordinary differential equation.

The *order* of an ordinary differential equation the highest derivative of y in the equation.

Definition [1]. The *explicit solution* of a first-order differential equation is a function

$$y = g(x), \;\; a < x < b, \tag{4}$$

defined and differentiable on $(a,\, b)$, with the property that the equation becomes an identity when y and $y^{'}$ are replaced by g and $g^{'}$, respectively. The solution of a differential equation $G(x, y) = 0$ it is called the *implicit solution.*

Example. The explicit solution of the first-order differential equation

$$y^{'}(x) = x\, y(x), \tag{5}$$

is

$$y(x) = c\, e^{x^2/2}, \tag{6}$$

where c is an arbitrary constant. The differential equation (5) has many solutions. The function (6), with arbitrary c, represents the *general solution* (the totality of all solutions of the equation). If we consider a definite value of c, for example $c = 1$, then the solution obtained $y(x) = e^{x^2/2}$ is called a *particular solution*.

2 First order differential equations

2.1 Separable equations

The equation

$$g(y)y^{'} = f(x), \tag{7}$$

or

$$g(y)dy = f(x)dx, \tag{8}$$

is called an *equation with separable variables*, or a *separable equation*. The variable x appears only on the right hand side and the function y appears only on the left hand side in Eq. (8). Integrating both sides we obtain

$$\int g(y)dy = \int f(x)dx + c. \tag{9}$$

If f and g are continuous functions the general solution of Eq. (7) is obtained evaluating Eq. (9).

Example. Solve the equation

$$(y^2 + 1)xdx + (x + 1)ydy = 0.$$

The above equation can be rewritten in the form

$$\frac{x}{x+1}dx + \frac{y}{y^2+1}dy = 0.$$

By integration we obtain

$$x - \ln|1+x| + \frac{1}{2}\ln(1+y^2) = c, \quad x+1 \neq 0.$$

With $x = 0$ and $y = 0$ we calculate $c = \dfrac{1}{2}\ln 2$ and

$$2x + \ln\frac{1+y^2}{2} = \ln(1+x)^2, \quad x \neq -1$$

.

Definition. A first-order differential equation together with an *initial condition* is called an *initial value problem.* The initial condition is the condition that at some point $x = x_0$ the solution $y(x)$ has a prescribed value $y(x_0) = y_0$.

2.2 Equations reducible to separable form

The first-order differential equation

$$y' = g\left(\frac{y}{x}\right), \tag{10}$$

where g is any given function of y/x ($g(x) = f(y/x)$), can be made separable equation by a simple change of variables. The change of variable is

$$\frac{y}{x} = u.$$

The function $y = u\,x$ and by differentiation we obtain

$$y' = u + u'x. \tag{11}$$

Combining the equations (11) and (10), and taking into account that $g(y/x) = g(u)$ we obtain

$$u + u'x = g(u).$$

By separating the variables u and x, the previous equation takes the form

$$\frac{du}{g(u)-u} = \frac{dx}{x}.$$

After integration and replacement of u by y/x the general solution of Eq. (10) is obtained.

Example. Solve the equation

$$\frac{dy}{dx} = \frac{2x+3y}{3x+2y} \quad \text{and} \;\; 3x+2y \neq 0.$$

With the change of function $y = ux$ we obtain

$$u^{'}x + u = \frac{2+3u}{3+2u}$$

or

$$u^{'}x = \frac{2-2u^2}{3+2u}$$

and

$$\frac{1}{2}\,\frac{3+2u}{1-u^2}du = \frac{dx}{x}\ .$$

Integrating

$$\int \frac{dx}{x} = \frac{1}{2}\int \frac{3+2u}{1-u^2}du$$

or

$$\ln|x| = -\frac{1}{2}\ln|u^2-1| - \frac{3}{4}\ln\left|\frac{1-u}{1+u}\right| + \ln|c|.$$

We obtain $x^4(u^2-1)^2\left(\frac{1-u}{1+u}\right)^3 = c$. Replacing u by y/x, the general integral will be

$$(x^2-y^2)^2(x-y)^3 = c(x+y)^3.$$

2.3 Exact differential equations

A first-order differential equation

$$M(x,y)dx + N(x,y)dy = 0, \tag{12}$$

is *exact* if the left hand side is an exact differential

$$d\,u(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy. \tag{13}$$

Equation (12) can be rewritten as

$$d\,u = 0.$$

and by integration the general solution is

$$u(x,y) = c. \tag{14}$$

If there is a function $u(x,y)$ with the properties

$$\text{(a)}\frac{\partial u}{\partial x} = M, \quad \text{(b)}\frac{\partial u}{\partial y} = N, \tag{15}$$

then $M(x,y)dx + N(x,y)dy = 0$ is an exact differential equation.

The necessary and sufficient condition for $M dx + N dy$ be an exact differential [1] is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{16}$$

To find $u(x,\, y)$ we have the following steps [1].

From Eq. (15.a) if we consider y to be a constant we obtain

$$u = \int M dx + k(y), \tag{17}$$

where and $k(y)$ is the "constant" of integration.

$k(y)$ is determined from Eq. (17) deriving $\partial u/\partial y$.

From Eq. (15.b) we get dk/dy.

Example. Solve the equation

$$-\frac{x}{2x-y}y' + \ln(2x-y) + \frac{2x}{2x-y} = 0 \quad 2x-y > 0.$$

Writing the equation in the form Eq. (12), we get

$$-\frac{x}{2x-y}dy + \left[\ln(2x-y) + \frac{2x}{2x-y}\right]dx = 0 \tag{18}$$

Equation (18) is exact. Consider $M = \ln(2x-y)+\dfrac{2x}{2x-y}$, and $N = -\dfrac{x}{2x-y}$. Then, by differentiation we obtain

$$\frac{\partial M}{\partial y} = \frac{-1}{2x-y} + \frac{2x}{(2x-y)^2},$$
$$\frac{\partial N}{\partial x} = \frac{-1}{2x-y} + \frac{2x}{(2x-y)^2}.$$

From Eq. (15.b) we have $N = \dfrac{\partial u}{\partial y}$ and by integration

$$u = x\ln(2x-y) + k(x).$$

To determine $k(x)$ we differentiate u and apply Eq. (15.a)

$$\frac{\partial u}{\partial x} = \ln(2x-y) + \frac{2x}{2x-y} + \frac{dk}{dx} = M.$$

By simple algebraic manipulations, we find that $\dfrac{dk}{dx} = 0$ and, consequently, $k(x) = c$, where c is an arbitrary constant. We obtain the final form

$$x\ln(2x-y) = c.$$

2.4 Linear differential equations

We consider the first-order differential equation

$$y^{'} + f(x)\,y = r(x), \tag{19}$$

which is linear in y and $y^{'}$ (f and r may be any given functions of x).

If $r(x) = 0,\ \forall x$ (for all x) the equation is *homogeneous*. For $r(x) \neq 0$ the equation is said to be *nonhomogeneous*.

Assuming that $f(x)$ and $r(x)$ are continuous for $x \in I$, we need to find a general formula for Eq. (19).

Case I. homogeneous equation
For the equation

$$y^{'} + f(x)y = 0, \tag{20}$$

separating variables we have

$$\frac{dy}{y} = -f(x)dx \quad \text{or} \quad \ln|y| = -\int f(x)dx + c^*,$$

and the solution is

$$y(x) = ce^{-\int f(x)dx} \quad (c = \pm e^{c^*} \text{ when } y < 0 \text{ or } y > 0). \tag{21}$$

Case II. nonhomogeneous equation
Multiplying Eq. (19) by

$$F(x) = e^{h(x)} \quad \text{where} \quad h(x) = \int f(x)dx.$$

we find

$$e^h(y' + fy) = e^h r.$$

Since $h' = f$, we obtain

$$\frac{d}{dx}(ye^h) = e^h r.$$

Integrating the above relation we have

$$ye^h = \int e^h r dx + c.$$

The general solution of Eq. (19) in the form of an integral may be written

$$y(x) = e^{-h}\left[\int e^h r dx + c\right], \quad h = \int f(x)dx. \tag{22}$$

Example. Solve the differential equation

$$xy' + (1 - x)y = xe^x.$$

We can rewrite the equation in the form

$$y' + \left(\frac{1}{x} - 1\right)y = e^x.$$

Comparing the previous equation to Eq. (22) we can identify

$$h = \int \left(\frac{1}{x} - 1\right)\, dx = \log x - x,$$

no constant being added in the integration. Thus, the solution will be

$$\begin{aligned} y &= e^{-\log x+x}\left(\int e^{\log x-x}e^{x}\, dx+c\right) \\ &= \frac{e^{x}}{x}\left(\int \frac{x}{e^{x}}e^{x}\, dx+c\right), \end{aligned}$$

or

$$y = e^{x}\left(\frac{x}{2}+\frac{c}{x}\right),$$

where c is an arbitrary constant.

2.5 Variation of parameters

Another way of finding the general solution of linear differential equation

$$y^{'} + f(x)y = r(x). \tag{23}$$

is the method of variation of parameters.
The solution corresponding to a homogeneous equation ($r(x) = 0$) is

$$v(x) = e^{-\int f(x)dx}. \tag{24}$$

With Eq. (24) we try to determine a function $u(x)$ such that

$$y(x) = u(x)v(x), \tag{25}$$

is the general solution of Eq. (23). This approach is called the method of variation of parameters [1].

Equations (25) and Eq. (23) can be combined into

$$u^{'}v + u(v^{'} + fv) = r,$$

or $u^{'}v = r$, since $v^{'} + fv = 0$. We find $u^{'} = \frac{r}{v}$ and by integration

$$u = \int \frac{r}{v}dx + c.$$

We obtain the general solution

$$y = uv = v\left(\int \frac{r}{v}dx + c\right), \tag{26}$$

which is identical with Eq. (22) of the previous section.

3 Second order differential equation

3.1 Homogeneous linear equations

A second-order differential equation which can be written as

$$y^{''} + f(x)y^{'} + g(x)y = r(x) \tag{27}$$

is said to be *linear*. It is said to be *nonlinear* if it cannot be written in the form of Eq. (27). The functions f and g are called the *coefficients* of the equation (27).

If $r(x) \neq 0$, then Eq. (27) is said to be *nonhomogeneous*. Otherwise, it is said to be *homogeneous* and takes the form

$$y^{''} + f(x)y^{'} + g(x)y = 0. \tag{28}$$

It is called a *solution* of a differential equation of the second order on an interval J a function $y = \phi(x)$ which is defined and two times differentiable on J. Moreover, the equation becomes an identity if ϕ and its derivative replace the unknown function y and its derivatives, respectively. For the case of homogeneous equations, the following theorem states that solutions of Eq. (28) can be obtained from known solutions by multiplication by constants and by addition.

Fundamental Theorem [1]. *If a solution of the homogeneous linear differential equation (28) on the interval J is multiplied by any constant, the resulting function is also a solution of Eq. (28) on J. The sum of two solutions of Eq. (28) on J is also a solution of Eq. (28) on that interval.*

Proof. We assume that $\phi(x)$ obeys the conditions to be a solution of Eq. (28) on J. If we replace y by $c\phi(x)$ is into Eq. (28), we obtain

$$(c\phi)^{''} + f(c\phi)^{'} + gc\phi = c[\phi^{''} + f\phi^{'} + g\phi].$$

Since ϕ is a solution of Eq. (28), then $\phi^{''} + f\phi^{'} + g\phi = 0$ and we find that $c\phi$ is also a solution of Eq. (28). The second part of the theorem can be proved in the same way.

Example. The functions $y_1 = \phi_1 = x$ and $y_2 = \phi_2 = x^2$, $x \in \mathcal{R} - \{0\}$ ($J \equiv \mathcal{R} - \{0\}$), are two solutions of the equation

$$x^2y^{''} - 2xy^{'} + 2y = 0.$$

The function $y_3 = c_1\phi_1 + c_2\phi_2 = c_1x + c_2x^2$ is also a solution of the equation.

3.2 Homogeneous equations with constant coefficients

We consider the homogeneous equations of the form

$$y^{''} + ay^{'} + by = 0, \tag{29}$$

where $a, b \in \mathcal{R}$ are constants, and $x \in \mathcal{R}$. The solution of the first-order homogeneous linear equation with constant coefficients

$$y^{'} + ky = 0,$$

is an exponential function,

$$y = C\, e^{-kx}.$$

We assume that

$$y = e^{\lambda x}, \tag{30}$$

may be a solution of Eq. (29) if λ is properly chosen. Substituting Eq. (30) and its derivatives

$$y^{'} = \lambda e^{\lambda x} \quad \text{and} \quad y^{''} = \lambda^2 e^{\lambda x},$$

into Eq. (29), we obtain

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

So Eq. (30) is a solution of Eq. (29), if λ is a solution of the equation

$$\lambda^2 + a\lambda + b = 0. \tag{31}$$

Eq. (31) is called the *characteristic equation* of Eq. (29). Its roots are

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}), \quad \lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b}). \tag{32}$$

From derivation it follows that the functions

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}, \tag{33}$$

are solutions of Eq. (29). This result can be verified by substituting Eq. (33) into Eq. (29).
Elementary algebra states that, since a and b are real, the characteristic equation may have

Case I	two distinct real roots,
Case II	two complex conjugate roots, or
Case III	a real double root.

Example 1. Solve the equation

$$2y'' - 5y' + 2y = 0.$$

The characteristic equation of the given differential equation will be

$$2\lambda^2 - 5\lambda + 2 = 0$$

so that

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = 2.$$

Then the general solution is

$$y = c_1 e^{x/2} + c_2 e^{2x}.$$

Example 2. The equation

$$y'' + 2y' + 5y = 0$$

has the characteristic equation

$$\lambda^2 + 2\lambda + 5 = 0$$

from which

$$\lambda_{1,2} = -1 \pm 2i.$$

The general solution will be

$$y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x).$$

Example 3. The equation

$$y'' - 2y' + 1 = 0$$

has the characteristic equation

$$(\lambda - 1)^2 = 0$$

which gives

$$\lambda_{1,2} = 1.$$

We obtain the general solution

$$y = e^x.$$

3.3 General solution. Fundamental system

Definition. The *general solution* of a second order differential equation is a solution which contains two arbitrary independent constants, i.e. the solution cannot be reduced to a form containing only one arbitrary constant or none. A *particular solution* is a solution obtained from the general solution assigning specific values to the arbitrary constants.

We consider the general homogeneous linear equation

$$y^{''} + f(x)y^{'} + g(x)y = 0, \tag{34}$$

and two solutions $y_1(x)$ and $y_2(x)$ of this equation. The Fundamental Theorem states that

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x), \tag{35}$$

is a general solution of Eq. (34), where c_1 and c_2 are two arbitrary constants.

Two functions $y_1(x)$ and $y_2(x)$ are *linearly dependent* on an open interval I where both functions are defined, if they are proportional on I

$$\text{(a)}\, y_1 = m\, y_2 \quad \text{or} \quad \text{(b)}\, y_2 = n\, y_1, \tag{36}$$

for all $x \in I$, where m and n are numbers. If the functions are not proportional, they are *linearly independent* on I.

If at least one of the functions y_1 and y_2 is identically zero on I, then the functions are linearly dependent on I. In any other case the functions are linearly dependent on I if and only if the quotient y_1/y_2 is constant on I. Hence, if y_1/y_2 depends on x on I, then y_1 and y_2 are linearly independent on I [1].

Example 1. The functions

$$y_1 = 9x \quad \text{and} \quad y_2 = 3x$$

are linearly dependent, because the quotient $y_1/y_2 = 3 = \text{const}$ while the functions

$$y_1 = x^2 + x \quad \text{and} \quad y_2 = x$$

are linearly independent because $y_1/y_2 = x + 1 \neq \text{const}$.

Two linearly independent solutions of Eq. (34) on I constitute a *fundamental system* or a *basis* of solutions on I.

Theorem [1]. *The solution*

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x) \quad (c_1,\, c_2\, arbitrary)$$

is a general solution of the differential equation Eq. (34) on an interval I of the x-axis if and only if the functions y_1 and y_2 constitute a fundamental system of solutions of Eq. (34) on I. y_1 and y_2 constitute such a fundamental system if and only if their quotient y_1/y_2 is not constant on I but depends on x.

Example 2. The equation

$$y^{''} - 2y^{'} - 15y = 0$$

has the solutions

$$y_1 = e^{5x} \quad \text{and} \quad y_2 = e^{-3x}.$$

These solutions constitute a fundamental system because the ratio y_1/y_2 is not constant. The general solution is

$$y = c_1\, y_1 + c_2\, y_2 = c_1\, e^{5x} + c_2\, e^{-3x}.$$

3.4 Complex roots of the characteristic equation. Initial value problem

The solutions of the homogeneous linear equation with constant coefficients

$$y^{''} + ay^{'} + by = 0 \quad (a,\, b \text{ real}) \tag{37}$$

are

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}, \tag{38}$$

where λ_1 and λ_2 are the roots of the corresponding characteristic equation

$$\lambda^2 + a\lambda + b = 0. \tag{39}$$

In the case of $\lambda_1 \neq \lambda_2$, the quotient y_1/y_2 is not constant, and the solutions constitute a fundamental system for all x. The general solution is

$$y = c_1\, e^{\lambda_1 x} + c_2\, e^{\lambda_2 x}. \tag{40}$$

The solutions of the Eq. (38) are real if the distinct roots of the corresponding characteristic equation are real (Case I). If λ_1 and λ_2 are complex conjugate roots of the form (Case II)

$$\lambda_1 = p + i\,q, \quad \lambda_2 = p - i\,q,$$

then the solutions Eq. (38) are complex

$$y_1 = e^{(p+i\,q)x}, \quad y_2 = e^{(p-i\,q)x}.$$

The real solutions can be derived from the complex solutions by applying the *Euler formulas*

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad e^{-i\theta} = \cos\theta - i\sin\theta,$$

for $\theta = qx$. The first solution becomes

$$y_1 = e^{(p+iq)x} = e^{px}e^{iqx} = e^{px}(\cos qx + i\sin qx),$$

while the second one is

$$y_2 = e^{(p-iq)x} = e^{px}e^{-iqx} = e^{px}(\cos qx - i\sin qx).$$

From Fundamental Theorem we can conclude that they are solutions of the differential equation Eq. (37). The corresponding general solution is

$$y(x) = e^{px}(A\cos qx + B\sin qx) \tag{41}$$

where A and B are arbitrary constants.

Example 1. Let us consider the second order differential equation with constant coefficients

$$y^{''} - 4y^{'} + 5y = 0$$

The corresponding characteristic equation is

$$\lambda^2 - 4\lambda + 5 = 0,$$

with the roots

$$\lambda_1 = p + iq = 2 + i \quad \text{and} \quad \lambda_2 = p - iq = 2 - i.$$

For this example $p = 2$, $q = 1$, and from Eq. (41) the answer is

$$y = e^{2x}(A\cos x + B\sin x).$$

Let us consider the values of the solution $y(x)$ and its derivative $y^{'}(x)$ at an initial point $x = x_0$

$$y(x_0) = K, \quad y^{'}(x_0) = L, \tag{42}$$

The conditions Eq. (42) and the equation Eq. (37) constitute an *initial value problem*. To solve such a problem we must find a particular solution of Eq. (37) satisfying Eq. (42). Such a problem has a unique solution.

Example 2. Let us consider the initial value problem

$$y^{''} - 4y^{'} + 5y = 0, \quad y(0) = 2, \quad y^{'}(0) = 0.$$

A fundamental system of solutions is

$$e^{2x}\cos x \quad \text{and} \quad e^{2x}\sin x,$$

and the corresponding general solution is

$$y(x) = e^{2x}(A\cos x + B\sin x),$$

with the initial condition $y(0) = A$. The derivative

$$y^{'} = e^{2x}[(2A + B)\cos x + (2B - A)\sin x)],$$

has the initial value $y^{'}(0) = 2A + B$. Solving the initial conditions system,

$$\begin{aligned} y(0) = A = 2, \\ y^{'}(0) = 2A + B = 0. \end{aligned}$$

we get $A = 4$, $B = -1$, and the general solution of the differential equation is

$$y = e^{2x}(2\cos x - 4\sin x). \tag{43}$$

3.5 Double root of the characteristic equation

Now we consider the case when the characteristic equation associated to a homogeneous linear differential equation with constant coefficients has a double root (*critical case*). If the differential equation takes the general form

$$y^{''} + ay^{'} + by = 0, \tag{44}$$

then the characteristic equation will be

$$\lambda^2 + a\lambda + b = 0. \tag{45}$$

A double root appears if an only if the discriminant of Eq. (45) is zero, that is

$$a^2 - 4b = 0, \quad \text{and, then,} \quad b = \frac{1}{4}a^2.$$

The double root of the characteristic equation is $\lambda = -a/2$. Then, the first solution of the differential equation is

$$y_1 = e^{ax/2}. \tag{46}$$

To find another solution $y_2(x)$ the method of variation of parameters may be applied. The second solution takes the form

$$y_2(x) = u(x)y_1(x) \quad \text{where} \quad y_1(x) = e^{-ax/2}.$$

Substituting y_2 in the differential equation with $b = a^2/4$ we obtain

$$u(y_1^{''} + ay_1^{'} + \frac{1}{4}a^2 y_1) + u^{'}(2y_1^{'} + ay_1) + u^{''}y_1 = 0.$$

The expression in the first parentheses is zero because y_1 is a solution. The second parentheses is also zero because

$$2y_1^{'} = 2\left(-\frac{a}{2}\right)e^{-ax/2} = -ay_1.$$

The equation reduces to $u^{''}y_1 = 0$, and a solution is $u = x$. Consequently, the second solution is

$$y_2(x) = xe^{\lambda x} \quad \left(\lambda = -\frac{a}{2}\right). \tag{47}$$

We can observe that the solutions y_1 and y_2 are linearly independent. This case can be summarized by the following theorem

Theorem (Double root) [1]. *In the case of a double root of Eq. (45) the functions (46) and (47) are solutions of Eq. (44). They constitute a fundamental system. The corresponding general solution is*

$$y = (c_1 + c_2 x)e^{\lambda x} \quad \left(\lambda = -\frac{a}{2}\right). \tag{48}$$

Example. Solve the following differential equation

$$y^{''} - 4y^{'} + 4y = O.$$

The double root of the characteristic equation is $\lambda = -4$. Then, the fundamental system of solutions is

$$e^{2x} \quad \text{and} \quad xe^{2x}$$

and the corresponding general solution is

$$y = (c_1 + c_2 x)e^{2x}.$$

All three cases are summarized in the following table:

Case	Roots of Eq. (45)	Fundamental system of Eq. (44)	General solution of Eq. (44)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}$, $e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Complex conjugate $\lambda_1 = p + iq$, $\lambda_2 = p - iq$	$e^{px} \cos qx$ $e^{px} \sin qx$	$y = e^{px}(A \cos qx + B \sin qx)$
III	Real double root $\lambda = -a/2$	$e^{\lambda x}$, $xe^{\lambda x}$	$y = (c_1 + c_2 x)e^{\lambda x}$

3.6 Series solutions

We consider the general homogeneous linear second-order equation

$$P(x)\frac{d^2 y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \tag{49}$$

with $P(x) \neq 0$ in the interval $\alpha < x < \beta$. We want to determine a polynomial solution $y(x)$ of Eq. (49).

Definition. A functions $f(x)$ can be expanded in power series so that

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots = \sum_{n=0}^{\infty} a_n(x - x_0)^n. \quad (50)$$

Such functions are said to be *analytic* at $x = x_0$ and the series (50) is called the Taylor series of f about $x = x_0$. The coefficients a_n can be computed with the formula $a_n = f^{(n)}(x_0)/\,n!$ where $f^{(n)}(x) = d^n f(x)/dx^n$.

We consider the functions $P(x)$, $Q(x)$, and $R(x)$ as power series about x_0

$$P(x) = p_0 + p_1(x - x_0) + \ldots, \quad Q(x) = q_0 + q_1(x - x_0) + \ldots,$$
$$R(x) = r_0 + r_1(x - x_0) + \ldots$$

and $y(x) = a_0 + a_1(x - x_0) + \ldots$.

Theorem [2]. *Let the functions $Q(x)/P(x)$ and $R(x)/P(x)$ have convergent Taylor series expansions about $x = x_0$ for $|x - x_0| < \rho$. Then, every solution $y(x)$ of the differential equation*

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \quad (51)$$

is analytic at $x = x_0$, and the radius of convergence of its Taylor series expansion about $x = x_0$ is at least ρ. The coefficients $a_2, a_3, \ldots$ in the Taylor series expansion

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots \quad (52)$$

are determined by plugging the series (52) into the differential equation (51) and setting the sum of the coefficients of the like powers of x in this expression equal to zero.

Example. Solve the equation

$$x^2\frac{d^2y}{dx^2} + (x^2 + x)\frac{dy}{dx} - y = 0.$$

Assuming a solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^k,$$

we obtain

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} k a_k x^{k-1}, \quad \frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2},$$

and hence

$$\sum_{k=0}^{\infty} k(k-1) a_k x^k + \sum_{k=0}^{\infty} k a_k x^{k+1} + \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k.$$

The first, third, and fourth summations may be combined to give

$$\sum_{k=0}^{\infty} [k(k-1) + k - 1] a_k x^k = \sum_{k=0}^{\infty} (k^2 - 1) a_k x^k,$$

and hence there follows

$$\sum_{k=0}^{\infty} (k^2 - 1) a_k x^k + \sum_{k=0}^{\infty} k a_k x^{k+1}.$$

In order to combine these sums, we replace k by n in the first and $(k+1)$ by n in the second, to obtain

$$\sum_{n=0}^{\infty} (n^2 - 1) a_n x^n + \sum_{n=1}^{\infty} (n-1) a_{n-1} x^n.$$

Since the ranges of summation differ, the term corresponding to $n = 0$ must be extracted from the first sum, after which the remainder of the first sum can be combined with the second. In this way we find

$$-a_0 + \sum_{n=1}^{\infty} [(n^2 - 1) a_n + (n-1) a_{n-1}] x^n.$$

In order that the previous relation may vanish identically, the constant term, as well as the coefficients of the successive powers of x, must vanish independently, giving the condition

$$a_0 = 0$$

and the recurrence formula

$$(n-1)[(n+1) a_n + a_{n-1}] = 0 \quad (n = 1, 2, 3, \ldots).$$

The recurrence formula is automatically satisfied when $n = 1$. When $n \geq 2$, it becomes

$$a_n = -\frac{a_{n-1}}{n+1} \quad (n = 2, 3, 4 \ldots).$$

Hence, we obtain

$$a_2 = -\frac{a_1}{3}, \quad a_3 = -\frac{a_2}{4} = \frac{a_1}{3 \cdot 4}, \quad a_4 = -\frac{a_3}{5} = -\frac{a_1}{3 \cdot 4 \cdot 5}, \ldots .$$

Thus, in this case $a_0 = 0$, a_1 is arbitrary, and all succeeding coefficients are determined in terms of a_1. The solution becomes

$$y = a_1 \left(x - \frac{x^2}{3} + \frac{x^3}{3 \cdot 4} - \frac{x^4}{3 \cdot 4 \cdot 5} + \ldots \right).$$

If this solution is put in the form

$$\begin{aligned} y &= \frac{2a_1}{x} \left(\frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots \right) \\ &= \frac{2a_1}{x} \left[x - 1 + \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots \right) \right], \end{aligned}$$

the series in parentheses in the final form is recognized as the expansion of e^{-x}, and, writing $2a_1 = c$, the solution obtained may be put in the closed from

$$y = c \left(\frac{e^{-x} - 1 + x}{x} \right).$$

In this case only one solution was obtained. This fact indicates that any linearly independent solutions cannot be expanded in power series near $x = 0$. That is, it is not regular at $x = 0$.

3.7 Regular singular points

We consider the differential equations

$$x^2 \frac{d^2 y}{dx^2} + \alpha x \frac{dy}{dx} + \beta y = 0 \qquad (53)$$

which can be rewritten in the form

$$\frac{d^2y}{dx^2} + \frac{\alpha}{x}\frac{dy}{dx} + \frac{\beta}{x^2}y = 0. \tag{54}$$

A generalization of Eq. (54) is the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{55}$$

where $p(x)$ and $q(x)$ can be expanded in series of the form

$$\begin{aligned} p(x) &= \frac{p_0}{x} + p_1 + p_2 x + p_3 x^2 + \ldots \\ q(x) &= \frac{q_0}{x^2} + \frac{q_1}{x} + q_2 + q_3 x + q_4 x^2 \ldots \end{aligned} \tag{56}$$

Definition [2]. Equation (55) is said to have a *regular singular point* at $x = 0$ if $p(x)$ and $q(x)$ have series expansions of the form (56). Equivalently, $x = 0$ is a regular singular point of Eq. (55) if the functions $x\,p(x)$ and $x^2\,q(x)$ are analytic at $x = 0$. Equation (55) is said to have a regular singular point at $x = x_0$ if the functions $(x - x_0)\,p(x)$ and $(x - x_0)^2\,q(x)$ are analytic at $x = x_0$. A singular point of Eq. (55) which is not regular is called *irregular*.

Example. Classify the singular points of Bessel's equation of order ν

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0, \tag{57}$$

where ν is a constant [1].

For $x = 0$ we have $P(x) = x^2 = 0$. Hence, $x = 0$ is the only singular point of Eq. (57). Dividing both sides of Eq. (57) by x^2 gives

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)y = 0.$$

The functions

$$x\,p(x) = 1 \quad \text{and} \quad x^2\,q(x) = x^2 - \nu^2$$

are both analytic at $x = 0$. Hence Bessel's equation of order ν has a regular singular point at $x = 0$.

3.8 Nonhomogeneous linear equations

Let us consider a second-order linear nonhomogeneous equation

$$y^{''} + f(x)y^{'} + g(x)y = r(x). \tag{58}$$

A general solution $y(x)$ of Eq. (58) can be obtained from a general solution $y_h(x)$ of the corresponding homogeneous equation

$$y^{''} + f(x)y^{'} + g(x)y = 0,$$

by adding to $y_h(x)$ any particular solution $\tilde{y}$ of Eq. (58) involving no arbitrary constant [1]

$$y(x) = y_h(x) + \tilde{y}(x). \tag{59}$$

To show that $y(x)$ is a solution of the nonhomogeneous differential equation we substitute Eq. (59) into Eq. (58). Then the left-hand side of Eq. (58) becomes

$$(y_h + \tilde{y})^{''} + f(y_h + \tilde{y})^{'} + g(y_h + \tilde{y}).$$

or

$$(y_h^{''} + fy_h^{'} + gy_h) + \tilde{y}^{''} + f\tilde{y}^{'} + g\tilde{y}.$$

The expression in the parentheses is zero because y_h is a solution of Eq. (59). The sum of the other terms is equal to $r(x)$ because $\tilde{y}$ satisfies Eq. (58). Hence $y(x)$ is a general solution of the Eq. (58).

Theorem [1]. *Suppose that $f(x)$, $g(x)$, and $r(x)$ in Eq. (58) are continuous functions on an open interval I. Let $Y(x)$ be any solution of Eq. (58) on I containing no arbitrary constants. Then $Y(x)$ is obtained from Eq. (59) by assigning suitable values to the two arbitrary constants contained in the general solution $y_h(x)$ of Eq. (59). In Eq. (59), the function $\tilde{y}(x)$ is any solution of Eq. (58) on I containing no arbitrary constants.*

Proof. Let set $Y - \tilde{y} = y^*$. Then

$$y^{*''} + fy^{*'} + gy^* = (Y^{''} + fY^{'} + gY) - (\tilde{y}^{''} + f\tilde{y}^{'} + g\tilde{y}) = r - r = 0,$$

that is, y^* is a solution of Eq. (59) which does not contain arbitrary constants. It can be obtained from y_h by assigning suitable values to the arbitrary constants in y_h. From this, since $Y = y^* + \tilde{y}$, the statement follows.

Theorem [1]. *A general solution $y(x)$ of the linear nonhomogeneous differential equation Eq. (58) is the sum of a general solution $y_h(x)$ of the corresponding homogeneous equation Eq. (59) and an arbitrary particular solution $y_p(x)$ of Eq. (58):*

$$y(x) = y_h(x) + y_p(x) \tag{60}$$

.

Example. Solve the equation

$$y^{''} + y = \sec x.$$

The homogeneous equation $y^{''} + y = 0$ has the characteristic equation $\lambda^2 + 1 = 0$ with roots $\lambda_1 = i$ and $\lambda_2 = -i$, so, the general solution of the homogeneous equation is

$$y = c_1 \cos x + c_2 \sin x.$$

Using the method of variation of parameter we have the following system of equations

$$c_1^{'} \cos x + c_2^{'} \sin x = 0,$$
$$-c_1^{'} \sin x + c_2^{'} \cos x = \sec x,$$

with the solution

$$c_1^{'} = -\tan x, \quad c_2^{'} = 1.$$

Thus by integrating,

$$c_1 = -\ln \sec x + A_1, \quad c_2 = x + A_2,$$

and the general solution is of the nonhomogeneous equation is

$$y = A_1 \cos x + A_2 \sin x - \cos x \ln \sec x + x \sin x.$$

3.9 The method of variation of parameters

This method can be applied to solve the nonhomogeneous equation of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x), \tag{61}$$

once the solutions of the homogeneous equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{62}$$

are known. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous equation (62). We will try to find a particular solution $\psi(x)$ of the nonhomogeneous Eq. (61) of the form [2]

$$\psi(x) = u_1(x)y_1(x) + u_2(x)y_2(x). \tag{63}$$

The differential equation (61) imposes only one condition on the two unknown functions $u_1(x)$ and $u_2(x)$. We may impose an additional condition on $u_1(x)$ and $u_2(x)$ such that the left hand side of the nonomogeneous equation be as simple as possible. Computing

$$\begin{aligned} \frac{d}{dx}\psi(x) &= \frac{d}{dx}[u_1y_1 + u_2y_2] \\ &= [u_1y_1^{'} + u_2y_2^{'}] + [u_1^{'}y_1 + u_2^{'}y_2] \end{aligned}$$

we see that $d^2\psi/dx^2$ will contain no second-order derivatives of u_1 and u_2 if

$$y_1(x)u_1^{'}(x) + y_2(x)u_2^{'}(x) = 0. \tag{64}$$

Imposing the condition (64) on the functions $u_1(x)$ and $u_2(x)$ the left hand side of the Eq. (61) becomes

$$\begin{aligned} & [u_1y_1^{'} + u_2y_2^{'}]^{'} + p(x)[u_1y_1^{'} + u_2y_2^{'}] + q(x)[u_1y_1 + u_2y_2] \\ = \;& u_1^{'}y_1^{'} + u_2^{'}y_2{'} + u_1[y_1^{''} + p(x)y_1^{'} + q(x)y_1] + u_2[y_2^{''} + p(x)y_2^{'} + q(x)y_2] \\ = \;& u_1^{'}y_1^{'} + u_2^{'}y_2^{'}. \end{aligned}$$

If $u_1(x)$ and $u_2(x)$ satisfy the two equations

$$\begin{aligned} y_1(x)u_1^{'} + y_2(x)u_2^{'}(x) &= 0 \\ y_1^{'}(x)u_1^{'}(x) + y_2^{'}(x)u_2^{'}(x) &= g(x), \end{aligned}$$

then $\psi(x) = u_1y_1 + u_2y_2$ is a solution of the nonhomogeneous equation (61). We solve the above system of equations as follows

$$\left[y_1(x)y_2^{'}(x) - y_1^{'}(x)y_2(x)\right]u_1^{'}(x) = -g(x)y_2(x)$$
$$\left[y_1(x)y_2^{'}(x) - y_1^{'}(x)y_2(x)\right]u_2^{'}(x) = g(x)y_1(x).$$

The function $u_1^{'}(x)$ and $u_2^{'}(x)$ are

$$u_1^{'}(x) = -\frac{g(x)y_2(x)}{W[y_1, y_2](x)} \quad \text{and} \quad u_2^{'}(x) = \frac{g(x)y_1(x)}{W[y_1, y_2](x)}, \tag{65}$$

where $W[y_1, y_2](x)$ is the Wronskian of the solutions

$$W[y_1, y_2](x) = \begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix}.$$

Integrating the right-hand sides of Eqs. (65) we obtain $u_1(x)$ and $u_2(x)$.

Example.
(a) Find a particular solution $\psi(x)$ of the equation

$$\frac{d^2y}{dx^2} + 4y = 8 \sin x \tag{66}$$

(b) Find the solution $y(x)$ of Eq. (66) which satisfies the initial conditions $y(0) = 1, y^{'}(0) = 1$.

(a) The functions $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$ are two linearly independent solutions of the homogeneous equation $y^{''} + 4y = 0$ with

$$W[y_1, y_2](x) = y_1 y_2^{'} - y_1^{'} y_2 = (\cos x) \cos x - (-\sin x) \sin x = 1.$$

Thus, from Eqs. (65),

$$u_1^{'}(x) = -8 \sin^2 x \quad \text{and} \quad u_2^{'}(x) = 8 \sin x \cos x. \tag{67}$$

Integrating the first equation of (67) gives

$$\begin{aligned} u_1(x) &= -8 \int \sin^2 x \; dx = -4 \int (1 - \cos 2x) \; dx \\ &= -4 \int \; dx + 4 \int \cos 2x \; dx \\ &= -4x + 2 \sin 2x. \end{aligned}$$

while integrating the second equation of (67) gives

$$u_2(x) = \int 4 \sin 2x \; dx = 4 \int \sin 2x \; dx = -2 \cos 2x.$$

Consequently,

$$\psi(x) = \cos x[-4x + 2\sin 2x] + \sin x(-2\cos 2x)$$

is a particular solution of Eq. (66).
(b)

$$y(x) = c_1 \cos x + c_2 \sin x + \cos x(-4x + 2\sin 2x) - 2\sin x \cos 2x$$

for some choice of constants c_1, c_2. The constants c_1 and c_2 are determined from the initial conditions

$$1 = y(0) = c_1 \quad \text{and} \quad 1 = y^{'}(0) = c_2 - 2.$$

Hence, $c_1 = 1, c_2 = 3$ and

$$y(x) = \cos x + 3\sin x + \cos x(-4x + 2\sin 2x) - 2\sin x \cos 2x.$$

4 Differential equations of arbitrary order

4.1 Homogeneous linear equations

A *linear* differential equation of nth order can be written in the following general form

$$y^{(n)} + f_{n-1}(x)y^{(n-1)} + \ldots + f_1(x)y^{'} + f_0(x)y = r(x) \tag{68}$$

where the function r on the right-hand side and the coefficient $f_0, f_1, \ldots, f_{n-1}$ are any given functions of x, and $y^{(n)}$ is the nth derivative of y.

Eq. (68) is said to be *homogeneous* if $r(x) = 0$. Then, Eq. (68) becomes

$$y^{(n)} + f_{n-1}(x)y^{(n-1)} + \ldots + f_1(x)y^{'} + f_0(x)y = 0. \tag{69}$$

If $r(x) \neq 0$, Eq. (68) is said to be *nonhomogeneous*.

A function $y = \phi(x)$is called a *solution* of a differential equation of nth order on an interval I if $\phi(x)$ is defined and n times differentiable on I and is such that the equation becomes an identity when we replace the unspecified function y and its derivatives in the equation by ϕ and its corresponding derivatives [1].

Existence and uniqueness theorem [1], [3]. *If $f_0(x), \ldots, f_{n-1}(x)$ in Eq. (69) are continuous functions on an open interval I, then the initial value problem consisting of the equation Eq. (69) and the n initial conditions*

$$y(x_0) = K_1, y^{'}(x_0) = K_2, \ldots, y^{(n-1)}(x_0) = K_n,$$

has a unique solution $y(x)$ on I; here x_0 is any fixed point in I, and $K_1, \ldots, K_n$ are given numbers .

A set of functions, $y_1(x), \ldots, y_n(x)$ are *linearly dependent* on some interval I where they are defined, if one of them can be represented on I as a "*linear combination*" of the other $n-1$ functions. Otherwise the functions are *linearly independent* on I.

A *fundamental system* or a *basis* of solutions of the linear homogeneous equation Eq. (69) is a set of n linearly independent solutions $y_1(x), \ldots, y_n(x)$ of that equation.

If $y_1, \ldots, y_n$ is such a fundamental system, then

$$y(x) = c_1 y_1(x) + \ldots + c_n y_n(x) \quad (c_1, \ldots, c_n \text{ arbitrary}) \tag{70}$$

is a general solution of Eq. (69) on I. The test for linear dependence and independence of solutions can be generalized to nth order equations as follows

Theorem [1]. *Suppose that the coefficients $f_0(x), \ldots, f_{n-1}(x)$ of Eq. (69) are continuous on an open interval I. Then n solutions $y_1, \ldots, y_n$ of Eq. (69) on I are linearly dependent on I if and only if their Wronskian*

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{'} & y_2^{'} & \cdots & y_n^{'} \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \tag{71}$$

is zero for some $x = x_0$ in I. (If $W = 0$ at $x = x_0$, then $W \equiv 0$ on I).

Theorem [1]. *Let Eq. (70) be a general solution of Eq. (69) on an open interval I where $f_0(x), \ldots, f_{n-1}(x)$ are continuous, and let $Y(x)$ be any solution of Eq. (69) on I involving no arbitrary constants. Then $Y(x)$ is obtained from Eq. (70) by assigning suitable values to the arbitrary constants $c_1, \ldots, c_n$.*

Example. The equation

$$y^{'''} - 2y^{''} - y^{'} + 2y = 0. \tag{72}$$

has the solutions $y_1 = e^x, y_2 = e^{2x}$, and $y_3 = e^{3x}$.
The Wronskian is

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0,$$

which shows that the functions constitute a fundamental system of solutions of Eq. (72). The corresponding general solution is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

4.2 Homogeneous linear equations with constant coefficients

A linear homogeneous equation of order n with constant coefficients

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y^{'} + a_0 y = 0, \tag{73}$$

has the correspondent characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0. \tag{74}$$

If this equation has n distinct roots $\lambda_1, \ldots, \lambda_n$, then the n solutions

$$y_1 = e^{\lambda_1 x}, \quad \ldots, \quad y_n = e^{\lambda_n x} \tag{75}$$

constitute a fundamental system for all x, and the corresponding general solution of Eq. (73) is

$$y = c_1 e^{\lambda_1 x} + \ldots + c_n e^{\lambda_n} x. \tag{76}$$

If λ is a root of order m, then

$$e^{\lambda x}, x e^{\lambda x}, \ldots, x^{m-1} e^{\lambda x} \tag{77}$$

are m linearly independent solutions of Eq. (73) corresponding to that root.

Example. Consider the differential equation

$$y''' + 3y'' - 4y' - 12y = 0.$$

The characteristic equation

$$\lambda^3 + 3\lambda^2 - 4\lambda + 12 = 0$$

has the solutions $\lambda_1 = -2$, $\lambda_2 = 2$, and $\lambda_3 = -3$, and the corresponding general solution Eq. (76) is

$$y = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{-3x}.$$

4.3 Linear differential equations in state space form

The nth-order differential equation

$$a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0 y = 0,$$

can be transformed into a system of n first order equations.

With the notations

$$x_1(t) = y,\ x_2(t) = dy/dt,\ \ldots\ x_n(t) = d^{n-1}y/dt^{n-1},$$

we obtain the system

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = x_3, \ldots, \quad \frac{dx_{n-1}}{dt} = x_n,$$

and

$$\frac{dx_n}{dt} = -\frac{a_{n-1}(t)x_n + a_{n-2}(t)x_{n-1} + \ldots + a_0 x_1}{a_n(t)}.$$

A system of n first-order linear equations has the general form

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}(t)x_1 + \ldots + a_{1n}(t)x_n + g_1(t), \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + \ldots + a_{nn}(t)x_n + g_n(t), \end{aligned} \tag{78}$$

and is said to be *nonhomogeneous* ($g_i(t) \neq 0$, $i = 1, \ldots, n$).

The system

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}(t)x_1 + \ldots + a_{1n}(t)x_n, \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + \ldots + a_{nn}(t)x_n, \end{aligned} \tag{79}$$

is said to be *homogeneous* ($g_i(t) = 0,\ i = 1, \ldots, n$).

The homogeneous linear system with constant coefficients (a_{ij} do not depend on t)

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + \ldots + a_{1n}x_n, \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + \ldots + a_{nn}x_n, \end{aligned} \tag{80}$$

can be written in matrix notation as

$$\dot{\mathbf{x}} = A\,\mathbf{x}, \tag{81}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}.$$

Theorem (existence-uniqueness theorem) [2]. *There exists one, and only one, solution of the initial-value problem for* $-\infty < t < \infty$

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}. \tag{82}$$

The dimension of the space of all solutions of the homogeneous linear system of differential equations (81) is n.

4.3.1 Solution via the eigenvalue-eigenvector method

Consider the linear homogeneous differential system

$$\dot{\mathbf{x}} = A\mathbf{x}. \tag{83}$$

Assuming a solution of the form

$$\mathbf{x}(t) = e^{\lambda t}\,\mathbf{v}, \;\; \mathbf{v} = \text{constant vector.}$$

Eq. (83) becomes

$$\lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v},$$

or

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{84}$$

The solution of Eq. (83) is $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ if, and only if, λ and $\mathbf{v}$ satisfy Eq. (84). A vector $\mathbf{v} \neq \mathbf{0}$ satisfying Eq. (84) is called an *eigenvector* of A with *eigenvalue* λ.

The eigenvalues λ of A are the roots of the equation

$$\det(A - \lambda I) = \det \begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{bmatrix} = 0.$$

Case I. Distinct eigenvalues

The matrix A has n linearly independent eigenvectors $\mathbf{v}^1, \dots, \mathbf{v}^n$ with distinct eigenvalues $\lambda_1 \neq \lambda_2 \neq \dots \lambda_{n-1} \neq \lambda_n$. For each eigenvalue λ_j we have an eigenvector $\mathbf{v}^j$ and a solution of Eq. (83) is of the form

$$\mathbf{x}^j(t) = e^{\lambda_j t}\,\mathbf{v}^j.$$

There are n linearly independent solutions $\mathbf{x}^j(t)$ of Eq. (83). Then the general solution of Eq. (83) is given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}^1 + c_2 e^{\lambda_2 t}\mathbf{v}^2 + \dots + c_n e^{\lambda_n t}\mathbf{v}^n. \tag{85}$$

Case II. Complex eigenvalues

If $\lambda = \alpha + i\beta$ is a complex eigenvalue of A with eigenvector $\mathbf{v} = \mathbf{v}^1 + i\mathbf{v}^2$, then a complex-valued solution of Eq. (83) is $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$.

Lemma [2]. *Let* $\mathbf{x}(t) = \mathbf{y}(t) + i\mathbf{z}(t)$ *be a complex-valued solution of Eq. (83). Then both* $\mathbf{y}(t)$ *and* $\mathbf{z}(t)$ *are real-valued solutions of Eq. (83) .*

The function $\mathbf{x}(t)$ can be written as

$$\begin{aligned} \mathbf{x}(t) &= e^{(\alpha+i\beta)t}(\mathbf{v}^1 + i\mathbf{v}^2) \\ &= e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{v}^1 + i\mathbf{v}^2) \\ &= e^{\alpha t}[(\mathbf{v}^1\cos\beta t - \mathbf{v}^2\sin\beta t) + i(\mathbf{v}^1\sin\beta t + \mathbf{v}^2\cos\beta t)]. \end{aligned}$$

If $\lambda = \alpha + i\beta$ is an eigenvalue of A with eigenvector $\mathbf{v} = \mathbf{v}^1 + i\mathbf{v}^2$, then

$$\mathbf{y}(t) = e^{\alpha t}(\mathbf{v}^1\cos\beta t - \mathbf{v}^2\sin\beta t)$$

and

$$\mathbf{z}(t) = e^{\alpha t}(\mathbf{v}^1\sin\beta t + \mathbf{v}^2\cos\beta t)$$

are two real-valued solutions of Eq. (83).

Case III. Equal eigenvalues

If the matrix A does not have n distinct eigenvalues, then A may not have n linearly independent eigenvectors. Let us assume that the $n \times n$ matrix A has only $k < n$ linearly independent eigenvectors. In this case Eq. (83) has only k linearly independent solutions of the form $e^{\lambda t}\mathbf{v}$.

To find additional solutions we present the following method as described in [2]:

1. We pick an eigenvalue λ of A and find all vectors $\mathbf{v}$ for which $(A - \lambda I)^2\mathbf{v} = 0$, but $(A - \lambda I)\mathbf{v} \neq 0$. For each such vector $\mathbf{v}$

$$e^{At}\mathbf{v} = e^{\lambda t}e^{(A-\lambda I)t} = e^{\lambda t}[\mathbf{v} + t(A - \lambda I)\mathbf{v}]$$

is an additional solution of Eq. (83). The process is repeated for all eigenvalues of A.

2. If we still do not have enough solutions, then we find all vectors $\mathbf{v}$ for which $(A - \lambda I)^3\mathbf{v} = 0$, but $(A - \lambda I)^2\mathbf{v} \neq 0$. For each such vector $\mathbf{v}$,

$$e^{At}\mathbf{v} = e^{\lambda t}\left[\mathbf{v} + t(A - \lambda I)\mathbf{v} + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v}\right]$$

is an additional solution of Eq. (83).

3. We keep proceeding in this fashion until n linearly independent solutions are obtained.

4.3.2 Fundamental solution matrix

Definition. A matrix $\mathbf{X}(t)$ whose columns are $\mathbf{x}^1(t), \ldots, \mathbf{x}^n(t)$, the n linearly independent solutions of Eq. (81)

$$\mathbf{X}(t) = \left[\mathbf{x}^1(t)|\mathbf{x}^2(t)|\ldots|\mathbf{x}^n(t)\right].$$

is called the fundamental solution matrix of Eq. (81) Every solution $\mathbf{x}(t)$ can be written in the form

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \ldots + c_n\mathbf{x}^n(t) \tag{86}$$

In the matrix vector form, equation (86) can be written as $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c}$, where $\mathbf{c}$ is a constant vector.

Example [2]. Find a fundamental matrix solution of the system of differential equations

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \mathbf{x}.$$

It can be verified that the three linearly independent solutions of the system are given by

$$e^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \quad e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad e^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, the fundamental matrix solution for the system is

$$\mathbf{X}(t) = \begin{bmatrix} -e^t & e^{3t} & -e^{-2t} \\ 4e^t & 2e^{3t} & e^{-2t} \\ e^t & e^{3t} & e^{-2t} \end{bmatrix}.$$

Theorem [2]. *Let $\mathbf{X}(t)$ be a fundamental solution matrix of the differential equation $\dot{\mathbf{x}} = A\mathbf{x}$. Then*

$$e^{At} = \mathbf{X}(t)\mathbf{X}^{-1}(0). \tag{87}$$

where e^{At} is also a fundamental solution matrix.

We consider the example given in [2] and show as to how e^{At} can be computed. In Eq. (81) let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}.$$

The eigenvalues are computed from the relation

$$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 5-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda)(5-\lambda).$$

Thus we have 3 distinct eigenvalues $\lambda = 1$, $\lambda = 3$, and $\lambda = 5$. The eigenvectors corresponding to those eigenvalues, respectively, are

$$\mathbf{v}^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}^2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}^3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

The three linear independent solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ are

$$\mathbf{x}^1(t) = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{x}^2(t) = e^{3t} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{x}^3(t) = e^{5t} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

The fundamental solution matrix is

$$\mathbf{X}(\mathbf{t}) = \begin{bmatrix} e^t & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix}.$$

We compute

$$\mathbf{X}^{-1}(0) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix},$$

and from the theorem

$$e^{At} = \mathbf{X}(t)\mathbf{X}^{-1}(0) = \begin{bmatrix} e^t & -\frac{1}{2}e^t + \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ 0 & e^{3t} & -e^{3t} + e^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix}.$$

4.3.3 The nonhomogeneous equation

The initial-value problem for a nonhomogeneous equation is

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(t_0) = \mathbf{x}^0. \tag{88}$$

Applying variation of parameter method, the solution is assumed of the form

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{u}(t),$$

where $\mathbf{X}(t) = [\mathbf{x}^1(t), \ldots, \mathbf{x}^n(t)]$, and

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}.$$

Using this relation Eq. (88) yields

$$\dot{\mathbf{X}}(t)\mathbf{u}(t) + \mathbf{X}(t)\dot{\mathbf{u}}(t) = A\mathbf{X}(t)\mathbf{u}(t) + \mathbf{f}(t). \tag{89}$$

Since matrix $\mathbf{X}(t)$ satisfies

$$\dot{\mathbf{X}}(t) = A\mathbf{X}(t), \tag{90}$$

we obtain

$$\mathbf{X}(t)\dot{\mathbf{u}}(t) = \mathbf{f}(t). \tag{91}$$

Matrix $\mathbf{X}(t)$ is nonsingular ($\mathbf{X}^{-1}(t)$ exists) and therefore

$$\dot{\mathbf{u}}(t) = \mathbf{X}^{-1}(t)\mathbf{f}(t). \tag{92}$$

Integrating this expression between t_0 and t we have

$$\begin{aligned} \mathbf{u}(t) &= \mathbf{u}(t_0) + \int_{t_0}^{t} \mathbf{X}^{-1}(s)\mathbf{f}(s)\, ds & (93) \\ &= \mathbf{X}^{-1}(t_0)\mathbf{x}^0 + \int_{t_0}^{t} \mathbf{X}^{-1}(s)\mathbf{f}(s)\, ds. & (94) \end{aligned}$$

Consequently,

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}^0 + \mathbf{X}(t)\int_{t_0}^{t} \mathbf{X}^{-1}(s)\mathbf{f}(s)\, ds. \tag{95}$$

If $\mathbf{X}(t) = e^{At}$ then

$$\mathbf{x}(t) \quad = \quad e^{A(t-t_0)}\mathbf{x}^0 + \int_{t_0}^{t} e^{A(t-s)}\mathbf{f}(s)\, ds. \tag{96}$$

Example. Find the solution of the initial value problem

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}, \quad \mathbf{x}^0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

From the homogeneous problem we can easily show that the fundamental solution matrix is given by

$$\mathbf{X}(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}.$$

It is easily verified that $\dot{\mathbf{X}} = A\mathbf{X}$ and $\mathbf{X}(0) = \mathbf{I}$.

$$\mathbf{X}^{-1}(s) = \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} e^{-s}.$$

and

$$\int_{0}^{t} \mathbf{X}(t)\, \mathbf{X}^{-1}(s)\mathbf{f}(s)\, ds = \begin{bmatrix} \frac{1}{2}(e^t - e^{-t}) \\ 0 \end{bmatrix}.$$

Then from Eq. (95) the solution is given by

$$\mathbf{x}(t) = \begin{bmatrix} (t-1)e^t \\ e^t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(e^t - e^{-t}) \\ 0 \end{bmatrix}.$$

4.4 Equilibrium and stability

Consider the differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \tag{97}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix},$$

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt},$$

and

$$\mathbf{f}(t, \mathbf{x}) = \begin{bmatrix} f_1(t, x_1, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, \ldots, x_n) \end{bmatrix},$$

is a nonlinear function. In general, Eq. (97)cannot be solved explicitly. However, one can easily determine the qualitative properties of solution of Eq. (97) in the neighborhood of an equilibrium point.

The *equilibrium points* are the values

$$\mathbf{x}^0 = \begin{bmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

for which, $\mathbf{x}(t) = \mathbf{x}^0$ is a solution of Eq. (97).

Observe that $\dot{\mathbf{x}}(t)$ is identically zero if $\mathbf{x}(t) \equiv \mathbf{x}^0$. The value $\mathbf{x}^0$ is an equilibrium of Eq. (97), if, and only if,

$$\mathbf{f}(t, \mathbf{x}^0) \equiv 0. \tag{98}$$

Example.[6] Find all equilibrium values of the system of differential equations

$$\frac{dx_1}{dt} = 1 - x_2, \quad \frac{dx_2}{dt} = x_1^3 + x_2.$$

The value

$$\mathbf{x}^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$$

is an equilibrium value if, and only if, $1 - x_2^0 = 0$ and $(x_1^0)^3 + x_2^0 = 0$. This yields $x_2^0 = 1$ and $x_1^0 = -1$. Hence $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the only equilibrium solution of this system.

Stability: Let $\boldsymbol{\phi}(t)$ be a known solution of Eq. (97). Suppose that $\boldsymbol{\psi}(t)$ is a second solution with $\boldsymbol{\psi}(0)$ very close to $\boldsymbol{\phi}(0)$ such that $\beta(t) \equiv \boldsymbol{\psi}(t) - \boldsymbol{\phi}(t)$ can be viewed as the disturbance on $\boldsymbol{\phi}(t)$.

The concept of stability is important in many applications.

Consider the equation of motion of a simple pendulum of mass m and length l given by

$$\frac{d^2y}{dt^2} + \frac{g}{l}\sin y = 0,$$

where y is the angular displacement from the vertical axis and g is acceleration due to gravity. With the notation $x_1 = y$ and $x_2 = dy/dt$ we have

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -\frac{g}{l}\sin x_1. \tag{99}$$

The system of Eq. (99) has equilibrium solutions $\{x_1 = 0, x_2 = 0\}$, and $\{x_1 = \pi, x_2 = 0\}$.

If we disturb the pendulum slightly from the equilibrium position $\{x_1 = 0, x_2 = 0\}$, then it will oscillate with small amplitude about $x_1 = 0$.

If we disturb the pendulum slightly from the equilibrium position $\{x_1 = \pi, x_2 = 0\}$, then it will either oscillate with very large amplitude about $x_1 = 0$, or it will rotate around and around.

The two solutions have very different properties, and, intuitively, we would say that the equilibrium value $\{x_1 = 0, x_2 = 0\}$ is stable, while the equilibrium point $\{x_1 = \pi, x_2 = 0\}$ is unstable.

In the case when $\mathbf{f}(t, \mathbf{x})$ does not depend explicitly on t i.e. $\mathbf{f} = \mathbf{f}(\mathbf{x})$ the differential equations are called *autonomous*.

4.5 Phase-plane

Let us consider a two dimensional system

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y). \tag{100}$$

Every solution $x = x(t)$, and $y = y(t)$ of Eq. (100) defines a curve in the three-dimensional space $\{t, x, y\}$.
For example the solution of the system of differential equations

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x,$$

is $x = \cos t$, $y = \sin t$. This solution describes a helix in three-dimensional space $\{t, x, y\}$.

Every solution $x = x(t)$, and $y = y(t)$, of Eq. (100), for $t_0 \le t \le t_1$, also defines a curve in the $x-y$ plane. This curve is called the *orbit*, or *trajectory*, of the solution $x = x(t)$, $y = y(t)$, and the xy plane is called the *phase-plane* of the solutions of Eq. (100).

In the general case let $\mathbf{x}(t)$ be a solution of the vector differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix} \tag{101}$$

on the interval $t_0 \le t \le t_1$. As t runs from t_0 to t_1, the set of points $(x_1(t), \dots, x_n(t))$ trace out a curve C in the n-dimensional space $x_1, x_2, \dots, x_n$. This curve is called the orbit of the solution $\mathbf{x} = \mathbf{x}(t)$, for $t_0 \le t \le t_1$, and the n-dimensional space $x_1, \dots, x_n$ is called the "phase-space" or "state-space" of the solution of Eq. (101).

4.6 Linear approximation at equilibrium points [5]

Consider again the Eq. (100)

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y),$$

with $f(0, 0) = g(0, 0) = 0$ as the equilibrium point. Using Taylor expansion about this point, we can write

$$f(x, y) = ax + by + P(x, y), \quad g(x, y) = cx + dy + Q(x, y),$$

where $P(x, y) = \mathbf{O}(r^2)$ and $Q(x, t) = \mathbf{O}(r^2)$ as $r = \sqrt{x^2 + y^2} \to 0$, and

$$a = \frac{\partial f}{\partial x}(0, 0), \quad b = \frac{\partial f}{\partial y}(0, 0), \tag{102}$$

$$c = \frac{\partial g}{\partial x}(0, 0), \quad d = \frac{\partial g}{\partial y}(0, 0). \tag{103}$$

The *linear approximation* of Eq. (100) in the neighbourhood of the origin is defined as the system

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy,$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \tag{104}$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}. \tag{105}$$

The solutions of Eq. (104) are geometrically similar to those of Eq. (100) near the origin unless one (or more) of the eigenvalues of $\mathbf{A}$ is zero or has zero real part.

The two linearly independent solutions are of the form

$$\mathbf{x} = \mathbf{u}\,e^{\lambda t}, \tag{106}$$

where

$$\mathbf{u} = \begin{bmatrix} r \\ s \end{bmatrix} \neq \mathbf{0}. \tag{107}$$

Then

$$\dot{\mathbf{x}} = \lambda\,\mathbf{u}\,e^{\lambda t}$$

, and equations (104) and (106) yield

$$(A - \lambda I)\,\mathbf{u} = 0 \tag{108}$$

where I is the identity matrix. With $\mathbf{u} \neq \mathbf{0}$ and Eq. (108), we have

$$\det(A - \lambda I) = 0,$$

or

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0. \tag{109}$$

The two eigenvalues are given by the solution of the quadratic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0. \tag{110}$$

The solutions of Eq. (108) are the *eigenvectors*: $\mathbf{u}_1$ corresponding to λ_1, and $\mathbf{u}_2$ corresponding to λ_2. The general solution of Eq. (104) is

$$\mathbf{x} = C_1 \mathbf{u}_1 e^{\lambda_1 t} + C_2 \mathbf{u}_2 e^{\lambda_2 t}, \text{ for } \lambda_1 \neq \lambda_2. \tag{111}$$

Using the nonsingular linear transformation

$$\mathbf{x}_1 = S\mathbf{x}; \quad S = [\mathbf{u_1}\ \mathbf{u_2}], \tag{112}$$

Eq. (104) becomes

$$\dot{\mathbf{x}}_1 = SAS^{-1}\mathbf{x}_1 = B\mathbf{x}_1, \tag{113}$$

where B is diagonal or in Jordan form. The topological character of the transformed equilibrium point at the origin is not affected in the new variable $\mathbf{x}_1 = [x_1,\, y_1]^T$. The equations in the new coordinates are simpler.

Case I. $\lambda_1 \neq \lambda_2 \neq 0$ and $\lambda_1, \lambda_2 \in \mathcal{R}$ (real)
We can choose S so that

$$\dot{x}_1 = \lambda_1 x_1, \quad \dot{y}_1 = \lambda_2 y_1,$$

and then the equation for the phase paths is

$$\frac{dy_1}{dx_1} = \frac{\lambda_2}{\lambda_1}\,\frac{y_1}{x_1}.$$

The solutions are

$$y_1 = C\,|x_1|^{\lambda_2/\lambda_1}, \text{ where } C = \text{arbitrary}.$$

The origin is a *node* (Figure 1) when $\lambda_2/\lambda_1 > 0$. The node is stable when $\lambda_1, \lambda_2 < 0$ (Figure 1) and unstable when $\lambda_1, \lambda_2 > 0$.

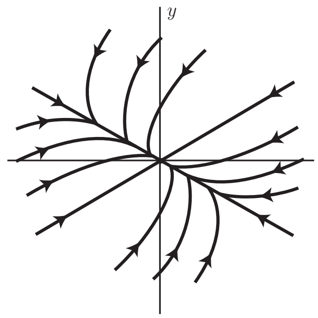

Figure 1: Stable node

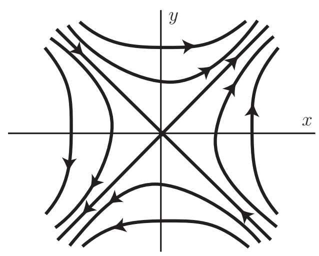

Figure 2: Saddle point

The origin is a *saddle-point* (Figure 2) when $\lambda_2/\lambda_1 < 0$.

Case II. $\lambda_1 = \lambda_2 = \lambda$ (b and c not both zero)
We can choose S so that

$$\dot{x}_1 = \lambda x_1 + y_1, \quad \dot{y}_1 = \lambda y_1, \quad \lambda \in \mathcal{R},$$

and then the equation for the phase paths is

$$\frac{dy_1}{dx_1} = \frac{\lambda y_1}{\lambda x_1 + y_1}.$$

The solutions are

$$y_1 = 0, \;\; x_1 = \frac{1}{\lambda} y_1 \log_e |y_1| + C y_1 \;\text{ where }\; C = \text{arbitrary}.$$

The origin is a *inflected node*, stable if $\lambda < 0$ (Figure 3) and unstable if $\lambda > 0$.

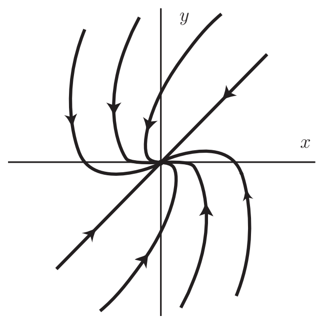

Figure 3: Stable inflected node

Case III. $\lambda_1 = \overline{\lambda}_2 = \alpha + \mathrm{i}\beta$ with $\beta \neq 0$
We can choose S so that the equations become

$$\dot{x}_1 = \alpha x_1 - \beta y_1, \quad \dot{y}_1 = \beta x_1 + \alpha y_1.$$

With $z(t) = x_1(t) + \mathrm{i}\, y_1(t) = r(t)e^{\mathrm{i}\theta(t)}$ we have $\dot{z} = (\alpha + \mathrm{i}\beta)z$, and $r(t) = |z(t)|$. The equations in polar coordinates are

$$\dot{r} = \alpha r, \quad \dot{\theta} = \beta.$$

The origin is a stable *spiral* (or *focus*) if $\alpha < 0, \beta \neq 0$ (Figure 4), and an unstable spiral if $\alpha > 0, \beta \neq 0$.

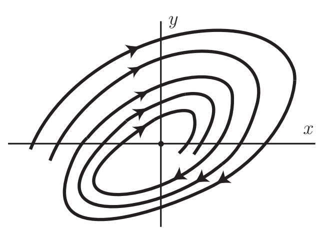

Figure 4: Stable spiral

The origin is a *center* if $\alpha = 0, \beta \neq 0$, (Figure 5).

We can sumarize all the above cases in the following table [5], Figure 6

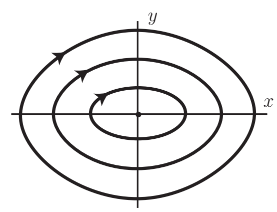

Figure 5: Center

(1) λ_1, λ_2	real, unequal, same sign	Node
(2) $\lambda_1 = \lambda_2$	(real)$b \neq 0, c \neq 0$	Inflected node
(3) λ_1, λ_2	complex, non-zero real part	Spiral
(4) $\lambda_1 \neq 0, \lambda_2 = 0$		Parallel lines
(5) λ_1, λ_2	real, different sign	Saddle point
(6) λ_1, λ_2	pure imaginary	Center

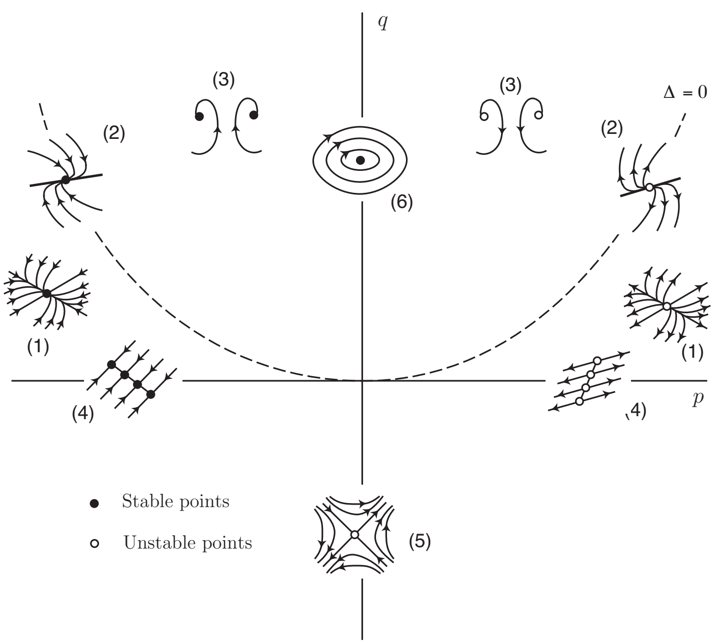

Figure 6: General classification

Example. Classify the equilibrium point at (0,0) for the system

$$\dot{x} = e^{-x-3y} - 1, \quad \dot{y} = -x(1-y^2).$$

Using Taylor expansion for the exponential function, the linearized system of equations about (0,0) is

$$\dot{x} = -x - 3y, \quad \dot{y} = -x,$$

or in matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues are $\lambda_{1,2} = \dfrac{-1 \pm \sqrt{17}}{2}$ are real with different sign. The equilibrium is a saddle point.

5 Partial differential equations

The word "ordinary" in *ordinary differential equation* distinguishes it from *partial differential equation* (PDE), involves partial derivatives of two or more independent variables. For a first order partial differential equation, a unified general theory exists; however, this a case for higher order partial differential equations. Generally speaking, the second order PDEs may be classified into three following categories, viz., elliptic, hyperbolic, and parabolic types.

5.1 Normal forms of elliptic, hyperbolic, and parabolic Equations

Consider a linear second order differential operator for the function $u(x, y)$ given by

$$L(u) = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2}, \tag{114}$$

where a, b, and c are either constants or functions of x and y. A corresponding quasilinear PDE may be represented by

$$L(u) + g(x, y, \partial u/\partial x, \partial u/\partial y) = L(u) + ... = 0, \tag{115}$$

where $g(x, y, \partial u/\partial x, \partial u/\partial y)$ is not necessarily linear and does not contain any second derivative.

Let us introduce the transformations

$$\begin{aligned}\xi &= \alpha x + \beta y,\\ \eta &= \gamma x + \delta y.\end{aligned} \tag{116}$$

Therefore, $L(u)$ in Eq. (114) takes the form

$$\begin{aligned}L(u) &= (a\alpha^2 + b\alpha\beta + c\beta^2)\frac{\partial^2 u}{\partial \xi^2}\\ &+(2a\alpha\gamma + b(\alpha\delta + \beta\gamma) + 2c\beta\delta)\frac{\partial^2 u}{\partial \xi \partial \eta}\\ &+(a\gamma^2 + b\gamma\delta + c\delta^2)\frac{\partial^2 u}{\partial \eta^2}\ .\end{aligned} \tag{117}$$

If the transformed operator is desired to be of the form $\dfrac{\partial^2 u}{\partial \xi \partial \eta}$, then we need

$$a\alpha^2 + b\alpha\beta + c\beta^2 = 0, \tag{118}$$

$$a\gamma^2 + b\gamma\delta + c\delta^2 = 0. \tag{119}$$

If $a = c = 0$, then the trivial transformation $\xi = y$ and $\eta = y$ provides the desired form. For the non-trivial case either a or c or both are non-zero. Let us say $a \neq 0$, thereby implying that $\alpha \neq 0, \gamma \neq 0$. Dividing Eq. (118) by β^2 and Eq. (119) by δ^2, we obtain two quadratic equations in (α/β) and (γ/δ). These yield

$$\alpha/\beta = \frac{1}{2a}\{-b \pm \sqrt{b^2 - 4ac}\}, \tag{120}$$

$$\gamma/\delta = \frac{1}{2a}\{-b \pm \sqrt{b^2 - 4ac}\}. \tag{121}$$

The ratios α/β and γ/δ must be different (by choosing positive sign in Eq. (120) and negative sign in Eq. (121)) so that the transformation given by Eq. (116) is non-singular. Further $b^2 - 4ac$ should be positive.

Therefore, $L(u)$ reduces to the form $\dfrac{\partial^2 u}{\partial \xi \partial \eta}$ if and only if

$$b^2 - 4ac > 0, \tag{122}$$

and this case is said to be "hyperbolic". Then the transformation Eq. (116) takes the form

$$\begin{aligned} \xi &= (-b + \sqrt{b^2 - 4ac})x + 2ay, \\ \eta &= (-b - \sqrt{b^2 - 4ac})x + 2ay. \end{aligned} \tag{123}$$

Then the PDE given by Eq. (115) reduces to

$$-4a(b^2 - 4ac)\frac{\partial^2 u}{\partial \xi \partial \eta} + g(\xi, \eta, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}) = 0. \tag{124}$$

If $b^2 - 4ac = 0$, then L is termed as "parabolic". In this case Eq. (120) and Eq. (121) reduce to a single equation and $\alpha/\beta = -b/2a$ forces the coefficient of $\partial^2 u/\xi^2$ in Eq. (117) to vanish. Further, since $b^2 = 4ac$ or $b/2a = 2c/b$, the coefficient of $\dfrac{\partial^2 u}{\partial \xi \partial \eta}$ also vanish. Thus the transformation (c.f. Eq. (123))

$$\begin{aligned} \xi &= -bx + 2ay, \\ \eta &= x \quad \text{(arbitrary)}, \end{aligned} \tag{125}$$

can be used to transform Eq. (115) into

$$a\frac{\partial^2}{\partial \eta^2} + g(\quad) = 0. \tag{126}$$

This is the normal form of a parabolic quasilinear PDE.

For the final case, $b^2 - 4ac < 0$, and the operator $L(u)$ is said to be "elliptic". In this case it is not possible to eliminate the coefficients of $\dfrac{\partial^2 u}{\partial \xi^2}$ or $\dfrac{\partial^2 u}{\partial \eta^2}$. Nevertheless, if we use the transformation

$$\xi = \frac{2ay - bx}{\sqrt{4ac - b^2}}, \quad \eta = t \quad \text{(arbitrary)}, \tag{127}$$

then $L(u) = a\left(\dfrac{\partial^2 u}{\partial \xi^2} + \dfrac{\partial^2 u}{\partial \eta^2}\right)$, and the general PDE has the form

$$a\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) + g(\xi, \eta, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}) = 0. \tag{128}$$

For the linear case

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = 0, \tag{129}$$

which is the well-known Laplace's equation.
Once a PDE has been reduced to its normal form, the method of characteristic may be effectively used to find its solution.

However, in the following we discuss the solution of a particular hyperbolic equation, known as the "wave equation" by the use of "separation of the variables" which is a popular approach in engineering.

The equation

$$c^2 \frac{\partial^2 u(x,t)}{\partial x^2} - \ddot{u}(x,t) = 0, \quad c = \text{constant}, \tag{130}$$

is a partial differential equation. The following notation was used

$$\ddot{u}(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2}.$$

The initial conditions are

$$u(x,0) = f(x), \quad \dot{u}(x,0) = g(x). \tag{131}$$

The boundary conditions are

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0. \tag{132}$$

We seek the solution of Eq. (130) in the form of a product of a function of time and a function of position

$$u(x,t) = U(x)\varphi(t). \tag{133}$$

Introducing (133) into (130), we replace Eq. (130) by the system of two ordinary equations

$$\ddot{\varphi} + \beta^2 c^2 \varphi = 0, \tag{134}$$

$$\frac{d^2 U}{dx^2} + \beta^2 U = 0, \tag{135}$$

where β is for the time being an undetermined parameter. The solution of Eqs. (134) and (135) is

$$\varphi(t) = A\sin\omega t + B\cos\omega t, \tag{136}$$

$$U(x) = C\sin\beta x + D\cos\beta x, \tag{137}$$

where $\omega = \beta c$.

We first consider the second boundary conditions (132). They imply that $C = 0$ and

$$D\beta\sin\beta l = 0.$$

The latter condition is satisfied if

$$\beta_n = \frac{n\pi}{l}, \qquad (n = 0, 1, 2, \ldots, \infty). \tag{138}$$

It is evident that every value of β_n is associated with a particular solution of Eq. (130), viz.

$$u_n(x,t) = (A_n\sin\omega_n t + B_n\cos\omega_n t)D\cos\beta_n x. \tag{139}$$

The general solution of (130) takes the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty}(A_n\sin\omega_n t + B_n\cos\omega_n t)D\cos\beta_n x. \tag{140}$$

The constants A_n, B_n are to be found from the initial conditions (131) i.e.

$$f(x) = \sum_{n=0}^{\infty} DB_n\cos\beta_n x, \qquad g(x) = \sum_{n=0}^{\infty} D\omega_n A_n\cos\beta_n x. \tag{141}$$

The functions

$$U_n(x) = D\cos\beta_n x, \tag{142}$$

are the eigenfunctions of the problem. They are orthogonal, i.e.

$$\int_0^l U_n(x)U_m(x)\,dx = 0, \text{ if } n \neq m,$$

$$\int_0^l U_n^2(x)\,dx = \frac{l}{2}D^2, \text{ if } n = m, \tag{143}$$

as can easily be verified by integration. The constant D is arbitrary.

Assume that $D^2 = 2/l$. Then $\int_0^l U_n^2(x)\,dx = 1$ and the eigenfunctions

$$U_n(x) = \sqrt{\frac{2}{l}}\cos\beta_n x = \sqrt{\frac{2}{l}}\cos\frac{n\pi x}{l}, \tag{144}$$

are called normalized eigenfunctions.

Making use of the normalized eigenfunctions we can rewrite relations (141) in the form

$$f(x) = \sqrt{\frac{2}{l}}\sum_{n=0}^{\infty} B_n\cos\beta_n x, \quad g(x) = \sqrt{\frac{2}{l}}\sum_{n=0}^{\infty}\omega_n A_n\cos\beta_n x. \tag{145}$$

To find the coefficient B_n we multiply the first equation (145) by $\cos\beta_n x$ and integrate with respect to x from 0 to l. Then, making use of the orthogonality relations, we obtain

$$\begin{aligned} B_n &= \sqrt{\frac{2}{l}}\int_0^l f(x)\cos\beta_n x\,dx, \quad (n = 1, 2, \ldots, \infty), \\ B_0 &= \frac{1}{2}\sqrt{\frac{2}{l}}\int_0^l f(x)\,dx. \end{aligned} \tag{146}$$

Similarly we have

$$\begin{aligned} A_n &= \frac{1}{c\beta_n}\sqrt{\frac{2}{l}}\int_0^l g(x)\cos\beta_n x\,dx, \quad (n = 1, 2, \ldots, \infty), \\ A_0 &= 0. \end{aligned} \tag{147}$$

Introducing the values of A_n, B_n into Eq. (140), we arrive at the final solution.

Example. We consider next the equation

$$c^2\frac{\partial^2 u}{\partial x^2} - \ddot{u} = 0, \tag{148}$$

with assuming homogeneous initial conditions ($u(x, 0) = \dot{u}(x, 0) = 0$) and the boundary conditions

$$u(0, t) = 0, \quad \frac{\partial u}{\partial x}(l, t) = P(t). \tag{149}$$

Performing the Laplace transform in Eq. (148) for the above boundary conditions, we obtain

$$c^2 \frac{d^2\overline{u}}{dx^2} - s^2\overline{u} = -su(x,0) - \dot{u}(x,0), \tag{150}$$

$$\overline{u}(0,s) = 0; \frac{d\overline{u}}{dx}(l,s) = \overline{P}(s), \tag{151}$$

where

$$\overline{u}(x,s) = \int_0^\infty e^{-st}u(x,t)\,dt, \quad \overline{P}(s) = \int_0^\infty e^{-st}P(t)\,dt.$$

The right-hand side of Eq. (150) vanishes in view of the homogeneous initial conditions, hence its solution can be represented in the form

$$\overline{u}(x,s) = A(s)e^{-sx/c} + B(s)e^{sx/c}. \tag{152}$$

The functions $A(s)$, $B(s)$ can be determined by means of the boundary conditions (151):

$$\begin{aligned} A(s) &= -B(s), \\ B(s) &= \frac{\overline{P}(s)c}{2s\,\cosh\dfrac{sl}{c}}. \end{aligned} \tag{153}$$

Hence

$$\overline{u}(x,s) = \frac{\overline{P}(s)l}{2}\,\frac{e^{sx/c} - e^{-sx/c}}{\dfrac{sl}{c}\cosh\dfrac{sl}{c}},$$

i.e.

$$\overline{u}(x,s) = \frac{\overline{P}(s)l\sinh\dfrac{sx}{c}}{\dfrac{sl}{c}\cosh\dfrac{sl}{c}}. \tag{154}$$

Now we invert the Laplace transform in (154). Taking into account that

$$\begin{aligned} L^{-1}\left(\frac{\sinh\dfrac{sx}{c}}{s\cosh\dfrac{sl}{c}}\right) &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n-\dfrac{1}{2}}\sin\left[\left(n-\frac{1}{2}\right)\frac{\pi x}{l}\right] \\ &\quad \sin\left[\left(n-\frac{1}{2}\right)\frac{\pi lt}{c}\right], \end{aligned}$$

and

$$L^{-1}\overline{P}(s) = P(t),$$

and making use of the convolution theorem was obtain

$$\begin{aligned} u(x,t) &= \frac{2c}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n-\dfrac{1}{2}}\sin\left[\left(n-\frac{1}{2}\right)\frac{\pi x}{l}\right] \\ &\quad \int_0^t P(\tau)\sin\left[\frac{2n-1}{2}\frac{\pi l}{c}(t-\tau)\right]d\tau. \end{aligned} \tag{155}$$

In the particular case

$$P(t) = P_0\, H(t),$$

where $H(t)$ is the Heaviside function, we have from Eq. (155)

$$\begin{aligned} u(x,t) &= \frac{8P_0c^2}{\pi^2 L}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin\frac{2n-1}{2}\frac{\pi x}{l} \\ &\quad \left[1-\cos\frac{(2n-1)\pi l t}{2c}\right]. \end{aligned} \tag{156}$$

Assume that $P(t) = P_0 e^{i\omega t}$ acts at the end $x = 0$ of the fixed rod. Taking into account that $u(x,t) = U(x)e^{i\omega t}$, we transform Eq. (148) to the form

$$c^2\frac{d^2U}{dx^2} + \omega^2 U = 0. \tag{157}$$

The boundary conditions take the form

$$U(0) = 0, \quad \frac{dU}{dx}(l) = P_0. \tag{158}$$

The constants A, B appearing in the solution

$$U(x) = A\sin\frac{\omega x}{c} + B\cos\frac{\omega x}{c}, \tag{159}$$

of Eq. (157) are determined from the boundary conditions (158). Finally we obtain

$$u(x,t) = \frac{P_0 c e^{i\omega t}}{\omega} \frac{\sin \dfrac{\omega x}{c}}{\cos \dfrac{\omega l}{c}}. \tag{160}$$

If the frequency ω approaches any of the eigenfrequency, the displacement u tends to infinity. Thus, we are faced with resonance.

6 Applications

Problem 1. A sphere of mass m falls on a vertical spring as shown in the Figure 7. The sphere makes contact with the spring and the spring compresses. The compression phase ends when the velocity of the sphere is zero. Next phase is the restitution phase when the spring is expanding and the sphere is moving upward. At the end of the restitution phase there is the separation of the sphere.

Find and solve the equation of motion for the sphere in contact with the spring.

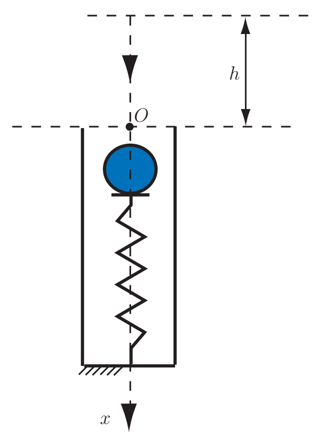

Figure 7: Sphere in contact with a spring

Solution

The x-axis selected downward as shown in the Figure 7.

At the moment $t = 0$ it is assumed that the sphere gets in contact with the spring and has the velocity $\mathbf{v}(t = 0) = \mathbf{v}_0 = v_0\,\boldsymbol{\imath}$.

Using Newton's second law, the equation of motion for the sphere in contact with the spring is:

$$m\,\mathbf{a} = \mathbf{G} + \mathbf{F}_e \quad \text{or} \quad m\,\ddot{x} = m\,g - k\,x. \tag{161}$$

The acceleration of the sphere is $\mathbf{a} = \ddot{x}\,\boldsymbol{\imath}$, where x is the linear displacement. The weight of the sphere is $\mathbf{G} = m\,g\,\boldsymbol{\imath}$, where g is the gravitational acceleration. The contact elastic force is $\mathbf{F}_e = -k\,x\,\boldsymbol{\imath}$, where k is the elastic constant of the spring. The initial conditions are

$$x(0) = 0 \quad \text{and} \quad \dot{x}(0) = v_0.$$

With the notation

$$\frac{k}{m} = \omega^2, \quad (\omega > 0),$$

Equation (161) becomes

$$\ddot{x} + \omega^2 x = g. \tag{162}$$

Assume the solution of Eq. (162) has the following expression

$$x = a\,\cos(\omega\,t - \varphi_0) + b. \tag{163}$$

Then

$$\dot{x} = -a\,\omega\,\sin(\omega\,t - \varphi_0) \quad \text{and} \quad \ddot{x} = -a\,\omega^2\,\cos(\omega\,t - \varphi_0).$$

Substituting Eq. (164) into Eq. (162)

$$-a\,\omega^2\,\cos(\omega\,t - \varphi_0) + \omega^2[a\,\cos(\omega\,t - \varphi_0) + b] = g,$$

the constant b is obtained

$$b = \frac{g}{\omega^2}. \tag{164}$$

Using the initial conditions ($x(0) = 0$ and $\dot{x}(0) = v_0$) the following expressions are obtained

$$x(0) = a\,\cos(-\varphi_0) + b = a\,\cos\varphi_0 + b = 0,$$
$$\dot{x}(0) = -a\,\omega\,\sin(-\varphi_0) = a\,\omega\sin\varphi_0 = v_0,$$

or

$$a\,\cos\varphi_0 = -b = -\frac{g}{\omega^2} \quad \text{and} \quad a\,\sin\varphi_0 = \frac{v_0}{\omega}.$$

It results

$$a = \sqrt{\frac{g^2}{\omega^4} + \frac{v_0^2}{\omega^2}},$$
$$\tan\,\varphi_0 = -\frac{v_0\,\omega}{g} \quad \text{or} \quad \varphi_0 = -\arctan\frac{v_0\,\omega}{g}. \tag{165}$$

The relation for the displacement of the sphere is

$$x - \frac{g}{\omega^2} = \left(\sqrt{\frac{g^2}{\omega^4} + \frac{v_0^2}{\omega^2}} \right) \cos\left(\omega\, t + \arctan \frac{v_0\, \omega}{g} \right). \tag{166}$$

If the sphere would be connected to the spring, it would oscillate around the position $x = \dfrac{g}{\omega^2}$.

The sphere reaches the maximum position on x-axis at $t = t_1$ when $\dot{x}(t_1) = 0$

$$\dot{x}(t_1) = -a\, \omega\, \sin(\omega\, t_1 - \varphi_0) = 0 \quad \Longrightarrow \quad \omega\, t_1 - \varphi_0 = \pi$$

or

$$t_1 = \frac{\pi}{\omega} + \frac{1}{\omega}\, \varphi_0 = \frac{\pi}{\omega} - \frac{1}{\omega} \arctan \frac{v_0\, \omega}{g}. \tag{167}$$

At the moment $t = t_2 = 2\, t_1$, the sphere attains again the reference O. At this moment, the sphere separates itself and moves upward, and the spring compresses. The velocity of the sphere at $t = t_2$ is

$$\dot{x}(t_2) = a\, \omega\, \sin(\omega\, t_2 - \varphi_0) = -v_0. \tag{168}$$

The contact time between the sphere and the spring is:

$$t_2 = 2\, t_1 = \frac{2\, \pi}{\omega} - \frac{2}{\omega} \arctan \frac{v_0\, \omega}{g}. \tag{169}$$

The jump in velocity is

$$\Delta v = \dot{x}(0) - \dot{x}(t_2) = v_0 - (-v_0) = 2\, v_0. \tag{170}$$

The displacement at t_1 is

$$x(t_1) = a\, \cos(\omega\, t_1 - \varphi_0) + b = a + b = \sqrt{\frac{g^2}{\omega^4} + \frac{v_0^2}{\omega^2}} + \frac{g}{\omega^2},$$

and the relative displacement is:

$$\lambda = x(0) - x(t_1) = 0 - x(t_1) = -\left(\frac{g}{\omega^2} + \sqrt{\frac{g^2}{\omega^4} + \frac{v_0^2}{\omega^2}} \right). \tag{171}$$

Numerical Example. The sphere with mass $m = 10$ kg falls falls from the height $h = 1$ m on the spring with the elastic constant $k = 294 \times 10^3$ N/m. The initial velocity of the sphere is $v_0 = \sqrt{2\,g\,h} = 4.42945$ m/s.
The total time of contact t_2 is calculated with Eq. (169), $t_2 = 0.0184728$ s. The relative displacement is $|\lambda| = 0.0261689$ m and the jump in velocity is calculated with Eq. (170), $\Delta = 8.85889$ m/s.
The maximum elastic force is $F_{e\,max} = k\,x(t_1) = k\,|\lambda| = 7693.65$ N. The maximum elastic force is approximative 76 greater then the weight of the sphere. For this dynamical problem the displacement of the sphere is very small, almost null, while the the jump in velocity is big.

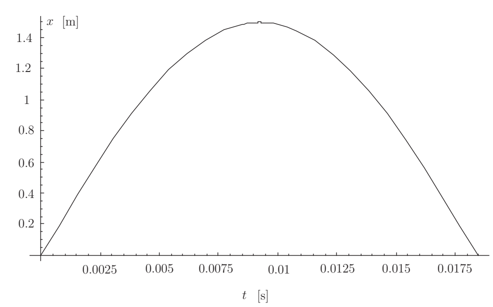

Figure 8: Displacement of the sphere

Figure 8 represents the dependence of the displacement of the sphere with respect to time calculated with Eq. (166). Figure 9 shows the variation in time of the velocity of the sphere in contact with the spring. At $t = 0$ the sphere gets in contact with the spring and at $t = t_2$ the sphere separates from the spring. Note that the initial velocity is equal with the absolute value of the final velocity. Figure 10 shows the variation of the elastic force with respect to time.

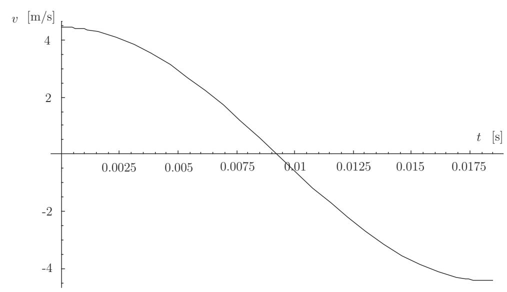

Figure 9: Velocity of the sphere

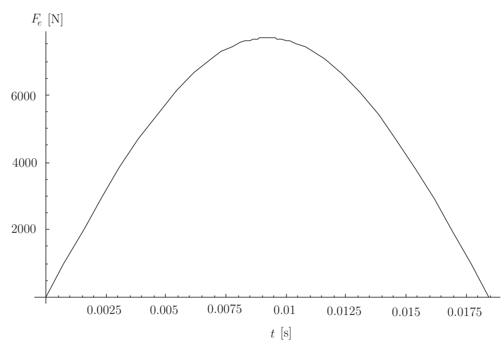

Figure 10: Elastic force

Problem 2. A rod AB with the mass M and the length $6\,a$ is connected to the ground at the pin joint O as shown in Figure 11.a. A mass m is attached to the rod at point A. The rod is connected to two springs, with the elastic constant k, as depicted in Figure 11.a.

Determine the equation of motion of the system for small oscillations if the initial angular velocity of the rod is ω_0. The gravitational acceleration is g.

Solution

At equilibrium the rod rotated around the pivot O with the angle θ_s (Figure 11.b). The sum of the moments of the forces acting on the rod with respect to O are

$$\sum M_0^{\text{equil}} \implies m\,g\,(2\,a) + k\,a\,\theta_s\,a - M\,g\,a + k\,(4\,a)\,\theta_s\,(4\,a) = 0,$$

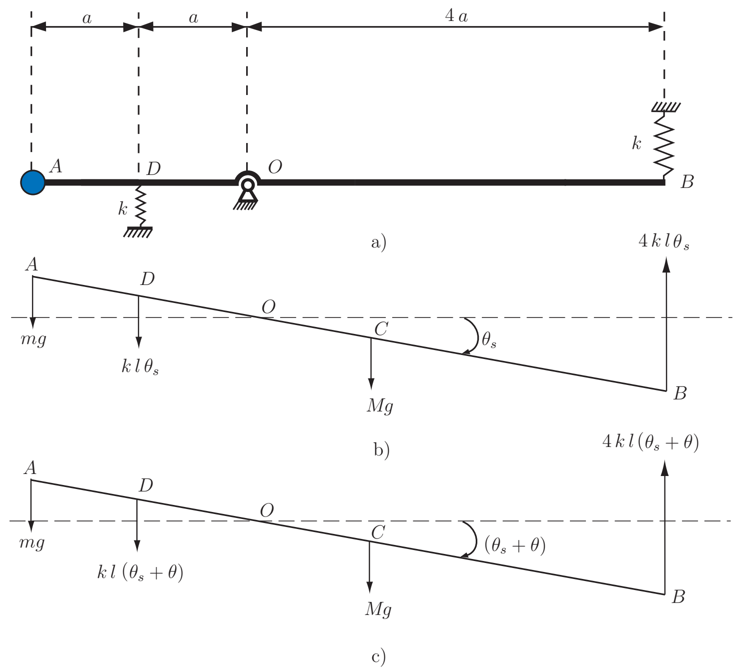

Figure 11: Small vibrations of a rod

or

$$a\,(2\,m\,g - M\,g + 17\,k\,a\,\theta_s) = 0. \tag{172}$$

The equation of motion of the rod in rotation is

$$-I_O\,\ddot{\theta} = M_O,$$

where I_O is the mass moment of inertia of the rod and mass m with respect to O

$$I_O = m\,(2a)^2 + \frac{M\,(6a)^2}{12} + M\,a^2 = 4\,a^2\,(m + M). \tag{173}$$

Consider the rod in a position defined by the angle $(\theta_s + \theta)$ (Figure 11.c). The sum of the moments with respect to the axis of rotation through O are

$$M_O = m\,g\,(2\,a) + k\,a\,(\theta_s + \theta)\,a - M\,g\,a + k\,(4\,a)\,(\theta_s + \theta)\,(4\,a).$$

With the equilibrium condition given by Eq. (172) the moment becomes

$$M_O = 17\,k\,a^2\,\theta, \tag{174}$$

and the equation of motion is

$$4\,a^2\,(m+M)\,\ddot{\theta} + 17\,k\,a^2\,\theta = 0,$$

or

$$\ddot{\theta} + \frac{17\,k}{4\,(m+M)}\,\theta = 0. \tag{175}$$

This is the equation of a free harmonic vibration (small oscillation) with the circular frequency

$$\omega = \sqrt{\frac{17\,k}{4\,(m+M)}} = \frac{1}{2}\sqrt{\frac{17\,k}{m+M}}.$$

The period of small oscillation is:

$$T = \frac{2\,\pi}{\omega} = 4\,\pi\,\sqrt{\frac{m+M}{17\,k}}.$$

The general solution of the differential equation Eq. (175) is:

$$\theta = C_1\,\cos\omega t + C_2\,\sin\omega t.$$

The initial conditions for $t=0$ are $\theta = 0$ and $\dot{\theta} = \omega_0$.
It results $C_1 = 0$ and $C_2 = \dfrac{\omega}{\omega_0}$.
The solution of the problem is

$$\theta = \frac{\omega_0}{\omega}\,\sin\omega t.$$

Problem 3. Two external forces acts on a body with the mass m: a force proportional with time (the proportionality factor is equal to k_1) and a medium resistant force which is proportional with the velocity of the body (the proportionality factor being equal to k_2). The gravity is neglected.

Find and solve the equation of motion of the body.

Solution

The differential equation of motion is $m\dfrac{dv}{dt} = k_1 t - k_2 v$.
The following notation is used $k_1 t - k_2 v = u$.

The derivative with respect to t gives $k_1 - k_2 \dfrac{dv}{dt} = \dfrac{du}{dt}$.
Multiplying by m the following relation is obtained

$$k_1 m - k_2 m \frac{dv}{dt} = m \frac{du}{dt} \quad \text{or} \quad k_1 m - k_2 u = m \frac{du}{dt}. \tag{176}$$

The previous relation is an equation with separable variables,

$$\frac{du}{k_1 m - k_2 u} = \frac{1}{m} dt.$$

After integration,

$$\int \frac{du}{k_1 m - k_2 u} = \frac{1}{m} \int dt + C \Rightarrow -\frac{1}{k_2} \ln |k_1 m - k_2 u| = \frac{t}{m} + C.$$

From the initial condition $v(0) = 0$ it results $u(0) = 0$, hence

$$-\frac{1}{k_2} \ln |k_1 m| = C.$$

Replacing the value of C, yields

$$-\frac{1}{k_2} \ln |k_1 m - k_2 u| = \frac{t}{m} - \frac{1}{k_2} \ln |k_1 m| .$$

Multiplying by $(-k_2)$

$$\ln |k_1 m - k_2 u| = \ln |k_1 m| - \frac{k_2}{m} t$$

hence

$$k_1 m - k_2 u = k_1 m \ e^{-\frac{k_2}{m} t} \Rightarrow k_2 u = k_1 m - k_1 m \ e^{-\frac{k_2}{m} t}.$$

Replacing u by its expression depending on v the following relation is obtained

$$k_2 k_1 t - k_2^2 v = k_1 m - k_1 m \ e^{-\frac{k_2}{m} t} \Rightarrow v(t) = \frac{k_1 m}{k_2^2} \ e^{-\frac{k_2}{m} t} + \frac{k_1}{k_2} t - \frac{k_1 m}{k_2^2}.$$

Next the dependence of the space in time is obtained using the equations

$$v(t) = \frac{ds(t)}{dt} \quad \text{or} \quad s(t) = \int v(t) dt + C, \quad \text{and} \quad s(0) = s_0.$$

Then yields,

$$s(t) = \int \left(\frac{k_1 m}{k_2^2} e^{-\frac{k_2}{m}t} + \frac{k_1}{k_2} t - \frac{k_1 m}{k_2^2} \right) dt + C = -\frac{k_1 m^2}{k_2^3} e^{-\frac{k_2}{m}t} + \frac{k_1}{2k_2} t^2 - \frac{k_1 m}{k_2^2} t + C.$$

The constant C is determined from the initial condition

$s(0) = s_0 \Rightarrow s_0 = C - \dfrac{k_1 m^2}{k_2^3}$ or $C = s_0 + \dfrac{k_1 m^2}{k_2^3}$.

The equation of the space is given by

$$s(t) = s_0 + \frac{k_1 m^2}{k_2^3} - \frac{k_1 m}{k_2^2} t + \frac{k_1}{2k_2} t^2 - \frac{k_1 m^2}{k_2^3} e^{-\frac{k_2}{m}t}.$$

Problem 4. (The emptying of a reservoir) A reservoir has the shape of a rotational surface about a vertical axis with a hole at the bottom. The hole has the area A. Find and solve the equations of motion for the liquid located in the reservoir.

The following particular cases are considered for the reservoir:

a) spherical shape of radius R;

b) conical frustum with the smaller radius, R_1, as base radius, the larger radius, R_2, as top radius, and the height is H;

c) conical frustum with the larger radius, R_2, as base radius, the smaller radius, R_1, as top radius, and the height is H;

d) right cone with the vertex at the bottom;

e) cylindric shape.

Solution

From hydrodynamics it is known the expression of the leakage velocity of a fluid through an orifice $v = k\sqrt{h}$, where h is the height of the free surface of the fluid.

The equation of the median radius of the reservoir is of the form $r = r(h)$. The volume of liquid that leaks during the elementary time dt is evaluated in the following way. Through the hole leaks the volume of liquid which fills a cylinder with base A and height $v\,dt$

$$dV = A\,v\,dt = A\,k\,\sqrt{h}\,dt.$$

On the other side, the differential volume which leaks is $dV = -\pi r^2 dh$. The following expression is obtained

$$A\,k\,\sqrt{h}\,dt = -\pi r^2 dh.$$

It results a differential equation with separable variables

$$dt = -\frac{\pi}{Ak}\frac{r^2(h)}{\sqrt{h}}dh.$$

Solving the integral it is found

$$t = -\frac{\pi}{Ak}\int\frac{r^2(h)}{\sqrt{h}}dh + C.$$

From the initial condition $h(0) = H$ the constant C can be determined.

a) In the case of a spherical shape (Figure 12) the median radius can be written as $r^2 = h(2R - h)$.

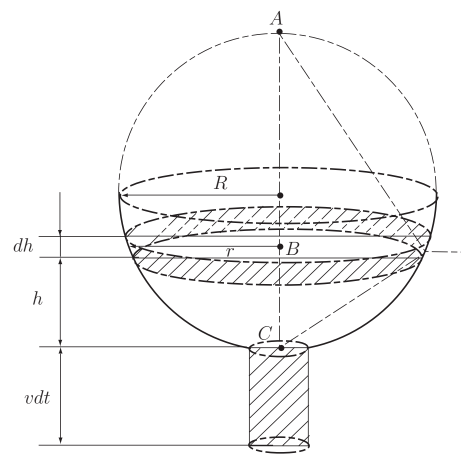

Figure 12: Spherical reservoir

Then,

$$t = -\frac{\pi}{Ak}\int\frac{h(2R-h)}{\sqrt{h}}dh + C,$$

or

$$t = -\frac{\pi}{Ak}\left[2R\int\sqrt{h}dh - \int h^{3/2}dh\right] + C = -\frac{\pi}{Ak}\left[\frac{4}{3}Rh^{3/2} - \frac{2}{5}h^{5/2}\right] + C.$$

Using the condition

$$h(0) \quad = \quad H,$$

yields

$$C = \frac{\pi}{Ak}\left[\frac{4}{3}RH^{3/2} - \frac{2}{5}H^{5/2}\right],$$

and hence

$$t = \frac{\pi}{Ak}\left[\frac{4}{3}R\left(H^{3/2} - h^{3/2}\right) - \frac{2}{5}\left(H^{5/2} - h^{5/2}\right)\right].$$

The time T for which $h(T) = 0$ is $T = \dfrac{\pi}{Ak}H^{3/2}\left(\dfrac{4}{3}R - \dfrac{2}{5}H\right)$.
For $H = R$ (the sphere is full) it results $T = \left(\dfrac{14}{15}\right)\dfrac{\pi R^{5/2}}{Ak}$.

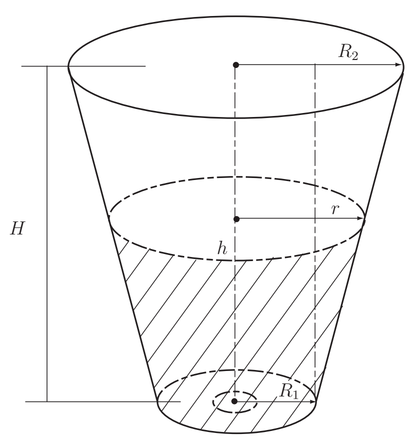

Figure 13: Spherical reservoir

b) From the geometry of the conical frustum (Figure 13), $\dfrac{r - R_1}{h} = \dfrac{R_2 - R_1}{H}$, and $r = R_1 + \dfrac{R_2 - R_1}{H}h$. Then, $\dfrac{r^2}{\sqrt{h}} = \dfrac{R_1^2}{\sqrt{h}} + \dfrac{2R_1(R_2 - R_1)}{H}\sqrt{h} + \left(\dfrac{R_2 - R_1}{H}\right)^2 h^{3/2}$

and substituting it in the expression of t, after the calculus of the integral, yields

$$t = -\frac{\pi}{Ak}\left[2R_1^2\sqrt{h} + \left(\frac{4}{3}\right)\frac{R_1(R_2-R_1)}{H}h^{3/2} + \left(\frac{2}{5}\right)\frac{R_2-R_1}{H}^2 h^{5/2}\right] + C.$$

Using the condition

$$h(0) = H,$$

it is found that

$$C = \frac{\pi}{Ak}\left[2R_1^2\sqrt{H} + \left(\frac{4}{3}\right)\frac{R_1(R_2-R_1)}{H}H^{3/2} + \left(\frac{2}{5}\right)\frac{R_2-R_1}{H}^2 H^{5/2}\right],$$

and hence,

$$t = \frac{\pi}{Ak}\left[2R_1^2(\sqrt{H}-\sqrt{h}) + \left(\frac{4}{3}\right)\frac{R_1(R_2-R_1)}{H}(H^{3/2}-h^{3/2})\right.$$

$$\left. + \left(\frac{2}{5}\right)\frac{R_2-R_1}{H}^2(H^{5/2}-h^{5/2})\right].$$

The condition $h(T) = 0$ implies

$$T = \frac{\pi\sqrt{H}}{Ak}\left[2R_1^2 + \frac{4}{3}R_1(R_2-R_1) + \frac{2}{5}(R_2-R_1)^2\right].$$

c) From Figure 14, $\dfrac{r-R_1}{H-h} = \dfrac{R_2-R_1}{H}$ and yields, $r = R_2 + \dfrac{R_1-R_2}{H}h$.

If in the expression of r from case b), R_1 is replaced by R_2, one can find the expression of r from case c). Consequently, the expressions of t and T for the case c) will be obtained from the corresponding expressions obtained at b), in which R_1 will be replaced by R_2 and R_2 by R_1

$$t = \frac{\pi}{Ak}\left[2{R_2}^2(\sqrt{H}-\sqrt{h}) + \left(\frac{4}{3}\right)\frac{R_2(R_1-R_2)}{H}(H^{3/2}-h^{3/2})\right.$$

$$\left. + \frac{5}{2}\left(\frac{R_1-R_2}{H}\right)^2(H^{5/2}-h^{5/2})\right],$$

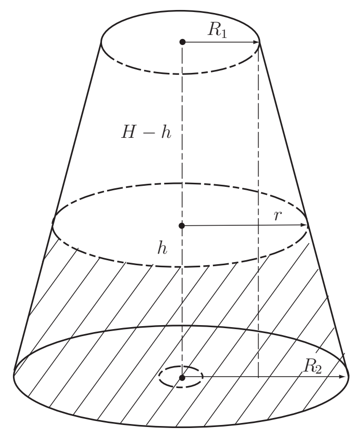

Figure 14: Conical frustum with larger radius as base radius

$$T = \frac{\pi\sqrt{H}}{Ak}\left[2R_2^2 + \frac{4}{3}R_2(R_1 - R_2) + \frac{2}{5}(R_1 - R_2)^2\right].$$

Comparing the expressions of T for the cases b) and c) and denoting by T' the expression in case c) it results

$$T' - T = \frac{\pi\sqrt{H}}{Ak}\left[2(R_2^2 - R_1^2) + \frac{4}{3}R_2R_1 - \frac{4}{3}R_2^2 - \frac{4}{3}R_1R_2 + \frac{4}{3}R_1^2\right.$$

$$\left.+\frac{2}{5}(R_1 - R_2)^2 - \frac{2}{5}(R_2 - R_1)^2\right] = \frac{\pi\sqrt{H}}{Ak}\frac{2}{3}(R_2^2 - R_1^2),$$

or,

$$T' = T + \frac{2}{3}\frac{\pi\sqrt{H}}{Ak}(R_2^2 - R_1^2).$$

d) It is obtained from case b), taking $R_1 = 0, R_2 = R$. Hence,

$$t = \frac{2\pi R^2}{5AkH^2}(H^{5/2} - h^{5/2}) \text{ and } T = \frac{2\pi R^2}{5Ak}\sqrt{H}.$$

e) It is obtained from case b), taking $R_1 = R_2 = R$. Then,

$$t = \frac{2\pi R^2}{Ak}(\sqrt{H} - \sqrt{h}) \text{ and } T = \frac{2\pi R^2}{Ak}\sqrt{H}.$$

Problem 5. Find the general solution of equation $\dfrac{y}{y'} = x + \sqrt{x^2+y^2}$.

Solution

The equation can be written in the form

$y\dfrac{dx}{dy} = x + \sqrt{x^2+y^2}$ or, $\dfrac{dx}{dy} = \dfrac{x}{y} + \sqrt{\dfrac{x^2}{y^2}+1}$.

Using the replacement $\dfrac{x}{y} = u$ or $x = yu$. It results

$\dfrac{dx}{dy} = u + y\dfrac{du}{dy} \Rightarrow u + y\dfrac{du}{dy} = u + \sqrt{u^2+1} \Rightarrow \dfrac{du}{\sqrt{u^2+1}} = \dfrac{dy}{y} \Rightarrow \displaystyle\int \dfrac{du}{\sqrt{u^2+1}} = \int \dfrac{dy}{y} + \ln c \Rightarrow \ln(u + \sqrt{u^2+1}) = \ln y + \ln c \Rightarrow u + \sqrt{u^2+1} = cy \Rightarrow$
$\dfrac{x}{y} + \sqrt{\dfrac{x^2}{y^2}+1} = cy \Rightarrow \sqrt{x^2+y^2} = cy^2 - x \Rightarrow x^2+y^2 = c^2y^4 - 2cxy^2 + x^2 \Rightarrow$
$c^2y^2 = 2cx + 1$ is the general solution.

Problem 6. (The problem of the swimmer) To cross a river, a swimmer starts from a point P on the bank. He wants to arrive at the point Q on the other side. The velocity of the river is constant and equal to $v_1 = k_1$ and the velocity of swimmer the is $v_2 = k_2$ where k_2 is constant. Find the trajectory described by the swimmer, knowing that the velocity of the swimmer is always directed toward Q.

Solution

Select Q as the origin of the system as shown in Figure 15. Consider that M is the swimmer position at time t. The components of the absolute velocity on the two axes Ox and Oy are

$$\frac{dx}{dt} = k_1 - k_2\frac{x}{\sqrt{x^2+y^2}},$$
$$\frac{dy}{dt} = -k_2\frac{y}{\sqrt{x^2+y^2}}.$$

Dividing the previous relation it results

$$\frac{dx}{dy} = \frac{x}{y} - k\sqrt{\frac{x^2}{y^2}+1},$$

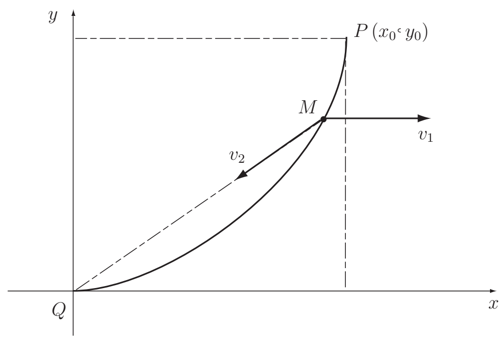

Figure 15: Path of the swimmer

where $k = \dfrac{k_1}{k_2}$.

The following notation is used $x = yu$ and $\dfrac{dx}{dy} = u + y\dfrac{du}{dy}$.

The differential equation becomes

$$y\frac{du}{dy} = -k\sqrt{u^2+1} \quad \text{or} \quad \frac{du}{\sqrt{u^2+1}} = -k\frac{dy}{y}.$$

After integration results

$$\ln(u + \sqrt{u^2+1}) = -k\ln y + \ln c \ (c > 0) \quad \text{or} \quad u + \sqrt{u^2+1} = cy^{-k}.$$

Then, yields

$$u = \frac{1}{2}\left(\frac{c}{y^k} - \frac{y^k}{c}\right).$$

Returning at x and y, $x = \dfrac{1}{2}y\left(\dfrac{c}{y^k} - \dfrac{y^k}{c}\right)$.

From the conditon for trajectory to pass through the initial point $P(x_0, y_0)$ the constant c is $c = y_0^{k-1}(x_0 + \sqrt{x_0^2 + y_0^2})$.

The condition for trajectory to pass through Q is written as

$\lim\limits_{y\to 0} \dfrac{1}{2}y\left(\dfrac{c}{y^k} - \dfrac{y^k}{c}\right) = 0$ and it is possible if $k < 1$.

For $k_1 = 0$, $k = 0$ and the trajectory has the equation $x = \dfrac{x_0}{y_0}y$, i.e., the linear segment between P and Q.

Problem 7. Determine the minimum velocity of a body thrown vertically upwards so that the body will not return to the Earth. The air resistance is neglected.

Solution

Denote the mass of the Earth by M and the mass of the body by m. Using Newton's law of gravitation, the force of attraction f acting on the body m is $f = k\dfrac{Mm}{r^2}$, where r is the distance between the center of the Earth and the center of gravity of the body and k is the gravitational constant. The differential equation of the motion for the body is

$$m\frac{d^2r}{dt^2} = -k\frac{Mm}{r^2} \quad \text{or} \quad \frac{d^2r}{dt^2} = -k\frac{M}{r^2}. \tag{177}$$

The minus sign indicates a negative acceleration. The differential Eq. (177) will be solved for the following initial conditions

$$r(0) = R \quad \text{and} \quad \frac{dr(0)}{dt} = v_0. \tag{178}$$

Here, R is the radius of Earth and v_0 is the launching velocity. The following notations are used $\dfrac{dr}{dt} = v \implies \dfrac{d^2r}{dt^2} = \dfrac{dv}{dt} = \dfrac{dv}{dr}\left(\dfrac{dr}{dt}\right) = v\dfrac{dv}{dr}$, where v is the velocity of motion. Substituting in Eq. (177), results $v\dfrac{dv}{dr} = -k\dfrac{M}{r^2}$. Separating variables, it is found $vdv = -kM\dfrac{dr}{r^2}$. Integrating this equation, yields $\dfrac{v^2}{2} = kM\dfrac{1}{r} + c_1$. From conditions (178), c_1 is found

$$\frac{v_0^2}{2} = kM\left(\frac{1}{R}\right) + c_1,$$

or,

$$c_1 = -\frac{kM}{R} + \frac{v_0^2}{2},$$

and

$$\frac{v^2}{2} = kM\frac{1}{r} + \left(\frac{v_0^2}{2} - \frac{kM}{R}\right). \tag{179}$$

The body should move so that the velocity is always positive, hence $\frac{v^2}{2} > 0$. Since for a boundless increase of r the quantity $\frac{kM}{R}$ becomes arbitrarily small, the condition $\frac{v^2}{2} > 0$ will be fulfilled for any r only for the case

$$\frac{v_0^2}{2} - \frac{kM}{R} \geq 0 \quad \text{or} \quad v_0 \geq \sqrt{\frac{2kM}{k}}.$$

Hence, the minimal velocity is determined by the equation

$$v_0 = \sqrt{\frac{2kM}{R}}, \tag{180}$$

where $k = 6.66(10^{-8})$ cm^3/(g s^2), $R = 63(10^7)$ cm. At the Earth's surface, for $r = R$, the acceleration of gravity is $g = 981$ cm/s^2. For this reason, from Eq. (177) yields $g = k\frac{M}{R^2}$ or $M = \frac{gR^2}{k}$. Substituting this value of M into Eq. (180) it results

$$v_0 = \sqrt{2gR} = \sqrt{2(981)(63)(10^7)} \approx 11.2(10^5) \text{ cm/s} = 11.2 \text{ km/s}.$$

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