Contents

7 Direct Dynamics
Newton-Euler Equations of Motion

The Newton-Euler equations of motion for a rigid body in plane motion are

$$m\ddot{\mathbf{r}}_C = \sum \mathbf{F} \quad \text{and} \quad I_{Czz}\,\boldsymbol{\alpha} = \sum \mathbf{M}_C,$$

or using the cartesian components

$$m\ddot{x}_C = \sum F_x, \;\; m\ddot{y}_C = \sum F_y, \;\; \text{and} \quad I_{Czz}\ddot{\theta} = \sum M_C.$$

The forces and moments are known and the differential equations are solved for the motion of the rigid body (direct dynamics).

7.1 Compound Pendulum

Figure 7.1(a) depicts a compound pendulum of mass m and length L. The pendulum is connnected to the ground by a pin joint and is free to swing in a vertical plane. The link is moving and makes an instant angle $\theta(t)$ with the horizontal. The local acceleration of gravity is g. Numerical application: $L = 3$ ft, g=32.2 ft/s^2, $G = m\,g$=12 lb. Find and solve the Newton-Euler equations of motion.

Solution

The system of interest is the link during the interval of its motion. The link in rotational motion is constrained to move in a vertical plane. First a reference frame will be introduced. The plane of motion will be designated the xy plane. The y-axis is vertical, with the positive sense directed vertically upward. The x-axis is horizontal and is contained in the plane of motion. The z-axis is also horizontal and is perpendicular to the plane of motion. These axes define an inertial reference frame. The unit vectors for the inertial reference frame are $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, and $\mathbf{k}$. The angle between the x and the link axis is denoted by θ. The link is moving and hence the angle is changing with time at the instant of interest. In the static equilibrium position of the link, the angle, θ, is equal to $-\pi/2$. The system has one degree of freedom. The angle, θ, is an appropriate generalized coordinate describing this degree of freedom. The system has a single moving body. The only motion permitted that body is rotation about a fixed horizontal axis (z-axis). The body is connected to the ground with the rotating pin joint (R) at O. The mass center of the link is at the point C. As the link is uniform, its mass center is coincident with its geometric center.

Kinematics

The mass center, C, is at a distance $L/2$ from the pivot point O and the position vector is

$$\mathbf{r}_{OC} = \mathbf{r}_C = x_C\mathbf{\imath} + y_C\mathbf{\jmath}, \tag{7.1}$$

where x_C and y_C are the coordinates of C

$$x_C = \frac{L}{2}\cos\theta \quad \text{and} \quad y_C = \frac{L}{2}\sin\theta. \tag{7.2}$$

The link is constrained to move in a vertical plane, with its pinned location, O, serving as a pivot point. The motion of the link is planar, consisting of pure rotation about the pivot point. The directions of the angular velocity and angular acceleration vectors will be perpendicular to this plane, in the z direction.The angular velocity of the link can be expressed as

$$\boldsymbol{\omega} = \omega\mathbf{k} = \frac{d\theta}{dt}\mathbf{k} = \dot{\theta}\mathbf{k}, \tag{7.3}$$

ω is the rate of rotation of the link. The positive sense is clockwise (consistent with the x and y directions defined above). This problem involves only a single moving rigid body and the angular velocity vector refers to that body. For this reason, no explicit indication of the body, 1, is included in the specification of the angular velocity vector $\boldsymbol{\omega} = \boldsymbol{\omega}_1$. The angular acceleration of the link can be expressed as

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \alpha\mathbf{k} = \frac{d^2\theta}{dt^2}\mathbf{k} = \ddot{\theta}\mathbf{k}, \tag{7.4}$$

α is the angular acceleration of the link. The positive sense is clockwise.

The velocity of the mass center can be related to the velocity of the pivot point using the relationship between the velocities of two points attached to the same rigid body

$$\mathbf{v}_C = \mathbf{v}_O + \boldsymbol{\omega}\times\mathbf{r}_{OC} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & \omega \\ x_C & y_C & 0 \end{vmatrix} = \omega(-y_C\mathbf{\imath} + x_C\mathbf{\jmath}) =$$
$$\frac{L\omega}{2}(-\sin\theta\mathbf{\imath} + \cos\theta\mathbf{\jmath}) = \frac{L\dot{\theta}}{2}(-\sin\theta\mathbf{\imath} + \cos\theta\mathbf{\jmath}). \tag{7.5}$$

The velocity of the pivot point, O, is zero.

The acceleration of the mass center can be related to the acceleration of the pivot point ($\mathbf{a}_O = \mathbf{0}$) using the relationship between the accelerations of two points attached to the same rigid body

$$\begin{aligned}
&\mathbf{a}_C = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{OC} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{OC}) = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{OC} - \omega^2 \mathbf{r}_{OC} = \\
&\begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & \alpha \\ x_C & y_C & 0 \end{vmatrix} - \omega^2 (x_C \boldsymbol{\imath} + y_C \boldsymbol{\jmath}) = \alpha(-y_C \boldsymbol{\imath} + x_C \boldsymbol{\jmath}) - \omega^2 (x_C \boldsymbol{\imath} + y_C \boldsymbol{\jmath}) = \\
&-(\alpha y_C + \omega^2 x_C)\boldsymbol{\imath} + (\alpha x_C - \omega^2 y_C)\boldsymbol{\jmath} = \\
&-\frac{L}{2}(\alpha \sin\theta + \omega^2 \cos\theta)\boldsymbol{\imath} + \frac{L}{2}(\alpha \cos\theta - \omega^2 \sin\theta)\boldsymbol{\jmath} = \\
&-\frac{L}{2}(\ddot{\theta} \sin\theta + \dot{\theta}^2 \cos\theta)\boldsymbol{\imath} + \frac{L}{2}(\ddot{\theta} \cos\theta - \dot{\theta}^2 \sin\theta)\boldsymbol{\jmath}.
\end{aligned} \tag{7.6}$$

It is also useful to define a set of body-fixed coordinate axes. These are axes that move with the link (body-fixed axe). The n-axis is along the length of the link, the positive direction running from the origin O toward the mass center C. The unit vector of the n-axis is $\mathbf{n}$. The t-axis will be perpendicular to the link and be contained in the plane of motion as shown in Fig. 7.1(a).The unit vector of the t-axis is $\mathbf{t}$ and $\mathbf{n} \times \mathbf{t} = \mathbf{k}$. The velocity of the mass center C in body-fixed reference frame is

$$\mathbf{v}_C = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{OC} = \begin{vmatrix} \mathbf{n} & \mathbf{t} & \mathbf{k} \\ 0 & 0 & \omega \\ \frac{L}{2} & 0 & 0 \end{vmatrix} = \frac{L\omega}{2}\mathbf{t} = \frac{L\dot{\theta}}{2}\mathbf{t}, \tag{7.7}$$

where $\mathbf{r}_{OC} = (L/2)\mathbf{n}$. The acceleration of the mass center C in body-fixed reference frame is

$$\mathbf{a}_C = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{OC} - \omega^2 \mathbf{r}_{OC} = \frac{L\alpha}{2}\mathbf{t} - \omega^2 \frac{L}{2}\mathbf{n} = \frac{L\ddot{\theta}}{2}\mathbf{t} - \dot{\theta}^2 \frac{L}{2}\mathbf{n}, \tag{7.8}$$

or

$$\mathbf{a}_C = \mathbf{a}_C^t + \mathbf{a}_C^n,$$

with the components

$$\mathbf{a}_C^t = \frac{L\ddot{\theta}}{2}\mathbf{t} \quad \text{and} \quad \mathbf{a}_C^n = -\frac{L\dot{\theta}^2}{2}\mathbf{n}.$$

Newton-Euler equation of motion

The link is rotating about a fixed axis. The mass moment of inertia of the link about the fixed pivot point O can be evaluated from the mass moment of inertia about the mass center C using the transfer theorem. Thus

$$I_O = I_C + m\left(\frac{L}{2}\right)^2 = \frac{mL^2}{12} + \frac{mL^2}{4} = \frac{mL^2}{3}. \tag{7.9}$$

The pin is frictionless and is capable of exerting horizontal and vertical forces on the link at O

$$\mathbf{F}_{01} = F_{01x}\boldsymbol{\imath} + F_{01y}\boldsymbol{\jmath}, \tag{7.10}$$

where F_{01x} and F_{01y} are the components of the pin force on the link in the fixed axis system.

The force driving the motion of the link is gravity. The weight of the link is acting through its mass center will cause a moment about the pivot point. This moment will give the link a tendency to rotate about the pivot point. This moment will be given by the cross product of the vector from the pivot point, O, to the mass center, C, crossed into the weight force $\mathbf{G} = -mg\boldsymbol{\jmath}$.

As the pivot point, O, of the link is fixed, the appropriate moment summation point will be about that pivot point. The sum of the moments about this point will be equal to the mass moment of inertia about the pivot point multiplied by the angular acceleration of the link. The only contributor to the moment is the weight of the link. Thus we should be able to directly determine the angular acceleration from the moment equation. The sum of the forces acting on the link should be equal to the product of the link mass and the acceleration of its mass center. This should be useful in determining the forces exerted by the pin on the link.

The free body diagram shows the link at the instant of interest, Fig. 7.1(b). The link is acted upon by its weight acting vertically downward through the mass center of the link. The link is acted upon by the pin force at its pivot point. The motion diagram shows the link at the instant of interest, Fig. 7.1(c). The motion diagram shows the relevant acceleration information. The Newton-Euler equations of motion for the link are

$$m\,\mathbf{a}_C = \Sigma\mathbf{F} = \mathbf{G} + \mathbf{F}_{01}, \tag{7.11}$$

$$I_C\,\boldsymbol{\alpha} = \Sigma\mathbf{M}_C = \mathbf{r}_{CO} \times \mathbf{F}_{01}. \tag{7.12}$$

Since the rigid body has a fixed point at O the equations of motion state that the moment sum about the fixed point must be equal to to the product of the link mass moment of inertia about that point and the link angular acceleration. Thus

$$I_O\,\boldsymbol{\alpha} = \Sigma\mathbf{M}_O = \mathbf{r}_{OC} \times \mathbf{G} \tag{7.13}$$

Using Eqs. (7.4),(7.10), and (7.13) the equation of motion is

$$\frac{mL^2}{3}\,\ddot{\theta}\mathbf{k} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ \frac{L}{2}\cos\theta & \frac{L}{2}\sin\theta & 0 \\ 0 & -mg & 0 \end{vmatrix}, \tag{7.14}$$

or

$$\ddot{\theta} = -\frac{3g}{2L}\cos\theta. \tag{7.15}$$

The equation of motion [Eq. (7.15)] is a nonlinear, second order, differential equation relating the second time derivative of the angle, θ, to the value of that angle and various problem parameters g and L. The equation is nonlinear due to the presence of the $\cos\theta$, where $\theta(t)$ is the unknown function of interest.

The force exerted by the pin on the link have are obtained from Eq. (7.12)

$$\mathbf{F}_{01} = m\,\mathbf{a}_C - \mathbf{G},$$

and the components of the force are

$$F_{01x} = m\,\ddot{x}_C = -\frac{mL}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta),$$
$$F_{01y} = m\,\ddot{y}_C + mg = \frac{mL}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + mg. \tag{7.16}$$

Using the moving reference frame (body-fixed) the components of the reaction force on n and t axes are

$$F_{01n} = m\,a_C^n - mg\sin\theta = -\frac{mL\dot{\theta}^2}{2} + mg\sin\theta,$$
$$F_{01t} = m\,a_C^t - mg\cos\theta = \frac{mL\ddot{\theta}}{2} + mg\cos\theta. \tag{7.17}$$

If the link is released from rest, then the initial value of the angular velocity is zero $\omega(t=0) = \omega(0) = \dot{\theta}(0) = 0$ rad/s. If the initial angle is $\theta(0) = 0$ radians, then the cosine of that initial angle is unity and the sine is zero. The initial angular acceleration can be determined from Eq. (7.15)

$$\ddot{\theta}(0) = \alpha(0) = -\frac{3g}{2L}\cos\theta(0) = -\frac{3g}{2L} = 16.1 \text{ rad/s}^2. \tag{7.18}$$

The negative sign indicates that the initial angular acceleration of the link is counterclockwise, as one would expect.

The initial reaction force components can be evaluated from Eq. (7.19)

$$\begin{aligned} F_{01x}(0) &= 0 \text{ lb},\\ F_{01y}(0) &= \frac{mL}{2}\ddot{\theta}(0) + mg = \frac{mg}{4} = 3 \text{ lb}. \end{aligned} \tag{7.19}$$

The equation of motion [Eq. (7.15)] is obtained symbolically using the MATLAB commands

```
syms L m g t
omega = [0 0 diff('theta(t)',t)];
alpha = diff(omega,t);
c = cos(sym('theta(t)'));
s = sin(sym('theta(t)'));
xC = (L/2)*c;
yC = (L/2)*s;
rC = [xC yC 0];
G = [0 -m*g 0];
IC = m*L^2/12;
IO = IC + m*(L/2)^2;
MO = cross(rC,G);
eq = -IO*alpha+MO;
eqz = eq(3);
```

The MATLAB statement `diff(X,'t')` or `diff(X,sym('t'))` differentiates a symbolic expression `X` with respect to `t`, and the statement `diff(X,'t',n)` and `diff(X,n,'t')` differentiates `X` `n` times where `n` is a positive integer.

An analytical solution to the differential equation is difficult to obtain. Numerical approaches have the advantage of being simple to apply even for complex mechanical systems. The MATLAB function `ode45` is used to solve the differential equation.

The differential equation $\ddot{\theta} = -\dfrac{3g}{2L}\cos\theta$ is of order 2. The equation has to be rewritten as a first order system. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$, this gives the first order system

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{3g}{2\,L}\cos x_1.$$

The MATLAB commands for the right hand side of the first order system are

```
eqI = subs(eqz,{L,m,g},{3,12/32.2,32.2});
eqI1 = subs(eqI,diff('theta(t)',t,2),'ddtheta');
eqI2 = subs(eqI1,diff('theta(t)',t),sym('x(2)'));
eqI3 = subs(eqI2,'theta(t)',sym('x(1)'));
dx1 = sym('x(2)');
dx2 = solve(eqI3,'ddtheta');
dx1dt = char(dx1);
dx2dt = char(dx2);
```

An `inline` function `g` is defined for the right hand side of the first order system. Note that `g` must be specified as a column vector using `[...;...]` (not a row vector using `[...,...]`). In the definition of `g`, `x(1)` was used for x_1 and `x(2)` was used for x_2. The definition of `g` should have the form

```
g = inline(sprintf('[%s; %s]', dx1dt, dx2dt), 't', 'x');
```

The statement has to have the form `inline(...,'t','y')`, even if `t` does not occur in your formula. The first component of `g` is `x(2)`. The statement `sprintf` writes formatted data to string.

The time `t` is going from an initial value `t0` to a final value `f`

```
t0 = 0;
tf = 10;
time = [0 tf];
```

The initial conditions at $t_0 = 0$ are $\theta(0) = \pi/4$ rad and $\dot{\theta}(0) = 0$ rad/s or in MATLAB

```
x0 = [pi/4; 0]; % define initial conditions
```

The numerical solution of all the components of the solution for `t` going from `t0` to `f` is obtained using the command

```
[t,xs] = ode45(g, time, x0);
```

where`x0` is the initial value vector at the starting point `t0`.

One can obtain a vector `t` and a matrix `xs` with the coordinates of these points using `ode45` command.

The vector of `x1` values in the first column of `xs` is obtained by using `xs(:,1)` and the vector of `x2` values in the second column of `xs` is obtained by using `xs(:,2)`

```
x1 = xs(:,1);
x2 = xs(:,2);
```

The plot of the solution curves are obtained using the commands

```
subplot(3,1,1),plot(t,x1,'r'),xlabel('t'),ylabel('\theta'),grid,...
subplot(3,1,2),plot(t,x2,'g'),xlabel('t'),ylabel('\omega'),grid,...
subplot(3,1,3),plot(x1,x2),xlabel('\theta'),ylabel('\omega'),grid
```

The plots using MATLAB are shown in Fig. 7.2. In general the error tends to grow as one goes further from the initial conditions.

To obtain numerical values at specific `t` values one can specify a vector `tp` of `t` values and use `[ts,xs] = ode45(g, tp, x0)`. The first element of the vector `tp` is the initial value and the vector `tp` must have at least 3 elements. To obtain the solution with the initial values at `t = 0, 0.5, 1.0, 1.5, ... , 10` one can use

```
[ts,xs] = ode45(g, 0:0.5:10, x0);
[ts,xs]
```

and the results are displayed as a table with 3 columns `ts, x1 = xs(:,1), x2 = xs(:,1)`.

A MATLAB computer program to solve the governing differential equation is given in the Program 7.1.

The differential equation can be solved numerically by m-file functions. First create a function file, `R.m` as shown below

```
function dx = R(t,x);
dx = zeros(2,1); % a column vector
W =12; L = 3; g = 32.2; m = W/g;
dx(1) = x(2);
dx(2) = -3*g*cos(x(1))/(2*L);
```

The ode solvers provided by Matlab (`ode45`) is used to solve the differential equation

```
tfinal=10;
time=[0 tfinal];
x0=[pi/4 0]; % x(1)(0)=pi/4 x(2)(0)=0
[t,x]=ode45(@R, time, x0);
```

The MATLAB program is given in the Program 7.2.

7.2 Double Pendulum

A two-link planar chain (double pendulum) is considered, Fig. 7.2(a). The links 1 and 2 have the masses m_1 and m_2 and the lengths $AB = L_1$ and $BD = L_2$. The system is free to move in a vertical plane. The local acceleration of gravity is g. Numerical application: $m_1 = m_2 = 1$ kg, $L_1 = 1$ m, $L_2 = 0.5$ m, and $g = 10$ m/s^2. Find and solve the equations of motion.

Solution

The plane of motion is xy plane with the y-axis vertical, with the positive sense directed upward. The origin of the reference frame is at A. The mass centers of the links are designated by $C_1(x_{C_1}, y_{C_1}, 0)$ and $C_2(x_{C_2}, y_{C_2}, 0)$. The number of degrees of freedom are computed using the relation

$$M = 3n - 2c_5 - c_4,$$

where n is the number of moving links, c_5 is the number of one degree of freedom joints, and c_4 is the number of two degrees of freedom joints. For the double pendulum $n = 2,\ c_5 = 2,\ c_4 = 0$, and the system has two degrees of freedom, $M = 2$, and two generalized coordinates. The angles $q_1(t)$ and $q_2(t)$ are selected as the generalized coordinates as shown in Fig. 7.3(a).

Kinematics

The position vector of the center of the mass C_1 of the link 1 is

$$\mathbf{r}_{C_1} = x_{C_1}\mathbf{\imath} + y_{C_1}\mathbf{\jmath},$$

where x_{C_1} and y_{C_1} are the coordinates of C_1

$$x_{C_1} = \frac{L_1}{2}\cos q_1,\ \ y_{C_1} = \frac{L_1}{2}\sin q_1.$$

The position vector of the center of the mass C_2 of the link 2 is

$$\mathbf{r}_{C_2} = x_{C_2}\mathbf{\imath} + y_{C_2}\mathbf{\jmath},$$

where x_{C_2} and y_{C_2} are the coordinates of C_2

$$x_{C_2} = L_1\cos q_1 + \frac{L_2}{2}\cos q_2 \quad \text{and} \quad y_{C_2} = L_1\sin q_1 + \frac{L_2}{2}\sin q_2.$$

The velocity vector of C_1 is the derivative with respect to time of the position vector of C_1

$$\mathbf{v}_{C_1} = \dot{\mathbf{r}}_{C_1} = \dot{x}_{C_1}\mathbf{\imath} + \dot{y}_{C_1}\mathbf{\jmath},$$

where

$$\dot{x}_{C_1} = -\frac{L_1}{2}\dot{q}_1 \sin q_1 \quad \text{and} \quad \dot{y}_{C_1} = \frac{L_1}{2}\dot{q}_1 \cos q_1.$$

The velocity vector of C_2 is the derivative with respect to time of the position vector of C_2

$$\mathbf{v}_{C_2} = \dot{\mathbf{r}}_{C_2} = \dot{x}_{C_2}\mathbf{\imath} + \dot{y}_{C_2}\mathbf{\jmath},$$

where

$$\dot{x}_{C_2} = -L_1\dot{q}_1 \sin q_1 - \frac{L_2}{2}\dot{q}_2 \sin q_2,$$
$$\dot{y}_{C_2} = L_1\dot{q}_1 \cos q_1 + \frac{L_2}{2}\dot{q}_2 \cos q_2.$$

The acceleration vector of C_1 is the double derivative with respect to time of the position vector of C_1

$$\mathbf{a}_{C_1} = \ddot{\mathbf{r}}_{C_1} = \ddot{x}_{C_1}\mathbf{\imath} + \ddot{y}_{C_1}\mathbf{\jmath},$$

where

$$\ddot{x}_{C_1} = -\frac{L_1}{2}\ddot{q}_1 \sin q_1 - \frac{L_1}{2}\dot{q}_1^2 \cos q_1,$$
$$\ddot{y}_{C_1} = \frac{L_1}{2}\ddot{q}_1 \cos q_1 - \frac{L_1}{2}\dot{q}_1^2 \sin q_1.$$

The acceleration vector of C_2 is the double derivative with respect to time of the position vector of C_2

$$\mathbf{a}_{C_2} = \ddot{\mathbf{r}}_{C_2} = \ddot{x}_{C_2}\mathbf{\imath} + \ddot{y}_{C_2}\mathbf{\jmath},$$

where

$$\ddot{x}_{C_2} = -L_1\ddot{q}_1 \sin q_1 - L_1\dot{q}_1^2 \cos q_1 - \frac{L_2}{2}\ddot{q}_2 \sin q_2 - \frac{L_2}{2}\dot{q}_2^2 \cos q_2,$$
$$\ddot{y}_{C_2} = L_1\ddot{q}_1 \cos q_1 - L_1\dot{q}_1^2 \sin q_1 + \frac{L_2}{2}\ddot{q}_2 \cos q_2 - \frac{L_2}{2}\dot{q}_2^2 \sin q_2.$$

The MATLAB commands for the linear accelerations of the mass centers C_1 and C_2 are

```
L1 = 1; L2 = 0.5; m1 = 1; m2 = 1; g = 10;
t = sym('t','real');
```

```
xB = L1*cos(sym('q1(t)'));
yB = L1*sin(sym('q1(t)'));
rB = [xB yB 0];
rC1 = rB/2;
vC1 = diff(rC1,t);
aC1 = diff(vC1,t);
xD = xB + L2*cos(sym('q2(t)'));
yD = yB + L2*sin(sym('q2(t)'));
rD = [xD yD 0];
rC2 = (rB + rD)/2;
vC2 = diff(rC2,t);
aC2 = diff(vC2,t);
```

The angular velocity vectors of the links 1 and 2 are

$$\boldsymbol{\omega}_1 = \dot{q}_1\mathbf{k} \quad \text{and} \quad \boldsymbol{\omega}_2 = \dot{q}_2\mathbf{k}.$$

The angular acceleration vectors of the links 1 and 2 are

$$\boldsymbol{\alpha}_1 = \ddot{q}_1\mathbf{k} \quad \text{and} \quad \boldsymbol{\alpha}_2 = \ddot{q}_2\mathbf{k}.$$

The MATLAB commands for the angular accelerations of the links 1 and 2 are

```
omega1 = [0 0 diff('q1(t)',t)];
alpha1 = diff(omega1,t);
omega2 = [0 0 diff('q2(t)',t)];
alpha2 = diff(omega2,t);
```

Newton-Euler equations of motion The weight forces on the links 1 and 2 are

$$\mathbf{G}_1 = -m_1 g\mathbf{J} \quad \text{and} \quad \mathbf{G}_2 = -m_2 g\mathbf{J},$$

and in MATLAB

```
G1 = [0 -m1*g 0];
G2 = [0 -m2*g 0];
```

The mass moment of inertia of the link 1 with respect to the center of mass C_1 is

$$I_{C_1} = \frac{m_1 L_1^2}{12}.$$

The mass moment of inertia of the link 1 with respect to the fixed point of rotation A is

$$I_A = I_{C_1} + m_1 \left(\frac{L_1}{2}\right)^2 = \frac{m_1 L_1^2}{3}.$$

The mass moment of inertia of the link 2 with respect to the center of mass C_2 is

$$I_{C_2} = \frac{m_2 L_2^2}{12}.$$

The MATLAB commands for the mass moments of inertia are

```
IC1 = m1*L1^2/12;
IA = IC1 + m1*(L1/2)^2;
IC2 = m2*L2^2/12;
```

The equations of motion of the pendulum are inferred using the Newton-Euler method. There are two rigid bodies in the system and the Newton-Euler equations are written for each link using the free body diagrams shown in Fig. 7.3(b).

Link 1

The Newton-Euler equations for the link 1 are

$$m_1 \mathbf{a}_{C_1} = \mathbf{F}_{01} + \mathbf{F}_{21} + \mathbf{G}_1,$$
$$I_{C_1} \boldsymbol{\alpha}_1 = \mathbf{r}_{C_1 A} \times \mathbf{F}_{01} + \mathbf{r}_{C_1 B} \times \mathbf{F}_{21},$$

where $\mathbf{F}_{01}$ is the joint reaction of the ground 0 on the link 1 at point A, and $\mathbf{F}_{21}$ is the joint reaction of the link 2 on the link 1 at point B

$$\mathbf{F}_{01} = F_{01x}\boldsymbol{\imath} + F_{01y}\boldsymbol{\jmath} \quad \text{and} \quad \mathbf{F}_{21} = F_{21x}\boldsymbol{\imath} + F_{21y}\boldsymbol{\jmath}.$$

Since the link 1 has a fixed point of rotation at A the moment sum about the fixed point must be equal to to the product of the link mass moment of inertia about that point and the link angular acceleration. Thus

$$I_A \boldsymbol{\alpha}_1 = \mathbf{r}_{AC_1} \times \mathbf{G}_1 + \mathbf{r}_{AB} \times \mathbf{F}_{21}, \tag{7.20}$$

or

$$\frac{m_1 L_1^2}{3} \ddot{q}_1 \mathbf{k} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ x_{C_1} & y_{C_1} & 0 \\ 0 & -m_1 g & 0 \end{vmatrix} + \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ x_B & y_B & 0 \\ F_{21x} & F_{21y} & 0 \end{vmatrix}, \quad \text{or}$$

$$\frac{m_1 L_1^2}{3} \ddot{q}_1 \mathbf{k} = (-m_1 g x_{C_1} + F_{21y} x_B - F_{21x} y_B)\mathbf{k}.$$

The equation of motion for link 1 is

$$\frac{m_1 L_1^2}{3}\,\ddot{q}_1 = \left(-m_1 g \frac{L_1}{2}\cos q_1 + F_{21y} L_1 \cos q_1 - F_{21x} L_1 \sin q_1\right). \tag{7.21}$$

Link 2

The Newton-Euler equations for the link 2 are

$$m_2 \mathbf{a}_{C_2} = \mathbf{F}_{12} + \mathbf{G}_2, \tag{7.22}$$

$$I_{C_2} \boldsymbol{\alpha}_2 = \mathbf{r}_{C_2 B} \times \mathbf{F}_{12}, \tag{7.23}$$

where $\mathbf{F}_{12} = -\mathbf{F}_{21}$ is the joint reaction of the link 1 on the link 2 at B. Equation (7.23) becomes

$$\begin{aligned} m_2 \ddot{x}_{C_2} &= -F_{21x}, \\ m_2 \ddot{y}_{C_2} &= -F_{21y} - m_2\, g, \\ \frac{m L_2^2}{12}\,\ddot{q}_2 \mathbf{k} &= \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ x_B - x_{C_2} & y_B - y_{C_2} & 0 \\ -F_{21x} & -F_{21y} & 0 \end{vmatrix}, \end{aligned} \tag{7.24}$$

or

$$m_2 \left(-L_1 \ddot{q}_1 \sin q_1 - L_1 \dot{q}_1^2 \cos q_1 - \frac{L_2}{2}\ddot{q}_2 \sin q_2 - \frac{L_2}{2}\dot{q}_2^2 \cos q_2\right)$$
$$= -F_{21x}, \tag{7.25}$$

$$m_2 \left(L_1 \ddot{q}_1 \cos q_1 - L_1 \dot{q}_1^2 \sin q_1 + \frac{L_2}{2}\ddot{q}_2 \cos q_2 - \frac{L_2}{2}\dot{q}_2^2 \sin q_2\right)$$
$$= -F_{21y} - m_2\, g, \tag{7.26}$$

$$\frac{m_2 L_2^2}{12}\,\ddot{q}_2 = \frac{L_2}{2}\left(-F_{21y}\cos q_2 + F_{21x}\sin q_2\right). \tag{7.27}$$

The reaction components F_{21x} and F_{21y} are obtained from Eqs. (7.25)(7.26)

$$\begin{aligned} F_{21x} &= m_2 \left(L_1 \ddot{q}_1 \sin q_1 + L_1 \dot{q}_1^2 \cos q_1 + \frac{L_2}{2}\ddot{q}_2 \sin q_2 + \frac{L_2}{2}\dot{q}_2^2 \cos q_2\right), \\ F_{21y} &= -m_2 \left(L_1 \ddot{q}_1 \cos q_1 - L_1 \dot{q}_1^2 \sin q_1 + \frac{L_2}{2}\ddot{q}_2 \cos q_2 - \frac{L_2}{2}\dot{q}_2^2 \sin q_2\right) + \\ &\quad m_2\, g. \end{aligned} \tag{7.28}$$

The equations of motion are obtained substituting F_{21x} and F_{21y} in Eq. (7.21) and Eq. (7.27)

$$\frac{m_2 L_1^2}{3}\ddot{q}_1 = -m_1 g \frac{L_1}{2}\cos q_1 -$$
$$m_2\left(L_1\ddot{q}_1\cos q_1 - L_1\dot{q}_1^2\sin q_1 + \frac{L_2}{2}\ddot{q}_2\cos q_2 - \frac{L_2}{2}\dot{q}_2^2\sin q_2 - g\right)L_1\cos q_1 -$$
$$m_2\left(L_1\ddot{q}_1\sin q_1 + L_1\dot{q}_1^2\cos q_1 + \frac{L_2}{2}\ddot{q}_2\sin q_2 + \frac{L_2}{2}\dot{q}_2^2\cos q_2\right)L_1\sin q_1, \quad (7.29)$$

$$\frac{m_2 L_2^2}{12}\ddot{q}_2 =$$
$$\frac{m_2\,L_2}{2}\left(L_1\ddot{q}_1\cos q_1 - L_1\dot{q}_1^2\sin q_1 + \frac{L_2}{2}\ddot{q}_2\cos q_2 - \frac{L_2}{2}\dot{q}_2^2\sin q_2 - g\right)\cos q_2 +$$
$$\frac{m_2\,L_2}{2}\left(L_1\ddot{q}_1\sin q_1 + L_1\dot{q}_1^2\cos q_1 + \frac{L_2}{2}\ddot{q}_2\sin q_2 + \frac{L_2}{2}\dot{q}_2^2\cos q_2\right)\sin q_2. \quad (7.30)$$

The eqautions of motion represent two nonlinear differential equations. The initial conditions (Cauchy problem) are necessary to solve the equations. At $t = 0$ the initial conditions are

$$q_1(0) = q_{10},\ \dot{q}_1(0) = \omega_{10},\ q_2(0) = q_{20},\ \text{and}\ \dot{q}_2(0) = \omega_{20}.$$

The equations of motion for the mechanical system will be solved using MATLAB. First the reaction joint force $\mathbf{F}_{21}$ is calculated from Eq. (7.22)

```
F21 = -m2*aC2 + G2;
```

The moment equations for each link, Eqs. (7.20) and (7.23), using MATLAB are

```
EqA = -IA*alpha1 + cross(rB, F21) + cross(rC1, G1);
Eq2 = -IC2*alpha2 + cross(rB - rC2, -F21);
```

Two lists `slist` and `nlist` are created

```
slist={diff('q1(t)',t,2),diff('q2(t)',t,2),...
       diff('q1(t)',t),diff('q2(t)',t),'q1(t)','q2(t)'};
nlist = {'ddq1', 'ddq2', 'x(2)', 'x(4)', 'x(1)','x(3)';
```

```
% diff('q1(t)',t,2) will be replaced by 'ddq1'
% diff('q2(t)',t,2) will be replaced by 'ddq2'
% diff('q1(t)',t) will be replaced by 'x(2)'
% diff('q2(t)',t) will be replaced by 'x(4)'
% 'q1(t)' will be replaced by 'x(1)'
% 'q2(t)' will be replaced by 'x(3)'
```

In the equations of motion `EqA` and `Eq2` the symbolical variables in `slist` are replaced with the symbolical variables in `nlist`

```
eq1 = subs(EqA(3),slist,nlist);
eq2 = subs(Eq2(3),slist,nlist);
```

The previous equations are solved in terms of `'ddq1'` and `'ddq2'`

```
sol = solve(eq1,eq2,'ddq1, ddq2');
```

The second order ODE system of two equations has to be rewritten your as a first order system.
Let `x(1)`=$q_1(t)$, `x(2)`=$\dot{q}_1(t)$, `x(3)`=$q_2(t)$, and `x(4)`=$\dot{q}_2(t)$, this gives the first order system
`d[x(1)]/dt = x(2),`
`d[x(2)]/dt = ddq1,`
`d[x(3)]/dt = x(4),`
`d[x(4)]/dt = ddq2.`

The MATLAB commands for the first order ODE system are

```
dx1 = sym('x(2)');
dx2 = sol.ddq1;
dx3 = sym('x(4)');
dx4 = sol.ddq2;

dx1dt = char(dx1);
dx2dt = char(dx2);
dx3dt = char(dx3);
dx4dt = char(dx4);
```

The `inline` function `g` is defined for the right hand side of the first order system

```
g = inline(sprintf('[%s;%s;%s;%s]', dx1dt, dx2dt, dx3dt, dx4dt),'t','x');
```

The time `t` is going from an initial value `t0` to a final value `f`

```
t0 = 0; tf = 5; time = [0 tf];
```

The initial conditions at $t_0 = 0$ are $q_1(0) = -\pi/4$ rad, $\dot{q}_1(0) = 0$ rad/s, $q_2(0) = -\pi/3$ rad, $\dot{q}_2(0) = 0$ rad/s, or in MATLAB

```
x0 = [-pi/4; 0; -pi/3; 0]; % define initial conditions
```

The numerical solution of all the components of the solution for `t` going from `t0` to `f` is obtained using the command

```
[t,xs] = ode45(g, time, x0);
```

where `x0` is the initial value vector at the starting point `t0`.
The plot of the solution curves q_1 and q_2 are obtained using the commands

```
x1 = xs(:,1);
x3 = xs(:,3);
subplot(2,1,1),plot(t,x1*180/pi,'r'),...
xlabel('t (s)'),ylabel('q1 (deg)'),grid,...
subplot(2,1,2),plot(t,x3*180/pi,'b'),...
xlabel('t (s)'),ylabel('q2 (deg)'),grid
```

The plots using MATLAB are shown in Fig. 7.4 and the MATLAB program is given in the Program 7.3.

Instead of using the `inline` function `g` the system of differential equations can be solved numerically by m-file functions. The function file, `RR.m` is created using the statements

```
......
sol = solve(eq1,eq2,'ddq1, ddq2');
```

```
dx2 = sol.ddq1; dx4 = sol.ddq2;
dx2dt = char(dx2); dx4dt = char(dx4);

% create the function file RR.m

fid = fopen('RR.m','w+');
fprintf(fid,'function dx = RR(t,x)\n');
fprintf(fid,'dx = zeros(4,1);\n');
fprintf(fid,'dx(1) = x(2);\n');
fprintf(fid,'dx(2) = ');
fprintf(fid,dx2dt);
fprintf(fid,';\n');
fprintf(fid,'dx(3) = x(4);\n');
fprintf(fid,'dx(4) = ');
fprintf(fid,dx4dt);
fprintf(fid,';');
fclose(fid);
cd(pwd);
```

The terms `dx2dt` and `dx4dt` are calculated symbolically from the previous program (Program 7.3). The MATLAB command `fid = fopen(file,perm)` opens the file `file` in the mode specified by `perm`. The mode `'w+'` deletes the contents of an existing file, or creates a new file, and opens it for reading and writing. The statement `fclose(fid)` closes the file associated with file identifier `fid`.

The `ode45` solver is used for the system of differential equations

```
t0 = 0; tf = 5; time = [0 tf];
x0 = [-pi/4 0 -pi/3 0];
[t,xs] = ode45(@RR, time, x0);
```

The computing time for solving the system of differential equations is shorter using the function file `RR.m`. The MATLAB program is given in the Program 7.4.

7.3 One-Link Planar Robot Arm

The robot arm shown in Fig. 7.5 is characterized by the length $L = 1$ m. The mass of the rigid body is $m = 1$ kg and the gravitational acceleration is $g = 9.81$ m/s^2.
The initial conditions, at $t = 0$ s, are $\theta(0) = \pi/18$ rad and $\dot{\theta}(0) = 0$. The robot arm can be brought from an initial state of rest to a final state of rest in such a way that θ has the specified value $\theta_f = \pi/3$ rad.
In the case of the robot arm the set of contact forces transmitted from 0 to 1 in order to drive the link 1 can be replaced with a couple of torque $\mathbf{T}_{01}$. The expression of $\mathbf{T}_{01}$ is

$$\mathbf{T}_{01} = T_{01x}\,\imath + T_{01y}\,\jmath + T_{01z}\mathbf{k} = T_{01z}\,\mathbf{k}.$$

The following feedback control law is used

$$T_{01z} = -\beta\dot{\theta} - \gamma(\theta - \theta_f) + 0.5\,g\,L\,m\,\cos\theta.$$

The constant gains are: $\beta = 45$ N·m·s/rad and $\gamma = 30$ N·m/rad.

Write a MATLAB program for solving the equations of motion.

Solution

The equation of motion for the robot arm is obtained symbolically using the MATLAB commands

```
syms t
L = 1; m = 1; g = 9.81;
c = cos(sym('theta(t)')); s = sin(sym('theta(t)'));
xC = (L/2)*c; yC = (L/2)*s;
rC = [xC yC 0];
omega = [0 0 diff('theta(t)',t)];
alpha = diff(omega,t);
G = [0 -m*g 0];
IO = m*L^2/3;
beta = 45;
gamma = 30;
qf = pi/3;
T01z = -beta*diff('theta(t)',t)-gamma*(sym('theta(t)')-qf)+0.5*g*L*m*c;
T01 = [0 0 T01z];
```

```
eq = -IO*alpha + cross(rC,G) + T01;
eqz = eq(3);
```

The equation has to be rewritten as a first order system ($x_1 = \theta$ and $x_2 = \dot{\theta}$)

```
slist = diff('theta(t)',t,2),diff('theta(t)',t),'theta(t)';
nlist = 'ddtheta', 'x(2)' , 'x(1)';
eqI = subs(eqz,slist,nlist);
dx1 = sym('x(2)');
dx2 = solve(eqI,'ddtheta');

dx1dt = char(dx1);
dx2dt = char(dx2);
```

An `inline` function `g` is defined for the right hand side of the first order system

```
g = inline(sprintf('[%s; %s]', dx1dt, dx2dt), 't', 'x');
```

and the solution is obtained using the commands

```
time = [0 10];
x0 = [pi/18; 0]; % define initial conditions
[ts,xs] = ode45(g, 0:1:10, x0);
plot(ts,xs(:,1)*180/pi,'LineWidth',1.5),...
xlabel('t (s)'),ylabel('\theta (deg)'),grid,axis([0, 10, 0, 70])
fprintf('Results \n'); fprintf('\n');
fprintf(' t(s) theta(rad) omega(rad/s) \n');
[ts,xs]
```

The plot of θ for the considered time interval, using MATLAB, is shown in Fig. 7.6. The MATLAB program and the results are given in the Program 7.5.

The system of differential equations can be solved numerically by m-file functions. The m-file function `Rrobot.m` is created

```
function dx = Rrobot(t,x);
```

```
dx = zeros(2,1);
dx(1) = x(2);
dx(2) = -135*x(2)-90*x(1)+30*pi;
```

The `ode45` solver is used to solve the differential equations

```
time = [0 10]; x0 = [pi/18 0];
[ts,xs] = ode45(@Rrobot, 0:1:10, x0);
```

and MATLAB program is given in Program 7.6.

7.4 Two-Link Planar Robot Arm

A two-link planar robot arm is shown in Fig. 7.7. The length of the links are $L_1 = 1$ m and $L_2 = 1$ m. The masses of the rigid links are $m_1 = 1$ kg and $m_2 = 1$ kg. The gravitational acceleration is $g = 9.81$ m/s^2.
The generalized coordinates are $q_1(t)$ and $q_2(t)$ as shown in Fig. 7.7.

The initial conditions, at $t = 0$ s, are $q_1(0) = -\pi/18$ rad, $\dot{q}_1(0) = 0$ rad/s, $q_2(0) = \pi/6$ rad, and $\dot{q}_2(0) = 0$ rad/s.

The robot arm can be brought from an initial state of rest to a final state of rest in such a way that q_1 and q_2 have the specified values $q_{1f} = \pi/6$ rad and $q_{2f} = \pi/3$ rad.

The set of contact forces transmitted from 0 to 1 can be replaced with a couple of torque $\mathbf{T}_{01} = T_{01z}\,\mathbf{k}$ applied to 1 at A. Similarly, the set of contact forces transmitted from 1 to 2 can be replaced with a couple of torque $\mathbf{T}_{12} = T_{12z}\,\mathbf{k}$ applied to 2 at B. The law of action and reaction then guarantees that the set of contact forces transmitted from 1 to 2 is equivalent to a couple of torque $-\mathbf{T}_{12}$ to 1 at B. The following feedback control laws are given

$$
\begin{aligned}
T_{01z} &= -\beta_{01}\,\dot{q}_1 - \gamma_{01}\,(q_1 - q_{1f}) + 0.5\,g\,L_1\,m_1\,\cos(q_1) + g\,L1\,m_2\,\cos(q_1),\\
T_{12z} &= -\beta_{12}\,\dot{q}_2 - \gamma_{12}\,(q_2 - q_{2f}) + 0.5\,g\,L_2\,m_2\,\cos(q_2).
\end{aligned}
$$

The constant gains are: $\beta_{01} = 450$ N·m·s/rad, $\gamma_{01} = 300$ N·m/rad, $\beta_{12} = 200$ N·m·s/rad, and $\gamma_{12} = 300$ N·m/rad.

Write a MATLAB program for solving the equations of motion.

Solution
The MATLAB commands for the kinematics of the robot arm are

```
L1 = 1; L2 = 1; m1 = 1; m2 = 1; g = 9.81;
t = sym('t','real');
xB = L1*cos(sym('q1(t)')); yB = L1*sin(sym('q1(t)'));
rB = [xB yB 0];
rC1 = rB/2; vC1 = diff(rC1,t); aC1 = diff(vC1,t);
xD = xB + L2*cos(sym('q2(t)')); yD = yB + L2*sin(sym('q2(t)'));
rD = [xD yD 0];
rC2 = (rB + rD)/2; vC2 = diff(rC2,t); aC2 = diff(vC2,t);
omega1 = [0 0 diff('q1(t)',t)]; alpha1 = diff(omega1,t);
omega2 = [0 0 diff('q2(t)',t)]; alpha2 = diff(omega2,t);
```

The weight forces on the links and the mass moment of inertia of the links are

```
G1 = [0 -m1*g 0]; G2 = [0 -m2*g 0];
IC1 = m1*L1^2/12; IA = IC1 + m1*(L1/2)^2; IC2 = m2*L2^2/12;
```

The joint reaction force $\mathbf{F}_{21}$ is calculated with

```
F21 = -m2*aC2 + G2;
```

The control torques are given by

```
b01 = 450; g01 = 300;
b12 = 200; g12 = 300;
q1f = pi/6;
q2f = pi/3;
T01z = -b01*diff('q1(t)',t)-g01*(sym('q1(t)')-q1f)...
    +0.5*g*L1*m1*cos(sym('q1(t)'))+g*L1*m2*cos(sym('q1(t)'));
T01 = [0 0 T01z];
T12z = -b12*diff('q2(t)',t)-g12*(sym('q2(t)')-q2f)...
   +0.5*g*L2*m2*cos(sym('q2(t)'));
T12 = [0 0 T12z];
```

The moment equations for each link, Eqs. (7.20) and (7.23), using MATLAB are

```
EqA = -IA*alpha1 + cross(rB, F21) + cross(rC1, G1) + T01 - T12;
Eq2 = -IC2*alpha2 + cross(rB - rC2, -F21) + T12;
slist = {diff('q1(t)',t,2),diff('q2(t)',t,2),...
        diff('q1(t)',t),diff('q2(t)',t),'q1(t)','q2(t)'};
nlist = {'ddq1', 'ddq2', 'x(2)', 'x(4)', 'x(1)','x(3)'};
eq1 = subs(EqA(3),slist,nlist);
eq2 = subs(Eq2(3),slist,nlist);
sol = solve(eq1,eq2,'ddq1, ddq2');
dx2 = sol.ddq1;
dx4 = sol.ddq2;

dx2dt = char(dx2);
dx4dt = char(dx4);
```

The equations of motion are complex and a m-file function `RRrobot.m` is constructed with the commands

```
fid = fopen('RRrobot.m','w+');
fprintf(fid,'function dx = RRrobot(t,x)\n');
fprintf(fid,'dx = zeros(4,1);\n');
fprintf(fid,'dx(1) = x(2);\n');
fprintf(fid,'dx(2) = ');
fprintf(fid,dx2dt);
fprintf(fid,';\n');
fprintf(fid,'dx(3) = x(4);\n');
fprintf(fid,'dx(4) = ');
fprintf(fid,dx4dt);
fprintf(fid,';');
fclose(fid);
cd(pwd);
```

The system of differential equations is solved using `ode45`

```
t0 = 0; tf = 15; time = [0 tf];
x0 = [-pi/18 0 pi/6 0];
```

```
[t,xs] = ode45(@RRrobot, time, x0);
x1 = xs(:,1);
x2 = xs(:,2);
x3 = xs(:,3);
x4 = xs(:,4);
subplot(2,1,1),plot(t,x1*180/pi,'r'),...
xlabel('t (s)'),ylabel('q1 (deg)'),grid,...
subplot(2,1,2),plot(t,x3*180/pi,'b'),...
xlabel('t (s)'),ylabel('q2 (deg)'),grid
[ts,xs] = ode45(@RRrobot,0:1:5,x0);
```

The plots of q_1 and q_2 for the considered time interval, using MATLAB, are shown in Fig. 7.8 and the MATLAB program is given in the Program 7.7.