1 Dynamic Force Analysis

1.1 Joint Forces Analysis

The R-RTR mechanism shown in Fig. 1.1 has the dimensions: $AB = 0.14$ m, $AC = 0.06$ m, and $CF = 0.2$ m. The driver link 1 makes an angle $\phi = \phi_1 = \dfrac{\pi}{3}$ rad with the horizontal axis and rotates with a constant speed of $n = n_1 = 30\,\pi$ rpm. The position vectors of the points A, B, C, and F are

$$\mathbf{r}_A = 0\imath + 0\jmath \text{ m},$$
$$\mathbf{r}_B = \mathbf{r}_{C_2} = x_B\imath + y_B\jmath = 0.07\imath + 0.121\jmath \text{ m},$$
$$\mathbf{r}_C = x_C\imath + y_C\jmath = 0\imath + 0.06\jmath \text{ m},$$
$$\mathbf{r}_F = x_F\imath + y_F\jmath = 0.150\imath + 0.191\jmath \text{ m}.$$

The position vectors of the mass centers of links 1 and 3 are

$$\mathbf{r}_{C_1} = x_{C_1}\imath + y_{C_1}\jmath = \frac{x_B}{2}\imath + \frac{y_B}{2}\jmath = 0.035\imath + 0.06\jmath \text{ m},$$
$$\mathbf{r}_{C_3} = x_{C_3}\imath + y_{C_3}\jmath = \frac{x_C + x_F}{2}\imath + \frac{y_C + y_F}{2}\jmath = 0.075\imath + 0.125\jmath \text{ m}.$$

The mass center of the slider 2 is at B ($B = C_2$): $\mathbf{r}_{C_2} = \mathbf{r}_B$.

The height of the links 1 and 3 is $h = 0.01$ m. The width of the slider 2 is $w_{Slider} = 0.05$ m, and the height is $h_{Slider} = 0.020$ m. All three moving links are rectangular prisms with the depth $d = 0.01$ m.

The density of the material is $\rho_{Steel} = \rho = 8000$ kg/m^3. The gravitational acceleration is $g = 9.807$ m/s^2.

Forces and moments on each link

Link 1

The mass of the link 1 is

$$m_1 = \rho \; AB \; h \; d = 0.112 \text{ kg}.$$

The acceleration of C_1 is

$$\mathbf{a}_{C_1} = -3.40932\,\imath - 5.90511\,\jmath \text{ m/s}^2.$$

The term $m_1\,C_1$ is

$$m_1\,\mathbf{a}_{C_1} = -0.381844\,\imath - 0.661373\,\jmath \text{ N}.$$

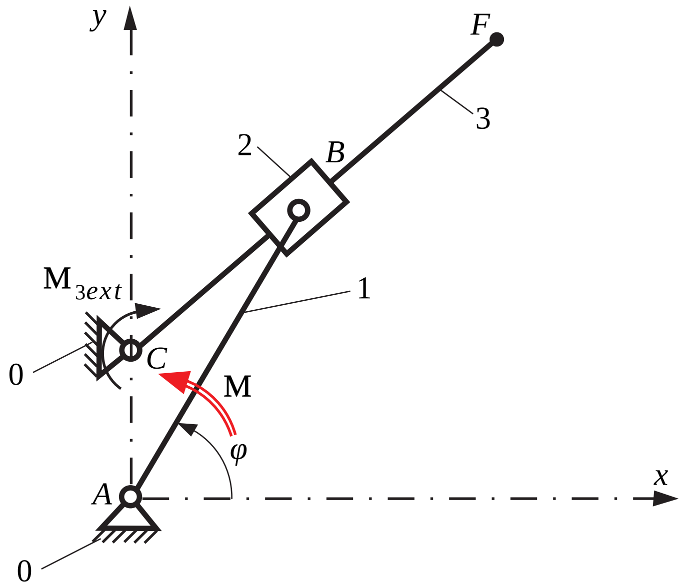

Fig. 1

The gravitational force on link 1 is

$$\mathbf{G}_1 = -m_1\ g\ \jmath = -1.09838\ \jmath\ \text{N}.$$

The mass moment of inertia of link 1 about C_1 is

$$I_{C_1} = m_1\ (AB^2 + h^2)/12 = 0.000183867\ \text{kg} \cdot \text{m}^2.$$

The term $I_{C_1}\ \boldsymbol{\alpha}_1$ is

$$I_{C_1}\ \boldsymbol{\alpha}_1 = \mathbf{0}.$$

Link 2

The mass of the link 2 is

$$m_2 = \rho_2\ h_{Slider}\ w_{Slider}\ d = 0.08\ \text{kg}.$$

The acceleration of $C_2 = B$ is

$$\mathbf{a}_{C_2} = -6.81864\ \imath - 11.8102\ \jmath\ \text{m/s}^2.$$

The term $m_2\, C_2$ is

$$m_2\, \mathbf{a}_{C_2} = -0.545491\ \imath - 0.944818\ \jmath\ \text{N}.$$

The gravitational force is

$$\mathbf{G}_2 = -m_2\ g\ \jmath = -0.78456\ \jmath\ \text{N}.$$

The mass moment of inertia of link 2 about C_2 is

$$I_{C_2} = m_2\ (h^2_{Slider} + w^2_{Slider})/12 = 0.0000193333\ \text{kg} \cdot \text{m}^2.$$

The term $I_{C_2}\ \boldsymbol{\alpha}_2$ is

$$I_{C_2}\ \boldsymbol{\alpha}_2 = 0.00169109\ \mathbf{k}\ \text{N} \cdot \text{m}.$$

where $\boldsymbol{\alpha}_2 = 87.47\ \mathbf{k}\ \text{rad/s}^2$.

Link 3

The mass of the link 3 is

$$m_3 = \rho\ CF\ h\ d = 0.112\ \text{kg}.$$

The acceleration of C_3 is

$$\mathbf{a}_{C_3} = -20.6416\ \imath - 6.4373\ \jmath\ \text{m/s}^2.$$

The term $m_3\, C_3$ is

$$m_3\, \mathbf{a}_{C_3} = -3.30266\ \imath - 1.02997\ \jmath\ \text{N}.$$

The gravitational force is

$$\mathbf{G}_3 = -m_3\ g\ \jmath = -1.56912\ \jmath\ \text{N}.$$

The mass moment of inertia of link 3 about C_3 is

$$I_{C_3} = m_3\ (CF^2 + h^2)/12 = 0.000534667\ \text{kg} \cdot \text{m}^2.$$

The term $I_{C_3}\ \boldsymbol{\alpha}_3$ is

$$I_{C_3}\ \boldsymbol{\alpha}_3 = 0.0467673\ \mathbf{k}\ \text{N} \cdot \text{m}.$$

where $\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_2 = 87.47\ \mathbf{k}\ \text{rad/s}^2$.

The external moment on link 3 is

$$\mathbf{M}_{3ext} = -\text{Sign}(\boldsymbol{\omega}_3)|M_{ext}|\mathbf{k} = -1000\mathbf{k}\ \text{N} \cdot \text{m},$$

where $\boldsymbol{\omega}_3 = 14.0619\ \mathbf{k}$ rad/s and $M_{ext} = |M_{ext}| = 1000$ N

Determine the moment **M** that acts on link 1 required for dynamic equilibrium and the joint forces for the mechanism.

Newton-Euler equations of motion

For each moving link the Newton-Euler equations of motion are written

$$m_i\ \mathbf{a}_{C_i} = \sum \mathbf{F}^{(i)} \quad \text{and} \quad I_{C_i}\boldsymbol{\alpha}_i = \sum \mathbf{M}_{C_i}^{(i)},$$

where C_i is the center of mass of the link i.

The position vectors are

$$\begin{aligned}
\mathbf{r}_A &= 0\imath + 0\jmath \text{ m},\\
\mathbf{r}_B = \mathbf{r}_{C_2} &= x_B\imath + y_B\jmath = 0.07\imath + 0.121\jmath \text{ m},\\
\mathbf{r}_C &= x_C\imath + y_C\jmath = 0\imath + 0.06\jmath \text{ m},\\
\mathbf{r}_F &= x_F\imath + y_F\jmath = 0.150\imath + 0.191\jmath \text{ m},\\
\mathbf{r}_{C_1} &= x_{C_1}\imath + y_{C_1}\jmath = 0.035\imath + 0.06\jmath \text{ m},\\
\mathbf{r}_{C_3} &= x_{C_3}\imath + y_{C_3}\jmath = 0.075\imath + 0.125\jmath \text{ m}.
\end{aligned}$$

The terms $m_i\ \mathbf{a}_{C_i}$ are

$$\begin{aligned}
m_1\,\mathbf{a}_{C_1} &= -0.381844\imath - 0.661373\jmath \text{ N},\\
m_2\,\mathbf{a}_{C_2} &= -0.545491\imath - 0.944818\jmath \text{ N},\\
m_3\,\mathbf{a}_{C_3} &= -3.30266\imath - 1.02997\jmath \text{ N}.
\end{aligned}$$

The terms $I_{C_i}\boldsymbol{\alpha}_i$ are

$$\begin{aligned}
I_{C_1}\ \boldsymbol{\alpha}_1 &= \mathbf{0} \text{ N}\cdot\text{m},\\
I_{C_2}\ \boldsymbol{\alpha}_2 &= 0.00169109\mathbf{k} \text{ N}\cdot\text{m},\\
I_{C_3}\ \boldsymbol{\alpha}_3 &= 0.0467673\mathbf{k} \text{ N}\cdot\text{m}.
\end{aligned}$$

The force analysis will start with link 3 because the moment $\mathbf{M}_{3ext}$ is known.

Link 3

For the link 3, Fig. 1.2(a), the equations of motion for planar motion give

$$\begin{aligned}
m_3\mathbf{a}_{C_3} &= \sum \mathbf{F}^{(3)} = \mathbf{F}_{03} + \mathbf{G}_3 + \mathbf{F}_{23},\\
I_{C_3}\ \boldsymbol{\alpha}_3 &= \sum \mathbf{M}_{C_3}^{(3)} = \mathbf{r}_{C_3C} \times \mathbf{F}_{03} + \mathbf{r}_{C_3Q} \times \mathbf{F}_{23} + \mathbf{M}_{3ext},
\end{aligned}$$

or

$$m_3\, a_{C_3x} = F_{03x} + F_{23x},$$
$$m_3\, a_{C_3y} = F_{03y} - m_3\, g + F_{23y},$$
$$I_{C_3}\, \alpha_3\, \mathbf{k} = \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ x_C - x_{C3} & y_C - y_{C3} & 0 \\ F_{03x} & F_{03y} & 0 \end{vmatrix} + \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ x_Q - x_{C3} & y_Q - y_{C3} & 0 \\ F_{23x} & F_{23y} & 0 \end{vmatrix}$$
$$+ M_{3ext}\mathbf{k},$$

or

$$m_3\, a_{C_3x} = F_{03x} + F_{23x},$$
$$m_3\, a_{C_3y} = F_{03y} - m_3\, g + F_{23y},$$
$$I_{C_3}\, \alpha_3 = (x_C - x_{C3})F_{03y} - (y_C - y_{C3})F_{03x}$$
$$+(x_Q - x_{C3})F_{23y} - (y_Q - y_{C3})F_{23x} + M_{3ext},$$

where the unknowns are

$$\mathbf{F}_{03} = F_{03x}\, \imath + F_{03y}\, \jmath, \quad \mathbf{F}_{23} = F_{23x}\, \imath + F_{23y}\, \jmath,$$

and the position vector $\mathbf{r}_Q = x_Q \imath + y_Q \jmath$ of the application point of the joint force $\mathbf{F}_{23}$.
Numerically the previous system of equations becomes

$$-3.30266 = F_{03x} + F_{23x}, \tag{1.1}$$
$$-1.02997 = -1.56912 + F_{03y} + F_{23y}, \tag{1.2}$$
$$0 = -0.065F_{03x} - 0.075F_{03y} + 0.125F_{23x} - 0.075F_{23y} +$$
$$F_{23y}x_Q - F_{23x}y_Q - 1000\ . \tag{1.3}$$

The application point Q of the joint force $\mathbf{F}_{23}$ is on the line BC:

$$\frac{y_B - y_C}{x_B - x_C} = \frac{y_Q - y_C}{x_Q - x_C} \quad \text{or} \quad 0.874 - \frac{y_Q - 0.06}{x_Q} = 0. \tag{1.4}$$

The joint force $\mathbf{F}_{23}$ is perpendicular to the sliding direction BC:

$$\mathbf{F}_{23} \cdot \mathbf{r}_{BC} = 0 \quad \text{or} \quad -0.07F_{23x} - 0.061F_{23y} = 0. \tag{1.5}$$

There are five scalar equations, Eqs. (1.1) throught (1.5), and six unknowns, F_{03x}, F_{03y}, F_{23x}, F_{23y}, x_Q, y_Q. The force analysis will continue with link 2.

Link 2

The Newton-Euler equations for slider 2, Fig. 1.2(b), give

$$m_2\,\mathbf{a}_{C_2} = \sum \mathbf{F}^{(2)} = \mathbf{F}_{12} + \mathbf{G}_2 + \mathbf{F}_{32},$$
$$I_{C_2}\,\boldsymbol{\alpha}_2 = \sum \mathbf{M}^{(2)}_{C_2} = \mathbf{r}_{BQ} \times \mathbf{F}_{32},$$

or

$$m_2\,a_{C_2x} = F_{12x} - F_{23x},$$
$$m_2\,a_{C_2y} = F_{12y} - m_2\,g - F_{23y},$$
$$I_{C_2}\,\alpha_2\,\mathbf{k} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ x_Q - x_B & y_Q - y_B & 0 \\ -F_{23x} & -F_{23y} & 0 \end{vmatrix},$$

or

$$m_2\,a_{C_2x} = F_{12x} - F_{23x},$$
$$m_2\,a_{C_2y} = F_{12y} - m_2\,g - F_{23y},$$
$$I_{C_2}\,\alpha_2 = (x_Q - x_B)(-F_{23y}) - (y_Q - y_B)(-F_{23x}),$$

where the new unknown is introduced (the reaction of link 1 on link 2):

$$\mathbf{F}_{12} = F_{12x}\,\boldsymbol{\imath} + F_{12y}\,\boldsymbol{\jmath}.$$

Numerically, the previous equations becomes

$$-0.545491 = F_{12x} - F_{23x}, \quad (1.6)$$
$$-0.944818 = -0.78456 + F_{12y} - F_{23y}, \quad (1.7)$$
$$0.00169109 = -0.121244\,F_{23x} + 0.07\,F_{23y} - x_Q\,F_{23y} + y_Q\,F_{23x} = 0. \quad (1.8)$$

Now there is a system of eight scalar equations, Eqs. (1.1) through (1.8), eight unknowns, and the solution is

$$\mathbf{F}_{03} = F_{03x}\,\boldsymbol{\imath} + F_{03y}\,\boldsymbol{\jmath} = 7078.41\,\boldsymbol{\imath} - 8093.7\,\boldsymbol{\jmath} \quad \text{N},$$
$$\mathbf{F}_{23} = F_{23x}\,\boldsymbol{\imath} + F_{23y}\,\boldsymbol{\jmath} = -7081.72\,\boldsymbol{\imath} + 8094.24\,\boldsymbol{\jmath} \quad \text{N},$$
$$\mathbf{F}_{12} = F_{12x}\,\boldsymbol{\imath} + F_{12y}\,\boldsymbol{\jmath} = -7082.26\,\boldsymbol{\imath} + 8094.08\,\boldsymbol{\jmath} \quad \text{N},$$
$$\mathbf{r}_Q = x_Q\,\boldsymbol{\imath} + y_Q\,\boldsymbol{\jmath} = 0.069\,\boldsymbol{\imath} + 0.121\,\boldsymbol{\jmath} \quad \text{m}.$$

Link 1

The force equation for the driver link 1, Fig. 1.2(c), gives

$$m_1\,\mathbf{a}_{C_1} = \sum \mathbf{F}^{(1)} = \mathbf{F}_{21} + \mathbf{G}_1 + \mathbf{F}_{01}.$$

The reaction of the ground 0 on the link 1 is

$$\mathbf{F}_{01} \quad = \quad m_1\,\mathbf{a}_{C_1} + \mathbf{F}_{12} - \mathbf{G}_1 = -7082.64\,\imath + 8094.52\,\jmath \quad \text{N}.$$

The sum of the moments about the mass center C_1 for link 1 gives the equilibrium moment

$$I_{C_1}\,\alpha_1\,\mathbf{k} = \sum \mathbf{M}^{(1)}_{C_1} = \mathbf{r}_{C_1B} \times \mathbf{F}_{21} + \mathbf{r}_{C_1A} \times \mathbf{F}_{01} + \mathbf{M},$$

or

$$\begin{aligned}\mathbf{M} \quad &= \quad \mathbf{r}_{C_1B} \times \mathbf{F}_{12} - \mathbf{r}_{C_1A} \times \mathbf{F}_{01} \\ &= \quad 712.632\,\mathbf{k} + 712.671\,\mathbf{k} = 1425.3\,\mathbf{k} \quad \text{N} \cdot \text{m}.\end{aligned}$$

Another way of calculating the moment $\mathbf{M}$ required for dynamic equilibrium is to write the moment equation of motion for link 1 about the fixed point A

$$I_A\,\alpha_1\,\mathbf{k} = \sum \mathbf{M}^{(1)}_{A} = \mathbf{r}_{AC_1} \times \mathbf{G}_1 + \mathbf{r}_{AB} \times \mathbf{F}_{21} + \mathbf{M} \implies \mathbf{M} = \mathbf{r}_B \times \mathbf{F}_{12} - \mathbf{r}_{C_1} \times \mathbf{G}_1,$$

where $I_A = I_{C_1} + m_1\,(AB/2)^2$.