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## 6 Dynamic Force Analysis

For a kinematic chain it is important to know how forces and moments are transmitted from the input to the output, so that the links can be properly designated. The friction effects are assumed to be negligible in the force analysis presented here.

### 6.1 Equation of Motion for General Planar Motion

The Newton's second law for a rigid body is obtained

$$m \mathbf{a}_C = \mathbf{F}$$

where  $\mathbf{a}_C$  is the acceleration of the mass center. The derivation of the equations of motion is valid for the general motion of a rigid body. The equation is equally applicable to planar and three-dimensional motions.

Resolving the sum of the external forces into cartesian rectangular components

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k},$$

and the position vector of the mass center

$$\mathbf{r}_C = x_C(t) \mathbf{i} + y_C(t) \mathbf{j} + z_C(t) \mathbf{k},$$

Newton's second law for the rigid body is

$$m \ddot{\mathbf{r}}_C = \mathbf{F}, \tag{6.1}$$

or

$$m \ddot{x}_C = F_x, \quad m \ddot{y}_C = F_y, \quad m \ddot{z}_C = F_z. \tag{6.2}$$

The summation  $\sum_i m_i r_i^2$  or the integration over the body  $\int r^2 dm$  is defined as mass moment of inertia  $I_{Czz}$  of the body about the  $z$ -axis through  $C$ . The mass moment of inertia  $I_{Czz}$  is a constant property of the body and is a measure of the rotational inertia or resistance to change in angular velocity due to the radial distribution of the rigid body mass around  $z$ -axis through  $C$ .

The angular momentum of the rigid body about  $z$ -axis through  $C$  is

$$H_C = I_{Czz} \omega \quad \text{or} \quad \mathbf{H}_C = I_{Czz} \omega \mathbf{k} = I_{Czz} \boldsymbol{\omega}.$$

The rotational equation of motion for the rigid body is

$$I_{Czz} \boldsymbol{\alpha} = \sum \mathbf{M}_C \quad \text{or} \quad I_{Czz} \alpha \mathbf{k} = \sum M_C \mathbf{k}. \quad (6.3)$$

For general planar motion the angular acceleration is  $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \ddot{\theta} \mathbf{k}$ , where the angle  $\theta$  describes the position, or orientation, of the rigid body about a fixed axis. If the rigid body is a plate moving in the plane of motion  $(X, Y)$ , the mass moment of inertia of the rigid body about  $z$ -axis through  $C$  becomes the polar mass moment of inertia of the rigid body about  $C$ ,  $I_{Czz} = I_C$ . For this case the Eq. (6.3) gives

$$I_C \boldsymbol{\alpha} = \sum \mathbf{M}_C. \quad (6.4)$$

A special application of Eq. (6.4) is for rotation about a fixed point. According to parallel-axis theorem

$$I_{Ozz} = I_{Czz} + m r_C^2,$$

where  $I_{Ozz}$  denotes the mass moment of inertia of the rigid body about  $z$ -axis through  $O$ . For the special case of rotation about a fixed point  $O$  one can use the formula

$$I_{Ozz} \boldsymbol{\alpha} = \sum \mathbf{M}_O. \quad (6.5)$$

The general equations of motion for a rigid body in plane motion are

$$\mathbf{F} = m \mathbf{a}_C \quad \text{or} \quad \mathbf{F} = m \ddot{\mathbf{r}}_C, \quad (6.6)$$

$$\sum \mathbf{M}_C = I_{Czz} \boldsymbol{\alpha}, \quad (6.7)$$

or using the cartesian components

$$m\ddot{x}_C = \sum F_x, \quad m\ddot{y}_C = \sum F_y, \quad I_{Czz}\ddot{\theta} = \sum M_C. \quad (6.8)$$

Equations (6.6) and (6.7) are interpreted in two ways

1. The forces and moments are known and the equations are solved for the motion of the rigid body (direct dynamics).
2. The motion of the *RB* is known and the equations are solved for the forces and moments (inverse dynamics).

The dynamic force analysis in this chapter is based on the known motion of the mechanism.

## 6.2 D'Alembert's Principle

Newton's second law can be written as

$$\mathbf{F} + (-m\mathbf{a}_C) = \mathbf{0}, \text{ or } \mathbf{F} + \mathbf{F}_{in} = \mathbf{0}, \quad (6.9)$$

where the term  $\mathbf{F}_{in} = -m\mathbf{a}_C$  is the *inertia force*. Newton's second law can be regarded as an "equilibrium" equation.

The total moment about a fixed point  $O$  is

$$\sum \mathbf{M}_O = (\mathbf{r}_C \times m\mathbf{a}_C) + I_{Czz}\boldsymbol{\alpha},$$

or

$$\sum \mathbf{M}_O + [\mathbf{r}_C \times (-m\mathbf{a}_C)] + (-I_{Czz}\boldsymbol{\alpha}) = \mathbf{0}. \quad (6.10)$$

The term  $\mathbf{M}_{in} = -I_{Czz}\boldsymbol{\alpha}$  is the *inertia moment*. The sum of the moments about any point, including the moment due to the inertial force  $-m\mathbf{a}$  acting at mass center and the inertial moment, equals zero.

The equations of motion for a rigid body are analogous to the equations for static equilibrium:

The sum of the forces equals zero and the sum of the moments about any point equals zero when the inertial forces and moments are taken into account. This is called *D'Alembert's principle*.

The dynamic force analysis is expressed in a form similar to static force analysis

$$\sum \mathbf{R} = \sum \mathbf{F} + \mathbf{F}_{in} = \mathbf{0}, \quad (6.11)$$

$$\sum \mathbf{T}_C = \sum \mathbf{M}_C + \mathbf{M}_{in} = \mathbf{0}, \quad (6.12)$$

where  $\sum \mathbf{F}$  is the vector sum of all external forces (resultant of external force), and  $\sum \mathbf{M}_C$  is the sum of all external moments about the center of mass  $C$  (resultant external moment).

For a rigid body in plane motion in the  $xy$  plane,

$$\mathbf{a}_C = \ddot{x}_C \mathbf{i} + \ddot{y}_C \mathbf{j}, \quad \boldsymbol{\alpha} = \alpha \mathbf{k},$$

with all external forces in that plane, Eqs. (6.11) and (6.12) become

$$\sum R_x = \sum F_x + F_{inx} = \sum F_x + (-m\ddot{x}_C) = 0, \quad (6.13)$$

$$\sum R_y = \sum F_y + F_{iny} = \sum F_y + (-m\ddot{y}_C) = 0, \quad (6.14)$$

$$\sum T_C = \sum M_C + M_{in} = \sum M_C + (-I_C \alpha) = 0. \quad (6.15)$$

With d'Alembert's principle the moment summation can be about any arbitrary point  $P$

$$\sum \mathbf{T}_P = \sum \mathbf{M}_P + \mathbf{M}_{in} + \mathbf{r}_{PC} \times \mathbf{F}_{in} = \mathbf{0}, \quad (6.16)$$

where

- $\sum \mathbf{M}_P$  is the sum of all external moments about  $P$ ,
- $\mathbf{M}_{in}$  is the inertia moment,
- $\mathbf{F}_{in}$  is the inertia force, and
- $\mathbf{r}_{PC}$  is a vector from  $P$  to  $C$ .

The dynamic analysis problem is reduced to a static force and moment balance problem where the inertia forces and moments are treated in the same way as external forces and moments.

### 6.3 Free-Body Diagrams

A free-body diagram is a drawing of a part of a complete system, isolated in order to determine the forces acting on that rigid body.

The following force convention is defined:  $\mathbf{F}_{ij}$  represents the force exerted by link  $i$  on link  $j$ .

Figure 6.1 shows various free-body diagrams that are considered in the analysis of a slider-crank mechanism Fig. 6.1(a).

In Fig. 6.1(b), the free body consists of the three moving links isolated from the frame 0. The forces acting on the system include an external driven force  $\mathbf{F}$ , and the forces transmitted from the frame at joint  $A$ ,  $\mathbf{F}_{01}$ , and at joint  $C$ ,  $\mathbf{F}_{03}$ . Figure 6.1(c) is a free-body diagram of the two links 1 and 2 and Fig. 6.1(d) is a free-body diagram of the two links 0 and 1. Figure 6.1(e) is a free-body diagram of crank 1 and Fig. 6.1(f) is a free-body diagram of slider 3.

The force analysis can be accomplished by examining individual links or a subsystem of links. In this way the joint forces between links as well as the required input force or moment for a given output load are computed.

## 6.4 Force Analysis Using Dyads

### RRR dyad

Figure 6.2 shows an RRR dyad with two links 2 and 3, and three pin joints,  $B$ ,  $C$ , and  $D$ . First, the exterior unknown joint reaction forces are considered

$$\mathbf{F}_{12} = F_{12x} \mathbf{i} + F_{12y} \mathbf{j} \quad \text{and} \quad \mathbf{F}_{43} = F_{43x} \mathbf{i} + F_{43y} \mathbf{j}.$$

To determine  $\mathbf{F}_{12}$  and  $\mathbf{F}_{43}$ , the following equations are written:

- sum of all forces on links 2 and 3 is zero

$$\begin{aligned} \sum \mathbf{F}^{(2\&3)} &\implies \\ m_2 \mathbf{a}_{C_2} + m_3 \mathbf{a}_{C_3} &= \mathbf{F}_{12} + \mathbf{G}_2 + \mathbf{G}_3 + \mathbf{F}_{43}, \end{aligned}$$

or

$$\begin{aligned} \sum \mathbf{F}^{(2\&3)} \cdot \mathbf{i} &\implies \\ m_2 a_{C_{2x}} + m_3 a_{C_{3x}} &= F_{12x} + F_{43x}, \end{aligned} \quad (6.17)$$

$$\begin{aligned} \sum \mathbf{F}^{(2\&3)} \cdot \mathbf{j} &\implies \\ m_2 a_{C_{2y}} + m_3 a_{C_{3y}} &= F_{12y} - m_2 g - m_3 g + F_{43y}. \end{aligned} \quad (6.18)$$

- sum of moments of all forces and moments on link 2 about  $C$  is zero

$$\begin{aligned} \sum \mathbf{M}_C^{(2)} &\implies \\ I_{C_2} \boldsymbol{\alpha}_2 + \mathbf{r}_{CC_2} \times m_2 \mathbf{a}_{C_2} &= \mathbf{r}_{CB} \times \mathbf{F}_{12} + (\mathbf{r}_{CC_2} \times \mathbf{G}_2. \end{aligned} \quad (6.19)$$

- sum of moments of all forces and moments on link 3 about  $C$  is zero

$$\begin{aligned} \sum \mathbf{M}_C^{(3)} &\implies \\ I_{C_3} \boldsymbol{\alpha}_3 + \mathbf{r}_{CC_3} \times m_3 \mathbf{a}_{C_3} &= \mathbf{r}_{CD} \times \mathbf{F}_{43} + \mathbf{r}_{CC_3} \times \mathbf{G}_3. \end{aligned} \quad (6.20)$$

The components  $F_{12x}$ ,  $F_{12y}$ ,  $F_{43x}$ , and  $F_{43y}$  are calculated from Eqs. (6.17), (6.18), (6.19), and (6.20).

The reaction force  $\mathbf{F}_{32} = -\mathbf{F}_{23}$  is computed from the sum of all forces on link 2

$$\sum \mathbf{F}^{(2)} \implies m_2 \mathbf{a}_{C_2} = \mathbf{F}_{12} + \mathbf{G}_2 + \mathbf{F}_{32} \quad \text{or} \quad \mathbf{F}_{32} = m_2 \mathbf{a}_{C_2} - \mathbf{F}_{12} - \mathbf{G}_2.$$

**RRT dyad**

Figure 6.3 shows an RRT dyad with the unknown joint reaction forces  $\mathbf{F}_{12}$ ,  $\mathbf{F}_{43}$ , and  $\mathbf{F}_{23} = -\mathbf{F}_{32}$ . The joint reaction force  $\mathbf{F}_{43}$  is perpendicular to the sliding direction  $\mathbf{F}_{43} \perp \Delta$  or

$$\mathbf{F}_{43} \cdot \Delta = (F_{43x}\mathbf{i} + F_{43y}\mathbf{j}) \cdot (\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = 0. \quad (6.21)$$

In order to determine  $\mathbf{F}_{12}$  and  $\mathbf{F}_{43}$  the following equations are written

- sum of all the forces on links 2 and 3 is zero

$$\sum \mathbf{F}^{(2\&3)} \implies m_2 \mathbf{a}_{C_2} + m_3 \mathbf{a}_{C_3} = \mathbf{F}_{12} + \mathbf{G}_2 + \mathbf{G}_3 + \mathbf{F}_{43},$$

or

$$\sum \mathbf{F}^{(2\&3)} \cdot \mathbf{i} \implies m_2 a_{C_{2x}} + m_3 a_{C_{3x}} = F_{12x} + F_{43x}, \quad (6.22)$$

$$\sum \mathbf{F}^{(2\&3)} \cdot \mathbf{j} \implies m_2 a_{C_{2y}} + m_3 a_{C_{3y}} = F_{12y} - m_2 g - m_3 g + F_{43y}. \quad (6.23)$$

- sum of moments of all the forces and the moments on link 2 about  $C$  is zero

$$\sum \mathbf{M}_C^{(2)} \implies I_{C_2} \alpha_2 + \mathbf{r}_{CC_2} \times m_2 \mathbf{a}_{C_2} = \mathbf{r}_{CB} \times \mathbf{F}_{12} + \mathbf{r}_{CC_2} \times \mathbf{G}_2. \quad (6.24)$$

The components  $F_{12x}$ ,  $F_{12y}$ ,  $F_{43x}$ , and  $F_{43y}$  are calculated from Eqs. (6.21), (6.22), (6.23), and (6.24).

The reaction force components  $F_{32x}$  and  $F_{32y}$  are computed from the sum of all the forces on link 2

$$\sum \mathbf{F}^{(2)} \implies m_2 \mathbf{a}_{C_2} = \mathbf{F}_{12} + \mathbf{G}_2 + \mathbf{F}_{32} \quad \text{or} \quad \mathbf{F}_{32} = m_2 \mathbf{a}_{C_2} - \mathbf{F}_{12} - \mathbf{G}_2.$$

**RTR dyad**

The unknown joint reaction forces  $\mathbf{F}_{12}$  and  $\mathbf{F}_{43}$  are calculated from the relations (Fig. 6.4)

- sum of all the forces on links 2 and 3 is zero

$$\sum \mathbf{F}^{(2\&3)} \implies m_2 \mathbf{a}_{C_2} + m_3 \mathbf{a}_{C_3} = \mathbf{F}_{12} + \mathbf{G}_2 + \mathbf{G}_3 + \mathbf{F}_{43},$$

or

$$\sum \mathbf{F}^{(2\&3)} \cdot \mathbf{i} \implies m_2 a_{C_2x} + m_3 a_{C_3x} = F_{12x} + F_{43x}, \quad (6.25)$$

$$\sum \mathbf{F}^{(2\&3)} \cdot \mathbf{j} \implies m_2 a_{C_2y} + m_3 a_{C_3y} = F_{12y} - m_2 g - m_3 g + F_{43y}. \quad (6.26)$$

- sum of the moments of all the forces and moments on links 2 and 3 about  $B$  is zero

$$\begin{aligned} \sum \mathbf{M}_B^{(2\&3)} \implies I_{C_2} \boldsymbol{\alpha}_2 + I_{C_3} \boldsymbol{\alpha}_3 + \mathbf{r}_{BC_2} \times m_2 \mathbf{a}_{C_2} + \mathbf{r}_{BC_3} \times m_3 \mathbf{a}_{C_3} = \\ \mathbf{r}_{BD} \times \mathbf{F}_{43} + \mathbf{r}_{BC_3} \times \mathbf{G}_3 + \mathbf{r}_{BC_2} \times \mathbf{G}_2. \end{aligned} \quad (6.27)$$

- sum of all the forces on link 2 projected onto the sliding direction  $\Delta = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$  is zero

$$\sum \mathbf{F}^{(2)} \cdot \Delta = (\mathbf{F}_{12} + \mathbf{F}_2) \cdot (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) = 0. \quad (6.28)$$

The components  $F_{12x}$ ,  $F_{12y}$ ,  $F_{43x}$ , and  $F_{43y}$  are calculated from Eqs. (6.25), (6.26), (6.27), and (6.28).

The force components  $F_{32x}$  and  $F_{32y}$  are computed from the sum of all the forces on link 2

$$\sum \mathbf{F}^{(2)} \implies m_2 \mathbf{a}_{C_2} = \mathbf{F}_{12} + \mathbf{G}_2 + \mathbf{F}_{32} \quad \text{or} \quad \mathbf{F}_{32} = m_2 \mathbf{a}_{C_2} - (\mathbf{F}_{12} + \mathbf{G}_2).$$

## 6.5 Force Analysis Using Contour Method

An analytical method to compute joint forces that can be applied for both planar and spatial mechanisms will be presented. The method is based on the decoupling of a closed kinematic chain and writing the dynamic equilibrium equations. The kinematic links are loaded with external forces and inertia forces and moments.

A general monocontour closed kinematic chain is considered in Fig. 6.5. The joint force between the links  $i - 1$  and  $i$  (joint  $A_i$ ) will be determined. When these two links  $i - 1$  and  $i$  are separated the joint forces  $\mathbf{F}_{i-1,i}$  and  $\mathbf{F}_{i,i-1}$  are introduced and

$$\mathbf{F}_{i-1,i} + \mathbf{F}_{i,i-1} = \mathbf{0}. \quad (6.29)$$

Table 6.1 shows the joint forces for one degree of freedom joints. It is helpful to “mentally disconnect” the two links ( $i - 1$ ) and  $i$ , which create joint  $A_i$ , from the rest of the mechanism. The joint at  $A_i$  will be replaced by the joint forces  $\mathbf{F}_{i-1,i}$  and  $\mathbf{F}_{i,i-1}$ . The closed kinematic chain has been transformed into two open kinematic chains, and two paths  $I$  and  $II$  are associated. The two paths start from  $A_i$ .

For the path  $I$  (counterclockwise), starting at  $A_i$  and following  $I$  the first joint encountered is  $A_{i-1}$ . For the link  $i - 1$  left behind, dynamic equilibrium equations are written according to the type of the joint at  $A_{i-1}$ . Following the same path  $I$ , the next joint encountered is  $A_{i-2}$ . For the subsystem ( $i - 1$  and  $i - 2$ ) equilibrium conditions corresponding to the type of joint at  $A_{i-2}$  can be specified, and so on. A similar analysis is performed for the path  $II$  of the open kinematic chain. The number of equilibrium equations written is equal to the number of unknown scalars introduced by joint  $A_i$  (joint forces at this joint). For a joint, the number of equilibrium conditions is equal to the number of relative mobilities of the joint.

## 6.6 R-RRT (slider-crank) Mechanism

Figure 6.6(a) is a schematic diagram of a R-RRT (slider-crank) mechanism comprised of a crank 1, a connecting rod 2, and a slider 3. The mechanism shown in the figure has the dimensions:  $AB = 1$  m and  $BC = 1$  m. The driver link 1 makes an angle  $\phi = \phi_1 = \pi/4$  rad with the horizontal axis and rotates with a constant speed of  $n = 30/\pi$  rpm. The point  $A$  is selected as the origin of the  $xyz$  reference frame. The position vectors of the joints  $B$  and  $C$  are

$$\mathbf{r}_B = x_B \mathbf{i} + y_B \mathbf{j} = \frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j} \text{ m} \quad \text{and} \quad \mathbf{r}_C = x_C \mathbf{i} + y_C \mathbf{j} = \sqrt{2} \mathbf{i} + 0 \mathbf{j} \text{ m}.$$

The angular velocities of links 1 and 2 are

$$\boldsymbol{\omega}_1 = \omega_1 \mathbf{k} = 1 \mathbf{k} \text{ rad/s} \quad \text{and} \quad \boldsymbol{\omega}_2 = \omega_2 \mathbf{k} = -1 \mathbf{k} \text{ rad/s}.$$

The angular accelerations of link 1 and 2 are  $\boldsymbol{\alpha}_1$  and  $\boldsymbol{\alpha}_2$ . For this particular configuration of the mechanism  $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = \mathbf{0}$ .

The velocity and acceleration of  $B$  are

$$\mathbf{v}_B = -\frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j} \text{ m/s} \quad \text{and} \quad \mathbf{a}_B = -\frac{\sqrt{2}}{2} \mathbf{i} - \frac{\sqrt{2}}{2} \mathbf{j} \text{ m/s}^2.$$

The velocity and acceleration of  $C$  are

$$\mathbf{v}_C = -\sqrt{2} \mathbf{i} \text{ m/s} \quad \text{and} \quad \mathbf{a}_C = -\sqrt{2} \mathbf{i} \text{ m/s}^2.$$

The center of mass of link 1 is  $C_1$ , the center of mass of link 2 is  $C_2$ , and the center of mass of slider 3 is  $C$ . The position vectors of the  $C_i$ ,  $i = 1, 2, 3$  are

$$\begin{aligned} \mathbf{r}_{C_1} &= \mathbf{r}_B/2 = x_{C_1} \mathbf{i} + y_{C_1} \mathbf{j} = \frac{\sqrt{2}}{4} \mathbf{i} + \frac{\sqrt{2}}{4} \mathbf{j} \text{ m}, \\ \mathbf{r}_{C_2} &= (\mathbf{r}_B + \mathbf{r}_C)/2 = x_{C_2} \mathbf{i} + y_{C_2} \mathbf{j} = \frac{3\sqrt{2}}{4} \mathbf{i} + \frac{\sqrt{2}}{4} \mathbf{j} \text{ m}, \\ \mathbf{r}_{C_3} &= \mathbf{r}_C = x_{C_3} \mathbf{i} + y_{C_3} \mathbf{j} = \sqrt{2} \mathbf{i} \text{ m}. \end{aligned}$$

The acceleration vectors of the  $C_i$ ,  $i = 1, 2, 3$  are

$$\begin{aligned} \mathbf{a}_{C_1} &= \mathbf{a}_B/2 = a_{C_1x} \mathbf{i} + a_{C_1y} \mathbf{j} = -\frac{\sqrt{2}}{4} \mathbf{i} - \frac{\sqrt{2}}{4} \mathbf{j} \text{ m/s}^2, \\ \mathbf{a}_{C_2} &= (\mathbf{a}_B + \mathbf{a}_C)/2 = a_{C_2x} \mathbf{i} + a_{C_2y} \mathbf{j} = -\frac{3\sqrt{2}}{4} \mathbf{i} - \frac{\sqrt{2}}{4} \mathbf{j} \text{ m/s}^2, \\ \mathbf{a}_{C_3} &= \mathbf{a}_C = a_{C_x} \mathbf{i} + a_{C_y} \mathbf{j} = -\sqrt{2} \mathbf{i} \text{ m/s}^2. \end{aligned}$$

The external driven force  $\mathbf{F}_{ext}$  applied on link 3 is opposed to the motion of the link (opposed to  $\mathbf{v}_C$ ). Because  $\mathbf{v}_C = -\sqrt{2} \mathbf{i}$  m/s, the external force vector will be

$$\mathbf{F}_{ext} = [-\text{Sign}(\mathbf{v}_C)] 100 \mathbf{i} = 100 \mathbf{i} \text{ N.}$$

The height of the links 1 and 2 is  $h = 0.01$  m. The width of the links 3 is  $w_{Slider} = 0.01$  m and the height is  $h_{Slider} = 0.01$  m [Fig. 6.6(b)]. All three moving links are rectangular prisms with the depth  $d = 0.001$  m. The acceleration of gravity is  $g = 10$  m/s<sup>2</sup>.

### Inertia forces and moments

#### Link 1

The mass of the crank 1 is

$$m_1 = \rho_1 AB h d,$$

where the density of the material is  $\rho_1$ . For the simplicity of calculations  $m_1 = 1$  kg.

The inertia force on link 1 at  $C_1$  is

$$\mathbf{F}_{in1} = -m_1 \mathbf{a}_{C_1} = \frac{\sqrt{2}}{4} \mathbf{i} + \frac{\sqrt{2}}{4} \mathbf{j} \text{ N.}$$

The gravitational force on crank 1 at  $C_1$  is

$$\mathbf{G}_1 = -m_1 g \mathbf{j} = -10 \mathbf{j} \text{ N.}$$

The mass moment of inertia of the link 1 is

$$I_{C_1} = m_1 (AB^2 + h^2)/12 = 0.0833417 \text{ kg} \cdot \text{m}^2.$$

The moment of inertia on link 1 is

$$\mathbf{M}_{in1} = -I_{C_1} \boldsymbol{\alpha}_1 = \mathbf{0}.$$

#### Link 2

The mass of connecting rod 2 is

$$m_2 = \rho_2 BC h d,$$

where the density of the material of link 2 is  $\rho_2$ . For the simplicity of calculations  $m_2 = 1$  kg.

The inertia force on link 2 at  $C_2$  is

$$\mathbf{F}_{in2} = -m_2 \mathbf{a}_{C_2} = \frac{3\sqrt{2}}{4} \mathbf{i} + \frac{\sqrt{2}}{4} \mathbf{j} \text{ N.}$$

The gravitational force on link 2 at  $C_2$  is

$$\mathbf{G}_2 = -m_2 g \mathbf{j} = -10 \mathbf{j} \text{ N.}$$

The mass moment of inertia of link 2 is

$$I_{C_2} = m_2 (BC^2 + h^2)/12 = 0.0833417 \text{ kg} \cdot \text{m}^2.$$

The moment of inertia on link 2 is

$$\mathbf{M}_{in2} = -I_{C_2} \boldsymbol{\alpha}_2 = \mathbf{0}.$$

### Link 3

The mass of the link 3 is

$$m_3 = \rho_3 h_{Slider} w_{Slider} d,$$

where the density of the material of link 3 is  $\rho_3$ . For the simplicity of calculations  $m_3 = 1$  kg.

The inertia force on link 3 at  $C_3 = C$  is

$$\mathbf{F}_{in3} = -m_3 \mathbf{a}_{C_3} = \sqrt{2} \mathbf{i} \text{ N.}$$

The gravitational force on link 3 at  $C_3 = C$  is

$$\mathbf{G}_3 = -m_3 g \mathbf{j} = -10 \mathbf{j} \text{ N.}$$

The mass moment of inertia of slider 3 is

$$I_{C_3} = m_3 (h_{Slider}^2 + w_{Slider}^2)/12 = 0.0000166667 \text{ kg} \cdot \text{m}^2.$$

The moment of inertia on slider 3 is

$$\mathbf{M}_{in3} = -I_{C_3} \boldsymbol{\alpha}_3 = \mathbf{0}.$$

For a given value of the crank angle  $\phi$  ( $\phi = \pi/4$ ) and a known driven force  $\mathbf{F}_{ext}$  find the joint reactions and the drive moment  $\mathbf{M}$  on the crank.

## Joint forces and drive moment

### 6.6.1 Newton-Euler Equations of Motion

Figure 6.6(c) shows the free-body diagrams of the crank 1, the connecting rod 2, and the slider 3. For each moving link the dynamic equilibrium equations are applied (Newton-Euler equations of motion)

$$m \mathbf{a}_C = \sum \mathbf{F} \quad \text{and} \quad I_C \boldsymbol{\alpha} = \sum \mathbf{M}_C,$$

where  $C$  is the center of mass of the link.

The force analysis starts with the link 3 because the external driven force  $\mathbf{F}_{ext}$  on the slider is given.

The reaction joint force of the ground 0 on the slider 3,  $\mathbf{F}_{03}$ , is perpendicular on the sliding direction,  $x$ -axis:  $\mathbf{F}_{03} \perp \mathbf{i}$  (Figure 6.7). The application point  $Q$  of the reaction force  $\mathbf{F}_{03}$  is determined using Euler moment equation

$$I_{C_3} \boldsymbol{\alpha}_3 = \mathbf{r}_{CQ} \times \mathbf{F}_{03} \quad \text{or} \quad \mathbf{0} = \mathbf{r}_{CQ} \times \mathbf{F}_{03} \quad \implies \quad \mathbf{r}_{CQ} = \mathbf{0} \quad \text{or} \quad C = Q.$$

It results the reaction force  $\mathbf{F}_{03}$  acts at  $C$ .

For the slider 3 the vector sum of the net forces (external forces  $\mathbf{F}_{ext}$ , gravitational force  $\mathbf{G}_3$ , joint forces  $\mathbf{F}_{23}$ ,  $\mathbf{F}_{03}$ ) is equal to  $m_3 \mathbf{a}_{C_3}$  (Fig. 6.7)

$$m_3 \mathbf{a}_{C_3} = \mathbf{F}_{23} + \mathbf{G}_3 + \mathbf{F}_{ext} + \mathbf{F}_{03},$$

where  $\mathbf{F}_{23} = F_{23x} \mathbf{i} + F_{23y} \mathbf{j}$  and  $\mathbf{F}_{03} = F_{03y} \mathbf{j}$ .

Projecting this vectorial equation onto  $x$  and  $y$  axes gives

$$\begin{aligned} m_3 a_{C_{3x}} &= F_{23x} + F_{ext}, \\ m_3 a_{C_{3y}} &= F_{23y} - m_3 g + F_{03y}, \end{aligned}$$

or numerically

$$(1)(-\sqrt{2}) = F_{23x} + 100, \quad (6.30)$$

$$0 = F_{23y} - (1)(10) + F_{03y}. \quad (6.31)$$

There are two equations Eqs. (6.30) and (6.31) and three unknowns  $F_{03y}$ ,  $F_{23x}$  and  $F_{23y}$  and that is why the analysis will continue with link 2.

For the connecting rod 2 (Fig. 6.8), Newton equation gives

$$m_2 \mathbf{a}_{C_2} = \mathbf{F}_{32} + \mathbf{G}_2 + \mathbf{F}_{12}.$$

The previous equation can be projected on  $x$  and  $y$  axes

$$\begin{aligned} m_2 a_{C_{2x}} &= F_{32x} + F_{12x}, \\ m_2 a_{C_{2y}} &= F_{32y} - m_2 g + F_{12y}, \end{aligned}$$

or numerically

$$(1)\left(-\frac{3\sqrt{2}}{4}\right) = F_{32x} + F_{12x}, \quad (6.32)$$

$$(1)\left(-\frac{\sqrt{2}}{4}\right) = F_{32y} - 1(10) + F_{12y}. \quad (6.33)$$

For the link 2 a moment equation can be written with respect to  $C_2$

$$I_{C_2} \alpha_2 = \mathbf{r}_{C_2C} \times \mathbf{F}_{32} + \mathbf{r}_{C_2B} \times \mathbf{F}_{12},$$

or

$$I_{C_2} \alpha_2 \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_C - x_{C_2} & y_C - y_{C_2} & 0 \\ F_{32x} & F_{32y} & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_B - x_{C_2} & y_B - y_{C_2} & 0 \\ F_{12x} & F_{12y} & 0 \end{vmatrix}$$

or

$$I_{C_2} \alpha_2 = (x_C - x_{C_2})F_{32y} - (y_C - y_{C_2})F_{32x} + (x_B - x_{C_2})F_{12y} - (y_B - y_{C_2})F_{12x},$$

or numerically

$$0 = \left(\sqrt{2} - \frac{3\sqrt{2}}{4}\right)F_{32y} - \left(-\frac{\sqrt{2}}{4}\right)F_{32x} + \left(\frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{4}\right)F_{12y} - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\right)F_{12x}. \quad (6.34)$$

Equations (6.30), (6.31), (6.32), (6.33), and (6.34) form a system of 5 equations with five scalar unknowns. The system can be solved using the `solve` statement.

The numerical values for the joint forces for the links 3 and 2 are

$$\begin{aligned} F_{03y} &= -85 - \frac{3\sqrt{2}}{2} \text{ N}, \\ F_{23x} &= -100 - \sqrt{2} \text{ N}, \quad F_{23y} = 95 + \frac{3\sqrt{2}}{2} \text{ N}, \\ F_{12x} &= -\frac{1}{4}(400 + 7\sqrt{2}) \text{ N}, \quad F_{12y} = \frac{5}{4}(84 + \sqrt{2}) \text{ N}, \end{aligned}$$

or

$$\begin{aligned} F_{03} &= |\mathbf{F}_{03}| = 85 + \frac{3\sqrt{2}}{2} \text{ N}, \\ F_{23} &= |\mathbf{F}_{23}| = \sqrt{F_{23x}^2 + F_{23y}^2} = \sqrt{\frac{38063}{2} + 485\sqrt{2}} \text{ N}, \\ F_{12} &= |\mathbf{F}_{12}| = \sqrt{F_{12x}^2 + F_{12y}^2} = \frac{1}{2}\sqrt{84137 + 2450\sqrt{2}} \text{ N}. \end{aligned}$$

For the crank 1 (Fig. 6.9), there are two vectorial equations

$$\begin{aligned} m_1 \mathbf{a}_{C_1} &= \mathbf{F}_{21} + \mathbf{G}_1 + \mathbf{F}_{01}, \\ I_{C_1} \boldsymbol{\alpha}_1 &= \mathbf{r}_{C_1 B} \times \mathbf{F}_{21} + \mathbf{r}_{C_1 A} \times \mathbf{F}_{01} + \mathbf{M}, \end{aligned}$$

where  $\mathbf{M}$ , is the input (motor) moment on the crank,  $\mathbf{F}_{21} = -\mathbf{F}_{12}$ , and  $\mathbf{F}_{01} = F_{01x}\mathbf{i} + F_{01y}\mathbf{j}$ .

The above vectorial equations give three scalar equations on  $x$ ,  $y$ , and  $z$

$$\begin{aligned} m_1 a_{C_{1x}} &= F_{21x} + F_{01x}, \\ m_1 a_{C_{1y}} &= F_{21y} - m_1 g + F_{01y}, \\ I_{C_1} \alpha_1 \mathbf{k} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_B - x_{C_1} & y_B - y_{C_1} & 0 \\ F_{21x} & F_{21y} & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_A - x_{C_1} & y_A - y_{C_1} & 0 \\ F_{01x} & F_{01y} & 0 \end{vmatrix} + \\ &+ M \mathbf{k} = \mathbf{0}, \end{aligned}$$

or

$$\begin{aligned} m_1 a_{C_{1x}} &= F_{21x} + F_{01x}, \\ m_1 a_{C_{1y}} &= F_{21y} - m_1 g + F_{01y} \\ I_{C_1} \alpha_1 &= (x_B - x_{C_1})F_{21y} - (y_B - y_{C_1})F_{21x} + \\ &+ (x_A - x_{C_1})F_{01y} - (y_A - y_{C_1})F_{01x} + M, \end{aligned}$$

or numerically

$$1\left(-\frac{\sqrt{2}}{4}\right) = \frac{1}{4}(400 + 7\sqrt{2}) + F_{01x}, \quad (6.35)$$

$$1\left(-\frac{\sqrt{2}}{4}\right) = -\frac{5}{4}(84 + \sqrt{2}) - 1(10) + F_{01y}, \quad (6.36)$$

$$0 = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\right) \left[-\frac{5}{4}(84 + \sqrt{2})\right] - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\right) \left[\frac{1}{4}(400 + 7\sqrt{2})\right] \\ -\frac{\sqrt{2}}{4}F_{01y} + \frac{\sqrt{2}}{4}F_{01y} + M = 0. \quad (6.37)$$

Equations (6.35), (6.36) and (6.37) give

$$F_{01x} = -2(50 + \sqrt{2}) \text{ N}, \quad F_{01y} = 115 + \sqrt{2} \text{ N}, \\ M = 3 + 105\sqrt{2} \text{ N} \cdot \text{m},$$

or

$$F_{01} = |\mathbf{F}_{01}| = \sqrt{F_{01x}^2 + F_{01y}^2} = \sqrt{23235 + 630\sqrt{2}} \text{ N}, \\ M = |\mathbf{M}| = 3 + 105\sqrt{2} \text{ N} \cdot \text{m}.$$

Another way of calculating the moment  $\mathbf{M}$  required for dynamic equilibrium is to write the moment equation of motion for link 1 about the fixed point  $A$

$$I_A \alpha_1 \mathbf{k} = \mathbf{r}_{AC_1} \times \mathbf{G}_1 + \mathbf{r}_{AB} \times \mathbf{F}_{21} + \mathbf{M} \implies \mathbf{M} = \mathbf{r}_B \times \mathbf{F}_{12} - \mathbf{r}_{C_1} \times \mathbf{G}_1,$$

where  $I_A = I_{C_1} + m_1 (AB/2)^2$ . The reaction force  $\mathbf{F}_{01}$  does not appear in this moment equation.

The MATLAB program using Newton-Euler equations and the results are given in Program 6.1.

### 6.6.2 Dyad Method

$B_R C_R C_T$  dyad

Figure 6.10 shows the dyad  $B_R C_R C_T$  with the unknown joint reactions  $\mathbf{F}_{12}$  and  $\mathbf{F}_{03}$ . The joint reaction  $\mathbf{F}_{03}$  is perpendicular to the sliding direction  $\mathbf{F}_{03} \perp \Delta = \mathbf{i}$  or  $\mathbf{F}_{03} = F_{03y}\mathbf{j}$ .

The following equations are written to determine  $\mathbf{F}_{12}$  and  $\mathbf{F}_{03}$

- Newton equation for links 2 and 3,  $\sum \mathbf{F}^{(2\&3)} \implies$

$$m_2 \mathbf{a}_{C_2} + m_3 \mathbf{a}_{C_3} = \mathbf{F}_{12} + \mathbf{G}_2 + \mathbf{G}_3 + \mathbf{F}_{03} + \mathbf{F}_{ext}$$

or

$$\begin{aligned} m_2 a_{C_{2x}} + m_3 a_{C_{3x}} &= F_{12x} + F_{ext}, \\ m_2 a_{C_{2y}} + m_3 a_{C_{3y}} &= F_{12y} - m_2 g - m_3 g + F_{03y} - m_2 a_{C_{2y}}. \end{aligned}$$

or

$$F_{12x} + 102.475 = 0, \quad (6.38)$$

$$F_{12y} + F_{03y} - 19.6464 = 0. \quad (6.39)$$

- Euler equation of moments for links 2 about  $C_R$ ,  $\sum \mathbf{M}_C^{(2)} \implies$

$$I_{C_2} \boldsymbol{\alpha}_2 + \mathbf{r}_{CC_2} \times m_2 \mathbf{a}_{C_2} = \mathbf{r}_{CB} \times \mathbf{F}_{12} + \mathbf{r}_{CC_2} \times \mathbf{G}_2,$$

or

$$-0.707105 F_{12y} - 0.707105 F_{12x} + 3.03552 = 0 \quad (6.40)$$

Equations (6.38), (6.39) and (6.40) give

$$F_{12x} = -102.475 \text{ N}, \quad F_{12y} = 106.768 \text{ N}, \quad \text{and} \quad F_{03y} = -87.1213 \text{ N}.$$

The joint reaction force  $\mathbf{F}_{32}$  is calculated from

$$\mathbf{F}_{32} = m_2 \mathbf{a}_{C_2} - (\mathbf{G}_2 + \mathbf{F}_{12}) = 101.414 \mathbf{i} - 97.1213 \mathbf{j} \text{ N}.$$

The moment  $\mathbf{M}$  required for dynamic equilibrium is calculated from the moment equation of for link 1 (Fig. 6.10) about the fixed point  $A$

$$\sum \mathbf{M}_A^{(1)} \implies I_{C_1} \boldsymbol{\alpha}_1 + \mathbf{r}_{C_1} \times m_1 \mathbf{a}_{C_1} = \mathbf{r}_B \times \mathbf{F}_{21} + \mathbf{G}_1 + \mathbf{M}.$$

It results  $M = 151.492 \text{ N m}$ . The joint reaction force  $\mathbf{F}_{01}$  is calculated from

$$\sum \mathbf{F}^{(1)} \implies m_1 \mathbf{a}_{C_1} = -\mathbf{F}_{12} + \mathbf{G}_1 + \mathbf{F}_{01},$$

and  $\mathbf{F}_{01} = -102.828 \mathbf{i} + 116.414 \mathbf{j} \text{ N}$ .

The MATLAB program dyad equations and the results are given in Program 6.2.

### 6.6.3 Contour Method

The diagram representing the mechanism is shown in Fig. 6.11 and has one contour, 0-1-2-3-0.

The dynamic force analysis can start with any joint.

Reaction  $\mathbf{F}_{03}$

The reaction force  $\mathbf{F}_{03}$  is perpendicular to the sliding direction of joint  $C_T$  ( $C_{Translation}$ ) as shown in Fig. 6.12

$$\mathbf{F}_{03} = F_{03y} \mathbf{J}.$$

The application point of the unknown reaction force  $\mathbf{F}_{03}$  is computed from a moment equation about  $C_R$  ( $C_{Rotation}$ ) for link 3 (path  $I$ ) (Fig. 6.12)

$$\sum \mathbf{M}_C^{(3)} = \mathbf{r}_{CP} \times \mathbf{F}_{03} = (\mathbf{r}_P - \mathbf{r}_C) \times \mathbf{F}_{03} = \mathbf{0},$$

or

$$x F_{03y} = 0 \Rightarrow x = 0.$$

The application point of the reaction force  $\mathbf{F}_{03}$  is at  $C$  ( $P \equiv C$ ).

The magnitude of the reaction force  $F_{03y}$  is obtained from a moment equation about  $B_R$  for the links 3 and 2 (path  $I$ )

$$\begin{aligned} \sum \mathbf{M}_B^{(3\&2)} = \mathbf{r}_{BC} \times (\mathbf{F}_{03} + \mathbf{F}_{in3} + \mathbf{G}_3 + \mathbf{F}_{ext}) + \\ \mathbf{r}_{BC_2} \times (\mathbf{F}_{in2} + \mathbf{G}_2) + \mathbf{M}_{in2} = \mathbf{0}, \end{aligned}$$

or

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_C - x_B & y_C - y_B & 0 \\ F_{in3x} + F_{ext} & F_{03y} + F_{in3y} - m_3 g & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{C_2} - x_B & y_{C_2} - y_B & 0 \\ F_{in2x} & F_{in2y} - m_2 g & 0 \end{vmatrix} + M_{in2} \mathbf{k} = \mathbf{0},$$

or numerically

$$\frac{3}{2} - \frac{5}{\sqrt{2}} + 45\sqrt{2} + \frac{F_{03y}}{\sqrt{2}} = 0.$$

The reaction  $F_{03y}$  is

$$F_{03y} = -85 - \frac{3\sqrt{2}}{2} \text{ N.}$$

Reaction  $\mathbf{F}_{23}$ 

The pin joint at  $C_R$ , between 2 and 3, is replaced with the reaction force (Fig. 6.13)

$$\mathbf{F}_{23} = -\mathbf{F}_{32} = F_{23x} \mathbf{i} + F_{23y} \mathbf{j}.$$

For the path  $I$ , an equation for the forces projected onto the sliding direction of the joint  $C_T$  is written for link 3

$$\begin{aligned} \sum \mathbf{F}^{(3)} \cdot \mathbf{i} &= (\mathbf{F}_{23} + \mathbf{F}_{in3} + \mathbf{G}_3 + \mathbf{F}_{ext}) \cdot \mathbf{i} = \\ F_{23x} + F_{in3x} + F_{ext} &= F_{23x} + 100 + \sqrt{2} = 0. \end{aligned} \quad (6.41)$$

For the path  $II$ , shown Fig. 6.13, a moment equation about  $B_R$  is written for link 2

$$\begin{aligned} \sum \mathbf{M}_B^{(2)} &= \mathbf{r}_{BC} \times (-\mathbf{F}_{23}) + \mathbf{r}_{BC_2} \times (\mathbf{F}_{in2} + \mathbf{G}_2) + \mathbf{M}_{in2} = \mathbf{0}, \\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_C - x_B & y_C - y_B & 0 \\ -F_{23x} & -F_{23y} & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{C_2} - x_B & y_{C_2} - y_B & 0 \\ F_{in2x} & F_{in2y} - m_2 g & 0 \end{vmatrix} + M_{in2} \mathbf{k} &= \mathbf{0}, \end{aligned}$$

or numerically

$$\frac{1}{2} - \frac{5\sqrt{2}}{2} - \frac{F_{23x}\sqrt{2}}{2} - \frac{F_{23y}\sqrt{2}}{2} = 0. \quad (6.42)$$

The joint force  $\mathbf{F}_{23}$  is obtained from the system of Eqs. (6.41) and (6.42)

$$F_{23x} = -100 - \sqrt{2} \text{ N and } F_{23y} = 95 + \frac{3\sqrt{2}}{2} \text{ N.}$$

Reaction  $\mathbf{F}_{12}$ 

The pin joint at  $B_R$ , between 1 and 2, is replaced with the reaction force (Fig. 6.14)

$$\mathbf{F}_{12} = -\mathbf{F}_{21} = F_{12x} \mathbf{i} + F_{12y} \mathbf{j}.$$

For the path  $I$ , shown Fig. 6.8, a moment equation about  $C_R$  is written for link 2

$$\begin{aligned} \sum \mathbf{M}_C^{(2)} &= \mathbf{r}_{CB} \times \mathbf{F}_{12} + \mathbf{r}_{CC_2} \times (\mathbf{F}_{in2} + \mathbf{G}_2) + \mathbf{M}_{in2} = \mathbf{0}, \\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_B - x_C & y_B - y_C & 0 \\ F_{12x} & F_{12y} & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{C_2} - x_C & y_{C_2} - y_C & 0 \\ F_{in2x} & F_{in2y} - m_2 g & 0 \end{vmatrix} + M_{in2} \mathbf{k} &= \mathbf{0}, \end{aligned}$$

or numerically

$$-\frac{1}{2} + \frac{5\sqrt{2}}{2} - \frac{F_{12x}\sqrt{2}}{2} - \frac{F_{12y}\sqrt{2}}{2} = 0. \quad (6.43)$$

Continuing on path  $I$ , an equation for the forces projected onto the sliding direction of the joint  $C_T$  is written for links 2 and 3

$$\begin{aligned} \sum \mathbf{F}^{(2\&3)} \cdot \mathbf{i} &= (\mathbf{F}_{12} + \mathbf{F}_{in2} + \mathbf{G}_2 + \mathbf{F}_{in3} + \mathbf{G}_3 + \mathbf{F}_{ext}) \cdot \mathbf{i} = \\ F_{12x} + F_{in2x} + F_{in3x} + F_{ext} &= F_{23x} + 100 + \sqrt{2} + \frac{3\sqrt{2}}{2} = 0. \end{aligned} \quad (6.44)$$

The joint force  $\mathbf{F}_{12}$  is obtained from the system of Eqs. (6.44) and (6.43)

$$F_{12x} = -\frac{1}{4}(400 + 7\sqrt{2}) \text{ N} \quad \text{and} \quad F_{12y} = \frac{5}{4}(84 + \sqrt{2}) \text{ N}.$$

Reaction  $\mathbf{F}_{01}$  and equilibrium moment  $\mathbf{M}$

The pin joint  $A_R$ , between 0 and 1, is replaced with the unknown reaction (Fig. 6.15)

$$\mathbf{F}_{01} = F_{01x} \mathbf{i} + F_{01y} \mathbf{j}.$$

The unknown equilibrium moment is  $\mathbf{M} = M \mathbf{k}$ . If the path  $I$  is followed (Fig. 6.15) for the pin joint  $B_R$ , a moment equation is written for link 1

$$\begin{aligned} \sum \mathbf{M}_B^{(1)} &= \mathbf{r}_{BA} \times \mathbf{F}_{01} + \mathbf{r}_{BC_1} \times (\mathbf{F}_{in1} + \mathbf{G}_1) + \mathbf{M}_{in1} + \mathbf{M} = \mathbf{0}, \\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -x_B & -y_B & 0 \\ F_{01x} & F_{01y} & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{C_1} - x_B & y_{C_1} - y_B & 0 \\ F_{in1x} & F_{in1y} - m_1 g & 0 \end{vmatrix} + M \mathbf{k} &= \mathbf{0}, \end{aligned}$$

or numerically

$$\frac{5\sqrt{2}}{2} + \frac{F_{01x}\sqrt{2}}{2} + \frac{F_{01y}\sqrt{2}}{2} + M = 0. \quad (6.45)$$

Continuing on path  $I$  the next joint encountered is the pin joint  $C_R$ , and a moment equation is written for links 1 and 2

$$\sum \mathbf{M}_C^{(1\&2)} = \mathbf{r}_{CA} \times \mathbf{F}_{01} + \mathbf{r}_{CC_1} \times (\mathbf{F}_{in1} + \mathbf{G}_1) + \mathbf{M}_{in1} + \mathbf{M}$$

$$\begin{aligned}
& +\mathbf{r}_{CC_2} \times (\mathbf{F}_{in2} + \mathbf{G}_2) + \mathbf{M}_{in2} = \mathbf{0}, \\
& \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -x_C & -y_C & 0 \\ F_{01x} & F_{01y} & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{C_1} - x_C & y_{C_1} - y_C & 0 \\ F_{in1x} & F_{in1y} - m_1 g & 0 \end{vmatrix} + M \mathbf{k} \\
& + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{C_2} - x_C & y_{C_2} - y_C & 0 \\ F_{in2x} & F_{in2y} - m_2 g & 0 \end{vmatrix} + M_{in2} \mathbf{k} = \mathbf{0},
\end{aligned}$$

or numerically

$$-\sqrt{2} F_{01y} + M - 1 + 10\sqrt{2} = 0. \quad (6.46)$$

Continuing on path *I* the next joint encountered is the slider joint  $C_T$ , and a force equation is written for links 1, 2, and 3

$$\begin{aligned}
\sum \mathbf{F}^{(1\&2\&3)} \cdot \mathbf{i} &= (\mathbf{F}_{01} + \mathbf{F}_{in1} + \mathbf{G}_1 + \mathbf{F}_{in2} + \mathbf{G}_2 + \mathbf{F}_{in3} + \mathbf{G}_3 + \mathbf{F}_{ext}) \cdot \mathbf{i} = \\
F_{01x} + F_{in1x} + F_{in2x} + F_{in3x} + F_{ext} &= F_{12x} + 100 + \sqrt{2} + \frac{3\sqrt{2}}{2} = 0. \quad (6.47)
\end{aligned}$$

From Eqs. (6.45), (6.46), and (6.47) the components  $F_{01x}$ ,  $F_{01y}$  and  $M$  are computed

$$\begin{aligned}
F_{01x} &= -2(50 + \sqrt{2}) \text{ N} \quad \text{and} \quad F_{01y} = 115 + \sqrt{2} \text{ N}, \\
M &= 3 + 105\sqrt{2} \text{ N} \cdot \text{m}.
\end{aligned}$$

The MATLAB program using contour equations and the results are given in Program 6.3.

## 6.7 R-RTR-RTR Mechanism

The planar R-RTR-RTR mechanism considered is shown in Fig. 6.16. The following numerical data are given:  $AC = 0.060$  m,  $AE = 0.250$  m and  $CD = 0.150$  m. The widths of the links 1, 3, and 5 are  $AB = 0.140$  m,  $DF = 0.400$  m, and respectively,  $EG = 0.500$  m. The height of the links 1, 3, and 5 is  $h = 0.010$  m. The width of the links 2 and 4 is  $w_{Slider} = 0.050$  m, and the height is  $h_{Slider} = 0.020$  m. All five moving links are rectangular prisms with the depth  $d = 0.001$  m. The angular velocity of the driver link 1 is  $n = 50$  rpm. The density of the material is  $\rho_{Steel} = \rho = 8000$  kg/m<sup>3</sup>. The gravitational acceleration is  $g = 9.807$  m/s<sup>2</sup>. The center of mass locations of the links  $i = 1, 2, \dots, 5$  are designated by  $C_i(x_{C_i}, y_{C_i}, 0)$ .

The external moment applied on link 5 is opposed to the motion of the link:  $\mathbf{M}_{5ext} = -\text{Sign}(\omega_5) |M_{ext}| \mathbf{k}$  where  $|M_{ext}| = 100$  N·m and  $\omega_5$  is the angular velocity of link 5.

Find the motor moment  $\mathbf{M}_m$  required for the dynamic equilibrium and the joint reaction forces when the driver link 1 makes an angle  $\phi = \frac{\pi}{6}$  rad with the horizontal axis.

### Solution

The position vectors (in meters) of the joints are

position of joint  $A$ :  $\mathbf{r}_A = \mathbf{0}$ ;

position of joint  $B$ :  $\mathbf{r}_B = x_B \mathbf{i} + y_B \mathbf{j} = 0.121 \mathbf{i} + 0.070 \mathbf{j}$ ;

position of joint  $C$ :  $\mathbf{r}_C = y_C \mathbf{j} = 0.060 \mathbf{j}$ ;

position of joint  $D$ :  $\mathbf{r}_D = x_D \mathbf{i} + y_D \mathbf{j} = -0.149 \mathbf{i} + 0.047 \mathbf{j}$ ;

position of joint  $E$ :  $\mathbf{r}_E = y_E \mathbf{j} = -0.250 \mathbf{j}$ ;

position of  $F$ :  $\mathbf{r}_F = x_F \mathbf{i} + y_F \mathbf{j} = 0.249 \mathbf{i} + 0.080 \mathbf{j}$ ; and

position of  $G$ :  $\mathbf{r}_G = x_G \mathbf{i} + y_G \mathbf{j} = -0.224 \mathbf{i} + 0.196 \mathbf{j}$ .

The angles of the links with the horizontal are  $\phi_2 = \phi_3 = 4.715^\circ$  and  $\phi_4 = \phi_5 = 16.666^\circ$ .

The position vector of the center of mass of link 1 is

$$\mathbf{r}_{C_1} = x_{C_1} \mathbf{i} + y_{C_1} \mathbf{j} = \frac{x_B}{2} \mathbf{i} + \frac{y_B}{2} \mathbf{j} = 0.060 \mathbf{i} + 0.035 \mathbf{j} \quad \text{m.}$$

The position vector of the center of mass of slider 2 is

$$\mathbf{r}_{C_2} = x_{C_2} \mathbf{i} + y_{C_2} \mathbf{j} = \mathbf{r}_B.$$

The position vector of the center of mass of link 3 is

$$\mathbf{r}_{C_3} = x_{C_3} \mathbf{i} + y_{C_3} \mathbf{j} = \frac{x_D + x_F}{2} \mathbf{i} + \frac{y_D + y_F}{2} \mathbf{j} = 0.049 \mathbf{i} + 0.064 \mathbf{j} \quad \text{m.}$$

The position vector of the center of mass of slider 4 is

$$\mathbf{r}_{C_4} = x_{C_4} \mathbf{i} + y_{C_4} \mathbf{j} = \mathbf{r}_D.$$

The position vector of the center of mass of link 5 is

$$\mathbf{r}_{C_5} = x_{C_5} \mathbf{i} + y_{C_5} \mathbf{j} = \frac{x_E + x_G}{2} \mathbf{i} + \frac{y_E + y_G}{2} \mathbf{j} = -0.112 \mathbf{i} - 0.026 \mathbf{j} \quad \text{m.}$$

The velocity and acceleration analysis are given by

$$\text{acceleration of joint } B: \mathbf{a}_{B_1} = \mathbf{a}_{B_2} = -3.323 \mathbf{i} - 1.919 \mathbf{j} \quad \text{m/s}^2;$$

$$\text{acceleration of joint } D: \mathbf{a}_{D_3} = \mathbf{a}_{D_4} = 4.617 \mathbf{i} - 1.811 \mathbf{j} \quad \text{m/s}^2;$$

$$\text{acceleration of joint } F: \mathbf{a}_F = -7.695 \mathbf{i} + 3.019 \mathbf{j} \quad \text{m/s}^2;$$

$$\text{acceleration of joint } G: \mathbf{a}_G = 2.767 \mathbf{i} + 0.919 \mathbf{j} \quad \text{m/s}^2;$$

$$\text{angular velocity of link 5: } \boldsymbol{\omega}_5 = 0.917 \mathbf{k} \quad \text{rad/s};$$

$$\text{angular acceleration of link 1: } \boldsymbol{\alpha}_1 = 0 \mathbf{k} \quad \text{rad/s}^2;$$

$$\text{angular acceleration of links 2 and 3: } \boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_3 = 14.568 \mathbf{k} \quad \text{rad/s}^2;$$

$$\text{angular acceleration of links 4 and 5: } \boldsymbol{\alpha}_4 = \boldsymbol{\alpha}_5 = -5.771 \mathbf{k} \quad \text{rad/s}^2.$$

The acceleration vector of the center of mass of link 1 is

$$\mathbf{a}_{C_1} = \frac{\mathbf{a}_{B_1}}{2} = -1.661 \mathbf{i} - 0.959 \mathbf{j} \quad \text{m/s}^2.$$

The acceleration vector of the center of mass of slider 2 is

$$\mathbf{r}_{C_2} = \mathbf{a}_{B_2} = -3.323 \mathbf{i} - 1.919 \mathbf{j} \quad \text{m/s}^2.$$

The acceleration vector of the center of mass of link 3 is

$$\mathbf{a}_{C_3} = \frac{\mathbf{a}_{D_3} + \mathbf{a}_F}{2} = -1.539 \mathbf{i} + 0.603 \mathbf{j} \quad \text{m/s}^2.$$

The acceleration vector of the center of mass of slider 4 is

$$\mathbf{a}_{C_4} = \mathbf{a}_{D_4} = 4.617 \mathbf{i} - 1.811 \mathbf{j} \quad \text{m/s}^2.$$

The acceleration vector of the center of mass of link 5 is

$$\mathbf{a}_{C_5} = \frac{\mathbf{a}_E + \mathbf{a}_G}{2} = 1.383 \mathbf{i} + 0.459 \mathbf{j} \quad \text{m/s}^2.$$

The external moment applied on link 5 is opposed to the motion of the link

$$\mathbf{M}_{5ext} = -\text{Sign}(\omega_5) |M_{ext}| \mathbf{k} = -\text{Sign}(0.917) (100) \mathbf{k} = -100 \mathbf{k} \quad \text{N m.}$$

### Inertia forces and moments

#### Link 1

The mass of the link is

$$m_1 = \rho AB h d = 8000(0.14)(0.01)(0.001) = 0.0112 \quad \text{kg.}$$

The inertia force of driver 1 at  $C_1$  is

$$\mathbf{F}_{in1} = -m_1 \mathbf{a}_{C_1} = -0.0112(-1.66198\mathbf{i} - 0.959545\mathbf{j}) = 0.0186142\mathbf{i} - 0.0107469\mathbf{j} \quad \text{N.}$$

The gravitational force on link 1 at  $C_1$  is

$$\mathbf{G}_1 = -m_1 g \mathbf{j} = -0.0112(9.807)\mathbf{j} = -0.109838\mathbf{j} \quad \text{N.}$$

The mass moment of inertia of link 1 with respect to  $C_1$  is

$$I_{C_1} = m_1 (AB^2 + h^2)/12 = 0.0112(0.14^2 + 0.01^2)/12 = 1.83867 \times 10^{-5} \quad \text{kg m}^2.$$

The moment of inertia of driver 1 is

$$\mathbf{M}_{in1} = -I_{C_1} \boldsymbol{\alpha}_1 = \mathbf{0}.$$

#### Link 2

The mass of the slider 2 is

$$m_2 = \rho h_{Slider} w_{Slider} d = 8000(0.02)(0.05)(0.001) = 0.008 \quad \text{kg.}$$

The inertia force of slider 2 at  $C_2$  is

$$\mathbf{F}_{in2} = -m_2 \mathbf{a}_{C_2} = -0.008(-3.323\mathbf{i} - 1.919\mathbf{j}) = 0.0265917\mathbf{i} + 0.0153527\mathbf{j} \quad \text{N.}$$

The gravitational force of slider 2 at  $C_2$  is

$$\mathbf{G}_2 = -m_2 g \mathbf{j} = -0.008(9.807)\mathbf{j} = -0.109838\mathbf{j} \quad \text{N.}$$

The mass moment of inertia of slider 2 with respect to  $C_2$  is

$$I_{C_2} = m_2 (h_{Slider}^2 + w_{Slider}^2)/12 = 0.008(0.02^2 + 0.05^2)/12 = 1.93333 \times 10^{-6} \text{ kg m}^2.$$

The moment of inertia of slider 2 is

$$\mathbf{M}_{in2} = -I_{C_2} \boldsymbol{\alpha}_2 = -1.93333 \times 10^{-6} (14.568) \mathbf{k} = -2.8165 \times 10^{-5} \mathbf{k} \text{ Nm.}$$

#### Link 3

The mass of the link is

$$m_3 = \rho F D h d = 8000(0.4)(0.01)(0.001) = 0.032 \text{ kg.}$$

The inertia force of link 3 is

$$\mathbf{F}_{in3} = -m_3 \mathbf{a}_{C_3} = -0.032(-1.539 \mathbf{i} + 0.603 \mathbf{j}) = 0.0492489 \mathbf{i} - 0.019326 \mathbf{j} \text{ N.}$$

The gravitational force of link 3 is

$$\mathbf{G}_3 = -m_3 g \mathbf{j} = -0.032(9.807) \mathbf{j} = -0.313824 \mathbf{j} \text{ N.}$$

The mass moment of inertia is

$$I_{C_3} = m_3 (FD^2 + h^2)/12 = 0.032(0.4^2 + 0.01^2)/12 = 0.000426933 \text{ kg m}^2.$$

The inertia moment on link 3 is

$$\mathbf{M}_{in3} = -I_{C_3} \boldsymbol{\alpha}_3 = -0.000426933(14.568) \mathbf{k} = -0.00621962 \mathbf{k} \text{ Nm.}$$

#### Link 4

The mass of the link is

$$m_4 = \rho h_{Slider} w_{Slider} d = 8000(0.02)(0.05)(0.001) = 0.008 \text{ kg.}$$

The inertia force is

$$\mathbf{F}_{in4} = -m_4 \mathbf{a}_{C_4} = -0.008(4.617 \mathbf{i} - 1.811 \mathbf{j}) = -0.0369367 \mathbf{i} + 0.0144946 \mathbf{j} \text{ N.}$$

The gravitational force is

$$\mathbf{G}_4 = -m_4 g \mathbf{j} = -0.008(9.807) \mathbf{j} = -0.109838 \mathbf{j} \text{ N.}$$

The mass moment of inertia is

$$I_{C_4} = m_4(h_{Slider}^2 + w_{Slider}^2)/12 = 0.008(0.02^2 + 0.05^2)/12 = 1.93333 \times 10^{-6} \text{ kg m}^2.$$

The moment of inertia is

$$\mathbf{M}_{in4} = -I_{C_4} \boldsymbol{\alpha}_4 = -1.93333 \times 10^{-6} (-5.771) \mathbf{k} = 1.11583 \times 10^{-5} \mathbf{k} \text{ N m.}$$

*Link 5*

The mass of the link is

$$m_5 = \rho EG h d = 8000(0.5)(0.01)(0.001) = 0.04 \text{ kg.}$$

The inertia force is

$$\mathbf{F}_{in5} = -m_5 \mathbf{a}_{C_5} = -0.04(1.383 \mathbf{i} + 0.459 \mathbf{j}) = -0.0553516 \mathbf{i} - 0.0183855 \mathbf{j} \text{ N.}$$

The gravitational force is

$$\mathbf{G}_5 = -m_5 g \mathbf{j} = -0.04(9.807) \mathbf{j} = -0.39228 \mathbf{j} \text{ N.}$$

The mass moment of inertia is

$$I_{C_5} = m_5 (EG^2 + h^2)/12 = 0.04(0.5^2 + 0.01^2)/12 = 0.000833667 \text{ kg m}^2.$$

The moment of inertia is

$$\mathbf{M}_{in5} = -I_{C_5} \boldsymbol{\alpha}_5 = -0.000833667 (-5.771) \mathbf{k} = 0.00481155 \mathbf{k} \text{ N m.}$$

## Joint forces and drive moment

### 6.7.1 Newton-Euler Equations of Motion

The force analysis starts with the link 5 because the external moment  $\mathbf{M}_{5ext}$  is given. Figure 6.17(a) shows the free body diagram of the link 5. The joint reaction force of the ground 0 on the link 5 at the joint  $F$  is  $\mathbf{F}_{05} = F_{05x}\mathbf{i} + F_{05y}\mathbf{j}$ . The joint reaction force of the link 4 on the link 5 is  $\mathbf{F}_{45} = F_{45x}\mathbf{i} + F_{45y}\mathbf{j}$ . The application point of the force  $\mathbf{F}_{45}$  is  $P(x_P, y_P)$  and the position vector of  $P$  is  $\mathbf{r}_P = x_P\mathbf{i} + y_P\mathbf{j}$ .

The point  $P$ , the application of the force  $\mathbf{F}_{45}$ , is located on the direction  $DE$ , that is

$$(\mathbf{r}_D - \mathbf{r}_E) \times (\mathbf{r}_P - \mathbf{r}_E) = \mathbf{0},$$

or

$$0.297670 x_P + 0.149492 y_P + 0.0373731 = 0, \quad (6.48)$$

The direction of the unknown joint force  $\mathbf{F}_{45}$  is perpendicular to the sliding direction  $\mathbf{r}_{DE}$

$$\mathbf{F}_{45} \cdot \mathbf{r}_{DE} = 0,$$

or

$$-0.149492 F_{45x} + 0.297670 F_{45y} = 0. \quad (6.49)$$

For the link 5 the vector sum of the net forces, gravitational force  $\mathbf{G}_5$ , joint forces  $\mathbf{F}_{05}$ ,  $\mathbf{F}_{45}$ , is equal to  $m_5 \mathbf{a}_{C_5}$  [Fig. 6.17(a)]

$$m_5 \mathbf{a}_{C_5} = \mathbf{F}_{05} + \mathbf{F}_{45} + \mathbf{G}_5.$$

Projecting the previous vectorial onto  $x$  and  $y$  axes gives

$$m_5 a_{C_5x} = F_{05x} + F_{45x},$$

$$m_5 a_{C_5y} = F_{05y} + F_{45y} - m_5 g,$$

or

$$F_{05x} + F_{45x} - 0.0553516 = 0, \quad (6.50)$$

$$F_{05y} + F_{45y} - 0.410666 = 0. \quad (6.51)$$

The vector sum of the moments that act on link 5 with respect to the center of mass  $C_5$  is equal to  $I_{C_5} \boldsymbol{\alpha}_5$  [Fig. 6.17(a)]

$$I_{C_5} \boldsymbol{\alpha}_5 = \mathbf{r}_{C_5E} \times \mathbf{F}_{05} + \mathbf{r}_{C_5P} \times \mathbf{F}_{45} + \mathbf{M}_{5ext},$$

or

$$\begin{aligned} 0.112198 F_{05y} + 0.223409 F_{05x} + (x_P + .112198) F_{45y} - \\ (y_P + 0.0265909) F_{45x} - 99.9952 = 0. \end{aligned} \quad (6.52)$$

There are five equations Eqs. (6.48)-(6.52) and six unknowns and that is why the analysis will continue with the slider 4. The free body diagram of the slider 4 is shown in Fig. 6.17(b).

The joint reaction force of the link 3 on the slider 4 at  $D = C_4$  is  $\mathbf{F}_{34} = F_{34x}\mathbf{i} + F_{34y}\mathbf{j}$  and the joint reaction force of the link 5 on the slider 4 is  $\mathbf{F}_{54} = -\mathbf{F}_{45} = -F_{45x}\mathbf{i} - F_{45y}\mathbf{j}$ .

For the slider 4, according to Newton's equations of motion, the vector sum of the net forces, gravitational force  $\mathbf{G}_4$ , joint forces  $\mathbf{F}_{34}, \mathbf{F}_{54}$ , is equal to  $m_4 \mathbf{a}_{C_4}$

$$m_4 \mathbf{a}_{C_4} = \mathbf{F}_{34} + \mathbf{F}_{54} + \mathbf{G}_4.$$

Projecting the previous vectorial onto  $x$  and  $y$  axes gives

$$\begin{aligned} m_4 a_{C_{4x}} &= F_{34x} + F_{54x}, \\ m_4 a_{C_{4y}} &= F_{34y} + F_{54y} - m_4 g, \end{aligned}$$

or

$$F_{34x} - F_{45x} - 0.0369367 = 0, \quad (6.53)$$

$$F_{34y} - F_{45y} - 0.0639614 = 0. \quad (6.54)$$

The vector sum of the moments that act on slider 4 with respect to the center of mass  $D = C_4$  is equal to  $I_{C_4} \boldsymbol{\alpha}_4$

$$I_{C_4} \boldsymbol{\alpha}_4 = \mathbf{r}_{C_4P} \times \mathbf{F}_{54},$$

or

$$-(x_P + 0.149492) F_{45y} + (y_P - 0.0476701) F_{45x} + (0.111583) 10^{-4} = 0. \quad (6.55)$$

There are eight equations Eqs. (6.48)-(6.55) with eight unknowns  $F_{05x}, F_{05y}, F_{45x}, F_{45y}, x_P, y_P, F_{34x}$ , and  $F_{34y}$ . The following numerical solution are obtained

$$\begin{aligned} \mathbf{F}_{05} &= 268.165 \mathbf{i} + 135.057 \mathbf{j} \text{ N}, \\ \mathbf{F}_{45} &= -268.109 \mathbf{i} - 134.647 \mathbf{j} \text{ N}, \\ \mathbf{F}_{34} &= -268.072 \mathbf{i} - 134.583 \mathbf{j} \text{ N}, \text{ and} \\ \mathbf{r}_P &= -0.149492 \mathbf{i} + 0.0476701 \mathbf{j} \text{ m}. \end{aligned}$$

The force analysis continues with the link 3. Figure 6.18(a) shows the free body diagram of the link 3. The joint reaction force of the link 4 on the link 3 is  $\mathbf{F}_{43} = -\mathbf{F}_{34} = 268.072\mathbf{i} + 134.583\mathbf{j}$  N. The joint reaction force of the ground 0 on the link 3 at the joint  $C$  is  $\mathbf{F}_{03} = F_{03x}\mathbf{i} + F_{03y}\mathbf{j}$ . The joint reaction force of the link 2 on the link 3 is  $\mathbf{F}_{423} = F_{23x}\mathbf{i} + F_{23y}\mathbf{j}$ . The application point of the force  $\mathbf{F}_{23}$  is  $Q(x_Q, y_Q)$  and the position vector of  $Q$  is  $\mathbf{r}_Q = x_Q\mathbf{i} + y_Q\mathbf{j}$ .

The point  $Q$ , the application of the force  $\mathbf{F}_{23}$ , is located on the direction  $BC$ , that is

$$(\mathbf{r}_B - \mathbf{r}_C) \times (\mathbf{r}_Q - \mathbf{r}_C) = \mathbf{0},$$

or

$$0.121244 y_Q - 0.01 x_Q - 0.00727461 = 0. \quad (6.56)$$

The direction of the unknown joint force  $\mathbf{F}_{23}$  is perpendicular to the sliding direction  $\mathbf{r}_{BC}$

$$\mathbf{F}_{23} \cdot \mathbf{r}_{BC} = 0,$$

or

$$0.121244 F_{23x} + 0.01 F_{23y} = 0. \quad (6.57)$$

For the link 3 the vector sum of the net forces, gravitational force  $\mathbf{G}_3$ , joint forces  $\mathbf{F}_{43}$ ,  $\mathbf{F}_{03}$ ,  $\mathbf{F}_{23}$ , is equal to  $m_3 \mathbf{a}_{C_3}$  [Fig. 6.18(a)]

$$m_3 \mathbf{a}_{C_3} = \mathbf{F}_{43} + \mathbf{F}_{03} + \mathbf{F}_{23} + \mathbf{G}_3,$$

Projecting the previous vectorial onto  $x$  and  $y$  axes gives

$$m_3 a_{C_{3x}} = F_{43x} + F_{03x} + F_{23x},$$

$$m_3 a_{C_{3y}} = F_{43y} + F_{03y} + F_{23y} - m_3 g,$$

or

$$F_{03x} + F_{23x} + 268.122 = 0, \quad (6.58)$$

$$F_{03y} + F_{23y} + 134.250 = 0. \quad (6.59)$$

The vector sum of the moments that act on link 3 with respect to the center of mass  $C_3$  is equal to  $I_{C_3} \boldsymbol{\alpha}_3$  [Fig. 6.19(a)]

$$I_{C_3} \boldsymbol{\alpha}_3 = \mathbf{r}_{C_3D} \times \mathbf{F}_{43} + \mathbf{r}_{C_3C} \times \mathbf{F}_{03} + \mathbf{r}_{C_5Q} \times \mathbf{F}_{23},$$

or

$$\begin{aligned} & -22.4246 - 0.0498308 F_{03y} + 0.00410997 F_{03x} + \\ & (x_Q - 0.0498308) F_{23y} - (y_Q - 0.0641100) F_{23x} = 0. \end{aligned} \quad (6.60)$$

There are five equations Eqs. (6.56)-(6.60) and six unknowns and that is why the analysis will continue with the slider 2. The free body diagram of the slider 2 is shown in Fig. 6.18(b).

The joint reaction force of the link 1 on the slider 2 at  $B$  is  $\mathbf{F}_{12} = F_{12x}\mathbf{i} + F_{12y}\mathbf{j}$  and the joint reaction force of the link 3 on the slider 2 is  $\mathbf{F}_{32} = -\mathbf{F}_{23} = -F_{23x}\mathbf{i} - F_{23y}\mathbf{j}$ .

For the slider 2 the vector sum of the net forces, gravitational force  $\mathbf{G}_2$ , joint forces  $\mathbf{F}_{32}$ ,  $\mathbf{F}_{12}$ , is equal to  $m_2 \mathbf{a}_{C_2}$

$$m_2 \mathbf{a}_{C_2} = \mathbf{F}_{32} + \mathbf{F}_{12} + \mathbf{G}_2.$$

Projecting the previous vectorial onto  $x$  and  $y$  axes gives

$$\begin{aligned} m_2 a_{C_{2x}} &= F_{32x} + F_{12x}, \\ m_2 a_{C_{2y}} &= F_{32y} + F_{12y} - m_2 g, \end{aligned}$$

or

$$-F_{23x} + F_{12x} + 0.0265917 = 0, \quad (6.61)$$

$$-F_{23y} + F_{12y} - 0.0631033 = 0. \quad (6.62)$$

The vector sum of the moments that act on slider 2 with respect to the center of mass  $B = C_2$  is equal to  $I_{C_2} \boldsymbol{\alpha}_2$

$$I_{C_2} \boldsymbol{\alpha}_2 = \mathbf{r}_{C_2Q} \times \mathbf{F}_{32},$$

or

$$-(x_Q - 0.121244) F_{23y} + (y_Q - 0.07) F_{23x} - (0.281650) 10^{-4} = 0. \quad (6.63)$$

There are eight equations Eqs. (6.56)-(6.63) with eight unknowns  $F_{03x}$ ,  $F_{03y}$ ,  $F_{23x}$ ,  $F_{23y}$ ,  $x_Q$ ,  $y_Q$ ,  $F_{12x}$ , and  $F_{12y}$ .

The following numerical solution are obtained

$$\begin{aligned} \mathbf{F}_{03} &= -256.745 \mathbf{i} - 272.179 \mathbf{j} \text{ N}, \\ \mathbf{F}_{23} &= -11.3762 \mathbf{i} + 137.93 \mathbf{j} \text{ N}, \\ \mathbf{F}_{12} &= -11.4028 \mathbf{i} + 137.993 \mathbf{j} \text{ N}, \text{ and} \\ \mathbf{r}_Q &= 0.121243 \mathbf{i} + 0.07 \mathbf{j} \text{ m}. \end{aligned}$$

The force analysis ends with the driver link 1. Figure 6.19 shows the free body diagram of the link 1. The joint reaction force of the link 2 on the link 1 is  $\mathbf{F}_{21} = -\mathbf{F}_{12} = 11.4028\mathbf{i} - 137.993\mathbf{j}$  N.

The joint reaction force of the ground 0 on the link 1 at the joint  $A$  is  $\mathbf{F}_{01} = F_{01x}\mathbf{i} + F_{01y}\mathbf{j}$ . For the link 1 the vector sum of the net forces, gravitational force  $\mathbf{G}_1$ , joint forces  $\mathbf{F}_{01}$ ,  $\mathbf{F}_{21}$ , is equal to  $m_1 \mathbf{a}_{C_1}$  (Fig. 6.19)

$$m_1 \mathbf{a}_{C_1} = \mathbf{F}_{01} - \mathbf{F}_{12} + \mathbf{G}_1 \quad \implies \quad \mathbf{F}_{01} = m_1 \mathbf{a}_{C_1} + \mathbf{F}_{12} - \mathbf{G}_1.$$

The vector sum of the moments that act on link 1 with respect to the center of mass  $C_1$  is equal to  $I_{C_1} \boldsymbol{\alpha}_1$

$$I_{C_1} \boldsymbol{\alpha}_1 = \mathbf{r}_{C_1A} \times \mathbf{F}_{01} - \mathbf{r}_{C_1B} \times \mathbf{F}_{12} + \mathbf{M}_{mot},$$

and the equilibrium moment (motor moment) is

$$\mathbf{M}_{mot} = I_{C_1} \boldsymbol{\alpha}_1 - \mathbf{r}_{C_1A} \times \mathbf{F}_{01} + \mathbf{r}_{C_1B} \times \mathbf{F}_{12}.$$

Another way of calculating the equilibrium moment is to take the sum of the moments that act on link 1 with respect  $A$

$$I_{C_1} \boldsymbol{\alpha}_1 + \mathbf{r}_{C_1} \times m_1 \mathbf{a}_{C_1} = \mathbf{r}_{C_1} \times \mathbf{G}_1 + \mathbf{r}_B \times (-\mathbf{F}_{12}) + \mathbf{M}_{mot},$$

and the equilibrium moment is

$$\mathbf{M}_{mot} = \mathbf{r}_B \times \mathbf{F}_{12} + \mathbf{r}_{C_1} \times (m_1 \mathbf{a}_{C_1} - \mathbf{G}_1) + I_{C_1} \boldsymbol{\alpha}_1.$$

The joint reaction force of the ground 0 on the link 1 is  $\mathbf{F}_{01} = -11.4214\mathbf{i} + 138.092\mathbf{j}$  N, and the equilibrium moment is  $\mathbf{M}_{mot} = 17.5356\mathbf{k}$  N m.

The MATLAB program using Newton-Euler equations and the results are given in Program 6.4.

### 6.7.2 Dyad Method

The dynamic force analysis starts with the last dyad (links 5 and 4) because the external moment  $\mathbf{M}_{5ext}$  on link 5 is known.

$E_R D_T D_R$  dyad

Figure 6.20 shows the forces and the moments that act on the dyad  $E_R D_T D_R$ . The unknown joint reaction forces are  $\mathbf{F}_{05} = F_{05x} \mathbf{i} + F_{05y} \mathbf{j}$ ,  $\mathbf{F}_{34} = F_{34x} \mathbf{i} + F_{34y} \mathbf{j}$ .

The Newton equation for links 5 and 4

$$\begin{aligned} m_5 \mathbf{a}_{C_5} + m_4 \mathbf{a}_{C_4} &= \mathbf{F}_{05} + \mathbf{G}_5 + \mathbf{G}_4 + \mathbf{F}_{34} \implies \\ \sum \mathbf{F}^{(5\&4)} &= \mathbf{F}_{05} + \mathbf{G}_5 + \mathbf{G}_4 + \mathbf{F}_{34} - m_5 \mathbf{a}_{C_5} - m_4 \mathbf{a}_{C_4} = \mathbf{0}. \end{aligned} \quad (6.64)$$

Equation (6.64) has a component on  $x$ -axis,  $\sum \mathbf{F}^{(5\&4)} \cdot \mathbf{i}$ , a component on  $y$ -axis,  $\sum \mathbf{F}^{(5\&4)} \cdot \mathbf{j}$ .

The Euler equation of moments for links 5 and 4 about  $D_R$  gives

$$\begin{aligned} I_{C_5} \boldsymbol{\alpha}_5 + \mathbf{r}_{DC_5} \times m_5 \mathbf{a}_{C_5} + I_{C_4} \boldsymbol{\alpha}_4 &= \mathbf{r}_{DE} \times \mathbf{F}_{05} + \mathbf{r}_{DC_5} \times \mathbf{G}_5 + \mathbf{M}_{5ext} \implies \\ \sum \mathbf{M}_D^{(5\&4)} &= (\mathbf{r}_E - \mathbf{r}_D) \times \mathbf{F}_{05} + (\mathbf{r}_{C_5} - \mathbf{r}_D) \times (\mathbf{G}_5 - m_5 \mathbf{a}_{C_5}) + \mathbf{M}_{5ext} \\ -I_{C_5} \boldsymbol{\alpha}_5 - I_{C_4} \boldsymbol{\alpha}_4 &= \mathbf{0}. \end{aligned} \quad (6.65)$$

The Newton equation for link 4 projected on the sliding direction  $ED$  is

$$\begin{aligned} (m_4 \mathbf{a}_{C_4}) \cdot \mathbf{r}_{ED} &= (\mathbf{F}_{34} + \mathbf{G}_4 + \mathbf{F}_{54}) \cdot \mathbf{r}_{ED} \implies \\ \sum \mathbf{F}^{(4)} \cdot \mathbf{r}_{ED} &= (\mathbf{F}_{34} + \mathbf{G}_4 - m_4 \mathbf{a}_{C_4}) \cdot (\mathbf{r}_D - \mathbf{r}_E) = 0. \end{aligned} \quad (6.66)$$

The force of the link 5 on link 4 is  $\mathbf{F}_{54}$  and  $\mathbf{F}_{54} \cdot \mathbf{r}_{ED} = 0$ .

There are four equations Eqs. (6.64)-(6.66) with four unknowns  $F_{05x}$ ,  $F_{05y}$ ,  $F_{34x}$ ,  $F_{34y}$ .

The force of the link 4 on link 5 is  $\mathbf{F}_{45}$  is calculated from Newton equation for link 5

$$\begin{aligned} m_5 \mathbf{a}_{C_5} &= \mathbf{F}_{05} + \mathbf{G}_5 + \mathbf{F}_{45} \implies \\ \mathbf{F}_{45} &= m_5 \mathbf{a}_{C_5} - \mathbf{G}_5 - \mathbf{F}_{05}. \end{aligned}$$

The application point of the joint force  $\mathbf{F}_{45}$  is  $P(x_P, y_P)$ . The point  $P$  is on the line  $ED$  or

$$\mathbf{r}_{ED} \times \mathbf{r}_{EP} = \mathbf{0} \quad \text{or} \quad (\mathbf{r}_D - \mathbf{r}_E) \times (\mathbf{r}_P - \mathbf{r}_E) = \mathbf{0}.$$

The second equation needed to calculate  $x_P$  and  $y_P$  is the moment equation on link 4 about  $D = C_4$

$$I_{C_4} \boldsymbol{\alpha}_4 = \mathbf{r}_{C_4P} \times (-\mathbf{F}_{45}).$$

$C_R B_T B_R$  dyad

Figure 6.21 shows the forces and the moments that act on the dyad  $C_R B_T B_R$  (links 3 and 2). The unknown joint reaction forces are  $\mathbf{F}_{03} = F_{03x} \mathbf{i} + F_{03y} \mathbf{j}$ ,  $\mathbf{F}_{12} = F_{12x} \mathbf{i} + F_{12y} \mathbf{j}$ .

The joint force  $\mathbf{F}_{43} = -\mathbf{F}_{34}$  was calculated from the previous dyad  $EDD$

$$F_{43} = -F_{34s}$$

The sum of all the forces that act on links 3 and 2 is

$$\begin{aligned} m_3 \mathbf{a}_{C_3} + m_2 \mathbf{a}_{C_2} &= \mathbf{F}_{43} + \mathbf{F}_{03} + \mathbf{G}_3 + \mathbf{G}_2 + \mathbf{F}_{12} \implies \\ \sum \mathbf{F}^{(3\&2)} &= \mathbf{F}_{43} + \mathbf{F}_{03} + \mathbf{G}_3 + \mathbf{G}_2 + \mathbf{F}_{12} - m_3 \mathbf{a}_{C_3} - m_2 \mathbf{a}_{C_2} = \mathbf{0}. \end{aligned} \quad (6.67)$$

Equation (6.67) has a component on  $x$ -axis,  $\sum \mathbf{F}^{(3\&2)} \cdot \mathbf{i}$ , a component on  $y$ -axis,  $\sum \mathbf{F}^{(3\&2)} \cdot \mathbf{j}$ .

The sum of moments of all the forces and moments on links 3 and 2 about  $B_R$  is zero

$$\begin{aligned} I_{C_3} \boldsymbol{\alpha}_3 + \mathbf{r}_{BC_3} \times m_3 \mathbf{a}_{C_3} + I_{C_2} \boldsymbol{\alpha}_2 &= \mathbf{r}_{BD} \times \mathbf{F}_{43} + \mathbf{r}_{BC} \times \mathbf{F}_{03} + \mathbf{r}_{BC_3} \times \mathbf{G}_3 \implies \\ \sum \mathbf{M}_B^{(3\&2)} &= (\mathbf{r}_D - \mathbf{r}_B) \times \mathbf{F}_{43} + (\mathbf{r}_C - \mathbf{r}_B) \times \mathbf{F}_{03} + (\mathbf{r}_{C_3} - \mathbf{r}_B) \times (\mathbf{G}_3 - m_3 \mathbf{a}_{C_3}) \\ -I_{C_3} \boldsymbol{\alpha}_3 - I_{C_2} \boldsymbol{\alpha}_2 &= \mathbf{0}. \end{aligned} \quad (6.68)$$

The sum of all the forces on link 2 projected on the sliding direction  $BC$  is

$$\begin{aligned} (m_2 \mathbf{a}_{C_2}) \cdot \mathbf{r}_{BC} &= (\mathbf{F}_{12} + \mathbf{G}_2 + \mathbf{F}_{32}) \cdot \mathbf{r}_{BC} \implies \\ \sum \mathbf{F}^{(2)} \cdot \mathbf{r}_{BC} &= (\mathbf{F}_{12} + \mathbf{G}_2 - m_2 \mathbf{a}_{C_2}) \cdot (\mathbf{r}_C - \mathbf{r}_B) = 0. \end{aligned} \quad (6.69)$$

The force of the link 3 on link 2 is  $\mathbf{F}_{32}$  and  $\mathbf{F}_{32} \cdot \mathbf{r}_{BC} = 0$ .

There are four equations Eqs. (6.67)-(6.69) with four unknowns  $F_{03x}$ ,  $F_{03y}$ ,  $F_{12x}$ ,  $F_{12y}$ .

The force of the link 3 on link 2 is  $\mathbf{F}_{32}$  is calculated from the sum of all the forces on link 2

$$\begin{aligned} m_2 \mathbf{a}_{C_2} &= \mathbf{F}_{32} + \mathbf{G}_2 + \mathbf{F}_{12} \implies \\ \mathbf{F}_{32} &= m_2 \mathbf{a}_{C_2} - \mathbf{G}_2 - \mathbf{F}_{12}. \end{aligned}$$

The application point of the joint force  $\mathbf{F}_{32}$  is  $Q(x_Q, y_Q)$ . The point  $Q$  is on the line  $BC$  or

$$\mathbf{r}_{BC} \times \mathbf{r}_{QC} = \mathbf{0} \quad \text{or} \quad (\mathbf{r}_C - \mathbf{r}_B) \times (\mathbf{r}_Q - \mathbf{r}_C) = \mathbf{0}.$$

The second equation needed to calculate  $x_Q$  and  $y_Q$  is the sum of all the moments on link 2 about  $B = C_2$

$$I_{C_2} \boldsymbol{\alpha}_2 = \mathbf{r}_{C_2Q} \times \mathbf{F}_{32}.$$

The joint reaction force of the ground on the link 1 and the equilibrium moment (drive moment) shown in Figure 6.19 are calculated using the procedure presented in the previous subsection.

TheMATLAB program using the dyad method and the results are given in Program 6.5.

### 6.7.3 Contour Method

The contour diagram representing the mechanism is shown in Fig. 6.22. It has two contours 0-1-2-3-0 and 0-3-4-5-0.

Reaction force  $\mathbf{F}_{05}$

The rotation joint  $E_R$  between the links 0 and 5 is replaced with the unknown reaction force  $\mathbf{F}_{05}$  (Fig. 6.23)

$$\mathbf{F}_{05} = F_{05x}\mathbf{i} + F_{05y}\mathbf{j}.$$

Following the path  $I$ , as shown in Fig. 6.23, a force equation is written for the translation joint  $D_T$ . The projection of all forces, that act on the link 5, onto the sliding direction  $\mathbf{r}_{DE}$  is zero

$$\sum \mathbf{F}^{(5)} \cdot \mathbf{r}_{DE} = (\mathbf{F}_{05} + \mathbf{G}_5 + \mathbf{F}_{in5}) \cdot \mathbf{r}_{DE} = 0, \quad (6.70)$$

where  $\mathbf{r}_{DE} = \mathbf{r}_E - \mathbf{r}_D$ .

Continuing on the path  $I$ , a moment equation is written for the rotation joint  $D_R$

$$\sum \mathbf{M}_D^{(4\&5)} = \mathbf{r}_{DE} \times \mathbf{F}_{05} + \mathbf{r}_{DC_5} \times (\mathbf{G}_5 + \mathbf{F}_{in5}) + \mathbf{M}_{in4} + \mathbf{M}_{in5} + \mathbf{M}_{5ext} = \mathbf{0}, \quad (6.71)$$

where  $\mathbf{r}_{DC_5} = \mathbf{r}_{C_5} - \mathbf{r}_D$ .

The following numerical solution is obtained

$$\mathbf{F}_{05} = 268.165\mathbf{i} + 135.057\mathbf{j} \text{ N.}$$

Reaction force  $\mathbf{F}_{45}$

The translation joint  $D_T$  between the links 4 and 5 is replaced with the unknown reaction force  $\mathbf{F}_{45}$  (Fig. 6.24)

$$\mathbf{F}_{45} = -\mathbf{F}_{54} = F_{45x}\mathbf{i} + F_{45y}\mathbf{j}.$$

The position of the application point  $P$  of the force  $\mathbf{F}_{45}$  is unknown

$$\mathbf{r}_P = x_P\mathbf{i} + y_P\mathbf{j},$$

where  $x_P$  and  $y_P$  are the plane coordinates of the point  $P$ .

Following the path  $I$  (Fig. 6.24), a moment equation is written for the rotation joint  $E_R$

$$\sum \mathbf{M}_E^{(5)} = \mathbf{r}_{EP} \times \mathbf{F}_{45} + \mathbf{r}_{EC_5} \times (\mathbf{G}_5 + \mathbf{F}_{in5}) + \mathbf{M}_{in5} + \mathbf{M}_{5ext} = \mathbf{0}, \quad (6.72)$$

where  $\mathbf{r}_{EP} = \mathbf{r}_P - \mathbf{r}_E$  and  $\mathbf{r}_{EC_5} = \mathbf{r}_{C_5} - \mathbf{r}_E$ .

Following the path  $II$  (Fig. 6.24), a moment equation is written for the rotation joint  $D_R$

$$\sum \mathbf{M}_D^{(4)} = \mathbf{r}_{DP} \times \mathbf{F}_{54} + \mathbf{M}_{in4} = \mathbf{0}, \quad (6.73)$$

where  $\mathbf{r}_{DP} = \mathbf{r}_P - \mathbf{r}_D$  and  $\mathbf{F}_{54} = -\mathbf{F}_{45}$ .

The direction of the unknown joint force  $\mathbf{F}_{45}$  is perpendicular to the sliding direction  $\mathbf{r}_{ED}$

$$\mathbf{F}_{45} \cdot \mathbf{r}_{ED} = 0. \quad (6.74)$$

The application point  $P$  of the force  $\mathbf{F}_{45}$  is located on the direction  $ED$ , that is

$$(\mathbf{r}_D - \mathbf{r}_E) \times (\mathbf{r}_P - \mathbf{r}_E) = \mathbf{0}. \quad (6.75)$$

The following numerical solutions are obtained

$$\mathbf{F}_{45} = -268.109\mathbf{i} - 134.646\mathbf{j} \text{ N and } \mathbf{r}_P = -0.149492\mathbf{i} + 0.0476701\mathbf{j} \text{ m.}$$

Reaction force  $\mathbf{F}_{34}$

The rotation joint  $D_R$  between the links 3 and 4 is replaced with the unknown reaction force  $\mathbf{F}_{34}$  (Fig. 6.25)

$$\mathbf{F}_{34} = -\mathbf{F}_{34} = F_{34x}\mathbf{i} + F_{34y}\mathbf{j}.$$

Following the path  $I$ , a force equation can be written for the translation joint  $D_T$ . The projection of all forces, that act on the link 4, onto the sliding direction  $ED$  is zero

$$\sum \mathbf{F}^{(4)} \cdot \mathbf{r}_{ED} = (\mathbf{F}_{34} + \mathbf{G}_4 + \mathbf{F}_{in4}) \cdot \mathbf{r}_{ED} = 0, \quad (6.76)$$

where  $\mathbf{r}_{ED} = \mathbf{r}_D - \mathbf{r}_E$ .

Continuing on the path  $I$  (Fig. 6.25), a moment equation is written for the rotation joint  $E_R$

$$\begin{aligned} \sum \mathbf{M}_E^{(4\&5)} &= \mathbf{r}_{EC_4} \times (\mathbf{G}_4 + \mathbf{F}_{in4}) + \mathbf{r}_{ED} \times \mathbf{F}_{34} + \mathbf{M}_{in4} + \\ &\mathbf{r}_{EC_5} \times (\mathbf{G}_5 + \mathbf{F}_{in5}) + \mathbf{M}_{in5} + \mathbf{M}_{5ext} = \mathbf{0}. \end{aligned} \quad (6.77)$$

where  $\mathbf{r}_{EC_5} = \mathbf{r}_{C_5} - \mathbf{r}_E$ , and  $\mathbf{r}_{EC_4} = \mathbf{r}_{C_4} - \mathbf{r}_E$ .

The following numerical solution is obtained

$$\mathbf{F}_{34} = -268.072\mathbf{i} - 134.583\mathbf{j} \text{ N.}$$

Reaction force  $\mathbf{F}_{03}$

The rotation joint  $C_R$  between the links 0 and 3 is replaced with the unknown reaction force  $\mathbf{F}_{03}$  (Fig. 6.26)

$$\mathbf{F}_{03} = F_{03x}\mathbf{i} + F_{03y}\mathbf{j}.$$

Following the path  $I$  (Fig. 6.26), a force equation is written for the translation joint  $B_T$ . The projection of all forces, that act on the link 3, onto the sliding direction  $CD$  is zero

$$\sum \mathbf{F}^{(3)} \cdot \mathbf{r}_{CD} = (\mathbf{F}_{03} + \mathbf{F}_{43} + \mathbf{G}_3 + \mathbf{F}_{in3}) \cdot \mathbf{r}_{CD} = 0, \quad (6.78)$$

where  $\mathbf{r}_{CD} = \mathbf{r}_D - \mathbf{r}_C$ .

Continuing on the path  $II$  (Fig. 6.26), a moment equation is written for the rotation joint  $B_R$

$$\sum \mathbf{M}_B^{(3\&2)} = \mathbf{r}_{BC_3} \times (\mathbf{G}_3 + \mathbf{F}_{in3}) + \mathbf{r}_{BC} \times \mathbf{F}_{03} + \mathbf{r}_{BD} \times \mathbf{F}_{43} + \mathbf{M}_{in2} + \mathbf{M}_{in3} = \mathbf{0}, \quad (6.79)$$

where  $\mathbf{r}_{BC_3} = \mathbf{r}_{C_3} - \mathbf{r}_B$ ,  $\mathbf{r}_{BC} = \mathbf{r}_C - \mathbf{r}_B$ , and  $\mathbf{r}_{BD} = \mathbf{r}_D - \mathbf{r}_B$ .

The following numerical solution is obtained

$$\mathbf{F}_{03} = -256.745\mathbf{i} - 272.179\mathbf{j} \text{ N.}$$

Reaction force  $\mathbf{F}_{23}$

The translation joint  $B_T$  between the links 2 and 3 is replaced with the unknown reaction force  $\mathbf{F}_{23}$  (Fig. 6.27)

$$\mathbf{F}_{23} = -\mathbf{F}_{32} = F_{23x}\mathbf{i} + F_{23y}\mathbf{j}.$$

The position of the application point  $Q$  of the force  $\mathbf{F}_{23}$  is unknown

$$\mathbf{r}_Q = x_Q\mathbf{i} + y_Q\mathbf{j},$$

where  $x_Q$  and  $y_Q$  are the plane coordinates of the point  $Q$ .

Following the path  $I$  (Fig. 6.27), a moment equation is written for the rotation joint  $C_R$

$$\sum \mathbf{M}_C^{(3)} = \mathbf{r}_{CQ} \times \mathbf{F}_{23} + \mathbf{r}_{CC_3} \times (\mathbf{G}_3 + \mathbf{F}_{in3}) + \mathbf{r}_{CD} \times \mathbf{F}_{43} + \mathbf{M}_{in3} = \mathbf{0}, \quad (6.80)$$

where  $\mathbf{r}_{CQ} = \mathbf{r}_Q - \mathbf{r}_C$ ,  $\mathbf{r}_{CC_3} = \mathbf{r}_{C_3} - \mathbf{r}_C$ , and  $\mathbf{r}_{CD} = \mathbf{r}_D - \mathbf{r}_C$ .

Following the path  $II$  (Fig. 6.27), a moment equation is written for the rotation joint  $B_R$

$$\sum \mathbf{M}_B^{(2)} = \mathbf{r}_{BQ} \times \mathbf{F}_{32} + \mathbf{M}_{in2} = \mathbf{0}, \quad (6.81)$$

where  $\mathbf{r}_{BQ} = \mathbf{r}_Q - \mathbf{r}_B$ .

The direction of the unknown joint force  $\mathbf{F}_{23}$  is perpendicular to the sliding direction  $BC$ . The following relation is written

$$\mathbf{F}_{23} \cdot \mathbf{r}_{BC} = 0,$$

The application point  $Q$  of the force  $\mathbf{F}_{23}$  is located on the direction  $BC$ , that is

$$(\mathbf{r}_B - \mathbf{r}_C) \times (\mathbf{r}_Q - \mathbf{r}_C) = \mathbf{0}. \quad (6.82)$$

The following numerical solutions are obtained

$$\mathbf{F}_{23} = -11.3762 \mathbf{i} + 137.93 \mathbf{j} \text{ N} \quad \text{and} \quad \mathbf{r}_Q = 0.121243 \mathbf{i} + 0.070 \mathbf{j} \text{ m}.$$

Reaction force  $\mathbf{F}_{12}$

The rotation joint  $B_R$  between the links 1 and 2 is replaced with the unknown reaction force  $\mathbf{F}_{12}$  (Fig. 6.28)

$$\mathbf{F}_{12} = -\mathbf{F}_{21} = F_{12x} \mathbf{i} + F_{12y} \mathbf{j}.$$

Following the path  $I$  (Fig. 6.28), a force equation is written for the translation joint  $B_T$ . The projection of all forces, that act on the link 2, onto the sliding direction  $BC$  is zero

$$\sum \mathbf{F}^{(2)} \cdot \mathbf{r}_{BC} = (\mathbf{F}_{12} + \mathbf{G}_2 + \mathbf{F}_{in2}) \cdot \mathbf{r}_{BC} = 0. \quad (6.83)$$

Continuing on the path  $I$ , a moment equation is written for the rotation joint  $C_R$

$$\begin{aligned} \sum \mathbf{M}_C^{(2\&3)} &= \mathbf{r}_{CB} \times \mathbf{F}_{12} + \mathbf{r}_{CC_2} \times (\mathbf{G}_2 + \mathbf{F}_{in2}) + \mathbf{M}_{in2} + \\ &\quad \mathbf{r}_{CC_3} \times (\mathbf{G}_3 + \mathbf{F}_{in3}) + \mathbf{r}_{CD} \times \mathbf{F}_{43} + \mathbf{M}_{in3} = \mathbf{0}, \end{aligned} \quad (6.84)$$

where  $\mathbf{r}_{CB} = \mathbf{r}_B - \mathbf{r}_C$ ,  $\mathbf{r}_{CC_2} = \mathbf{r}_{C_2} - \mathbf{r}_C$ ,  $\mathbf{r}_{CC_3} = \mathbf{r}_{C_3} - \mathbf{r}_C$ , and  $\mathbf{r}_{CD} = \mathbf{r}_D - \mathbf{r}_C$ . The following numerical solution is obtained

$$\mathbf{F}_{12} = -11.4028 \mathbf{i} + 137.993 \mathbf{j} \text{ N.}$$

The motor moment  $\mathbf{M}_m$

The motor moment needed for the dynamic equilibrium of the mechanism is  $\mathbf{M}_m = M_m \mathbf{k}$  (Fig. 6.29). Following the path *I* (Fig. 6.30), a moment equation is written for the rotation joint  $A_R$

$$\sum \mathbf{M}_A^{(1)} = \mathbf{r}_{AB} \times \mathbf{F}_{21} + \mathbf{r}_{AC_1} \times (\mathbf{G}_1 + \mathbf{F}_{in1}) + \mathbf{M}_{in1} + \mathbf{M}_m = \mathbf{0}. \quad (6.85)$$

The numerical solution is

$$\mathbf{M}_m = 17.5356 \mathbf{k} \text{ N} \cdot \text{m.}$$

Reaction force  $\mathbf{F}_{01}$

The rotation joint  $A_R$  between the links 0 and 1 is replaced with the unknown reaction force  $\mathbf{F}_{01}$  (Fig. 6.30)

$$\mathbf{F}_{01} = -\mathbf{F}_{10} = F_{01x} \mathbf{i} + F_{01y} \mathbf{j},$$

Following the path *I* (Fig. 6.30), a moment equation is written for the rotation joint  $B_R$

$$\sum \mathbf{M}_B^{(1)} = \mathbf{r}_{BA} \times \mathbf{F}_{01} + \mathbf{r}_{BC_1} \times (\mathbf{G}_1 + \mathbf{F}_{in1}) + \mathbf{M}_{in1} + \mathbf{M}_m = \mathbf{0}, \quad (6.86)$$

where  $\mathbf{r}_{BA} = -\mathbf{r}_B$ , and  $\mathbf{r}_{BC_1} = \mathbf{r}_{C_1} - \mathbf{r}_B$ .

Continuing on the path *I* (Fig. 6.31), a force equation is written for the translation joint  $B_T$ . The projection of all forces, that act on the links 1 and 2, onto the sliding direction  $BC$  is zero

$$\sum \mathbf{F}^{(1\&2)} \cdot \mathbf{r}_{BC} = (\mathbf{F}_{01} + \mathbf{G}_1 + \mathbf{F}_{in1} + \mathbf{G}_2 + \mathbf{F}_{in2}) \cdot \mathbf{r}_{BC} = 0. \quad (6.87)$$

The following numerical solution is obtained

$$\mathbf{F}_{01} = -11.4214 \mathbf{i} + 138.092 \mathbf{j} \text{ N.}$$

The MATLAB program for the dynamic force analysis is presented in Program 6.6.