```
(* RRT Robot*)
Apply [Clear,Names["Global`*"]];
Off[General::spell];
Off[General::spell1];

"transformation matrix from RF1 to RF0: R10="
R10={{1,0,0},
     {0,Cos[q1[t]],Sin[q1[t]]},
     {0,-Sin[q1[t]],Cos[q1[t]]}};
MatrixForm[R10]

"transformation matrix from RF2 to RF1: R21="
R21={{Cos[q2[t]],0,-Sin[q2[t]]},
     {0,1,0},
     {Sin[q2[t]],0,Cos[q2[t]]}};
MatrixForm[R21]

"angular velocity of link 1 in RF0
expressed in terms of RF1 {i1,j1,k1}: w10="
w10={q1'[t],0,0}

"angular velocity of link 2 in RF0
expressed  in terms of RF1 {i1,j1,k1}: w201="
w201={q1'[t],q2'[t],0}

"angular velocity of link 2 in RF0
expressed in terms of RF2 {i2,j2,k2}: w20="
w20=w201.Transpose[R21]

"angular acceleration of link 1 in RF0
expressed in terms of RF1 {i1,j1,k1}: α10="
α10=D[w10,t]

"angular acceleration of link 2 in RF0
expressed in terms of RF2 {i2,j2,k2}: α20="
α20=D[w20,t]

"position vector of mass center C1 of link 1
in RF0 expressed in terms of RF1 {i1,j1,k1}: rC1="
rC1={0,0,L1}

"linear velocity of mass center C1 of link 1
in RF0 expressed in terms of RF1 {i1,j1,k1}: vC1="
vC1=D[rC1,t]+Cross[w10,rC1]

"linear velocity of joint B in RF0
expressed in terms of RF1 {i1,j1,k1}: vB="
vB =D[{0,0, 2 L1},t]+Cross[w10,{0,0,2 L1}]

"position vector of mass center C2 of link 2
in RF0 expressed in terms of RF2 {i2,j2,k2}: rC2="
rC2={0,0,2 L1}.Transpose[R21]+{0,0,L2}

"linear velocity of mass center C2 of link 2 in RF0
expressed in terms of RF2 {i2,j2,k2}: vC2="
vC2 =D[rC2,t]+Cross[w20,rC2]

"position vector of mass center C3 of link 3 in RF0
expressed in terms of RF2 {i2,j2,k2}: rC3="
rC3=rC2+{0,0,q3[t]}

"linear velocity of mass center C3 of link 3 in RF0
expressed in terms of RF2 {i2,j2,k2}: vC3="
vC3=D[rC3,t]+Cross[w20,rC3];
```

```
vC3=Simplify[vC3]

"linear velocity of  C32 of link 2 in RF0
expressed in terms of RF2 {i2,j2,k2}
C32 of link 2 is superposed with C3 of link 3: vC32="
vC32=vC2+Cross[w20,{0,0,q3[t]}];
vC32=Simplify[vC32]

(* another way of computing vC3 is: *)
vC3'=vC32+D[{0,0,q3[t]},t];
(*vC3-vC3'={0,0,0};*)

"linear acceleration of mass center C1 of link 1
in RF0 expressed in terms of RF1 {i1,j1,k1}: aC1="
aC1 =D[vC1,t]+Cross[w10,vC1]

"linear acceleration of mass center C2 of link 2
in RF0 expressed in terms of RF2 {i2,j2,k2}: aC2="
aC2=D[vC2,t]+Cross[w20,vC2]

"linear acceleration of mass center C3 of link 3
in RF0 expressed in terms of RF2 {i2,j2,k2}: aC3="
aC3=D[vC3,t]+Cross[w20,vC3]

"gravitational  force that acts on link 1 at C1
in RF0 expressed in terms of RF1 {i1,j1,k1}: G1="
G1={ -m1 g ,0 , 0 }
"gravitational  force that acts on link 2 at C2
in RF0 expressed in terms of RF2 {i2,j2,k2}: G2="
G2={ -m2 g ,0 , 0 }.Transpose[R21]
"gravitational  force that acts on link 3 at C3
in RF0 expressed in terms of RF2 {i2,j2,k2}: G3="
G3={ -m3 g ,0 , 0 }.Transpose[R21]

"contact torque of 0 that acts on link 1 in RF0
expressed in terms of RF1 {i1,j1,k1}: T01="
T01={T01x,T01y,T01z}

"contact torque of link 1 that acts on link 2 in RF0
expressed in terms of RF2 {i2,j2,k2}: T12="
T12={T12x,T12y,T12x}

"contact force of link 2 that acts on link 3 at C3 in RF0
expressed in terms of RF2 {i2,j2,k2}: F23="
F23={F23x,F23y,F23z}

(*generalized active forces*)

"generalized active force Q1="
Q1=D[w10,q1'[t]].T01+
    D[vC1,q1'[t]].G1+
    D[w10,q1'[t]].Transpose[R21].(-T12)+
    D[w20,q1'[t]].T12+
    D[vC2,q1'[t]].G2+
    D[vC32,q1'[t]].(-F23)+
    D[vC3,q1'[t]].G3+
    D[vC3,q1'[t]].F23

"generalized active force Q2="
Q2=D[w10,q2'[t]].T01+
    D[vC1,q2'[t]].G1+
    D[w10,q2'[t]].Transpose[R21].(-T12)+
    D[w20,q2'[t]].T12+
    D[vC2,q2'[t]].G2+
    D[vC32,q2'[t]].(-F23)+
```

```
    D[vC3,q2'[t]].G3+
    D[vC3,q2'[t]].F23

"generalized active force Q3="
Q3=D[w10,q3'[t]].T01+
    D[vC1,q3'[t]].G1+
    D[w10,q3'[t]].Transpose[R21].(-T12)+
    D[w20,q3'[t]].T12+
    D[vC2,q3'[t]].G2+
    D[vC32,q3'[t]].(-F23)+
    D[vC3,q3'[t]].G3+
    D[vC3,q3'[t]].F23

(* inertia dyadics *)

"central inertia dyadic for link 1
expressed in terms of RF1 {i1,j1,k1}: I1="
I1={{m1(2 L1)^2/12,0,0},{0,m1(2 L1)^2/12,0},{0,0,0}}

"central inertia dyadic for link 2
expressed in terms of RF2 {i2,j2,k2}: I2="
I2={{m2(2 L2)^2/12,0,0},{0,m2(2 L2)^2/12,0},{0,0,0}}

"central inertia torque for link 3
expressed in terms of RF2 {i2,j2,k2}: I3="
I3={{I3x,0,0},{0,I3y,0},{0,0,I3z}}

(*===============================*)
(*LAGRANGE's equations of motion*)
(*===============================*)

"kinetic energy of link 1: T1="
T1=m1 vC1.vC1/2+w10.I1.w10/2
"kinetic energy of link 2: T2="
T2=m2 vC2.vC2/2+w20.I2.w20/2
"kinetic energy of link 3: T3="
T3=m3 vC3.vC3/2+w20.I3.w20/2
"total kinetic energy: T="
T=T1+T2+T3
T=Expand[T];

LHS1=D[D[T,q1'[t]],t]-D[T,q1[t]];
LHS2=D[D[T,q2'[t]],t]-D[T,q2[t]];
LHS3=D[D[T,q3'[t]],t]-D[T,q3[t]];

"First Lagrange's equation of motion"
"D[D[T,q1'[t]],t] - D[T,q1[t]] = Q1"
Lagr1=LHS1-Q1
"Second Lagrange's equation of motion"
"D[D[T,q2'[t]],t] - D[T,q2[t]] = Q2"
Lagr2=LHS2-Q2
"Third Lagrange's equation of motion"
"D[D[T,q3'[t]],t] - D[T,q3[t]] = Q3"
Lagr3=LHS3-Q3

(*==========================*)
(*Kane's dynamical equations*)
(*==========================*)

(*inertia moments*)
"inertia moment for link 1 expressed in terms of RF1 {i1,j1,k1}: M1in="
M1in= -α10.I1-Cross[w10,I1.w10]
"inertia moment for link 2 expressed in terms of RF2 {i2,j2,k2}: M2in="
M2in=-α20.I2-Cross[w20,I2.w20]
"inertia moment for link 3 expressed in terms of RF2 {i2,j2,k2}: M3in="
```

```
M3in=-α20.I3-Cross[w20,I3.w20]

(*generalized inertia forces*)

"generalized inertia forces corresponding to q1: Kin1="
Kin1=D[w10,q1'[t]].M1in+                    (*link1*)
     D[vC1,q1'[t]].(-m1 aC1)+               (*link1*)
     D[w20,q1'[t]].M2in+                    (*link2*)
     D[vC2,q1'[t]].(-m2 aC2)+               (*link2*)
     D[w20,q1'[t]].M3in+                    (*link3*)
     D[vC3,q1'[t]].(-m3 aC3)                (*link3*)

"generalized inertia forces corresponding to q2: Kin2="
Kin2=D[w10,q2'[t]].M1in+                    (*link1*)
     D[vC1,q2'[t]].(-m1 aC1)+               (*link1*)
     D[w20,q2'[t]].M2in+                    (*link2*)
     D[vC2,q2'[t]].(-m2 aC2)+               (*link2*)
     D[w20,q2'[t]].M3in+                    (*link3*)
     D[vC3,q2'[t]].(-m3 aC3)                (*link3*)

"generalized inertia forces corresponding to q3: Kin3="
Kin3=D[w10,q3'[t]].M1in+                    (*link1*)
     D[vC1,q3'[t]].(-m1 aC1)+               (*link1*)
     D[w20,q3'[t]].M2in+                    (*link2*)
     D[vC2,q3'[t]].(-m2 aC2)+               (*link2*)
     D[w20,q3'[t]].M3in+                    (*link3*)
     D[vC3,q3'[t]].(-m3 aC3)                (*link3*)

(*Kane's dynamical equations*)
"First Kane's dynamical equation"
kane1=Q1+Kin1
"Second Kane's dynamical equation"
kane2=Q2+Kin2
"Third Kane's dynamical equation"
kane3=Q3+Kin3

(*numerical data*)
Tp=15.; (*s*)
indata={L1→0.4,L2→0.4,I3x→5,I3y→4,I3z→1,
        m1→90,m2→60,m3→40,g→9.81,
        q1s→N[Pi/18],q2s→N[Pi/6], q3s→0.25,
        q1f→N[Pi/3], q2f→N[ Pi/3],q3f→0.3,
    b01→450,g01→300,
    b12→200,g12→300,
    b23→150,g23→50};

Print[" "];
Print[" "];

(*================*)
(*INVERSE DYNAMICS*)
(*================*)
Print["INVERSE DYNAMICS"];

q1t = q1s + (q1f-q1s)/Tp (t - Tp/(2 Pi) Sin[2 Pi/Tp t]);
q2t = q2s + (q2f-q2s)/Tp (t - Tp/(2 Pi) Sin[2 Pi/Tp t]);
q3t = q3s + (q3f-q3s)/Tp (t - Tp/(2 Pi) Sin[2 Pi/Tp t]);
gen= {
      q1[t] → q1t,
      q1'[t]→ D[q1t,t],
      q1''[t]→D[D[q1t,t],t],
      q2[t] → q2t,
      q2'[t]→D[q2t,t],
      q2''[t]→D[D[q2t,t],t],
      q3[t] → q3t,
```

```
        q3'[t]→D[q3t,t],
        q3''[t]→D[D[q3t,t],t]};

Plot[Evaluate[q1[t]*180/N[Pi]/.gen/.indata],{t,0,Tp},
    PlotRange→{All,All},AxesLabel→{"t[s]","q1[deg]"}];
Plot[Evaluate[q2[t]*180/N[Pi]/.gen/.indata],{t,0,Tp},
    PlotRange→{All,All},AxesLabel→{"t[s]","q2[deg]"}];
Plot[Evaluate[q3[t]/.gen/.indata],{t,0,Tp},PlotRange→{All,All},
    AxesLabel→{"t[s]","q3[m]"}];


Print["=== Lagrange ==="]

LE1 = (Lagr1/.gen/.indata)==0;
LE2 = (Lagr2/.gen/.indata)==0;
LE3 = (Lagr3/.gen/.indata)==0;

LSol = Solve[{LE1,LE2,LE3},{T01x,T12y,F23z}];
T01xL=T01x/.LSol[[1]];
T12yL=T12y/.LSol[[1]];
F23zL=F23z/.LSol[[1]];

Plot[Evaluate[T01xL],{t,0,15},PlotRange→{All,All},
    AxesLabel→{"t[s]","T01x[N m]"}];
Plot[Evaluate[T12yL],{t,0,15},PlotRange→{All,All},
    AxesLabel→{"t[s]","T12y[N m]"}];
Plot[Evaluate[F23zL],{t,0,Tp},PlotRange→{All,All},
    AxesLabel→{"t[s]","F23z[N]"}];

Print["=== Kane ==="];

KE1 = (kane1/.gen/.indata)==0;
KE2 = (kane2/.gen/.indata)==0;
KE3 = (kane3/.gen/.indata)==0;

KSol = Solve[{KE1,KE2,KE3},{T01x,T12y,F23z}];
T01xK=T01x/.KSol[[1]];
T12yK=T12y/.KSol[[1]];
F23zK=F23z/.KSol[[1]];

Plot[Evaluate[T01xK],{t,0,Tp},PlotRange→{All,All},
    AxesLabel→{"t[s]","T01x[N m]"}];
Plot[Evaluate[T12yK],{t,0,Tp},PlotRange→{All,All},
    AxesLabel→{"t[s]","T12y[N m]"}];
Plot[Evaluate[F23zK],{t,0,Tp},PlotRange→{All,All},
    AxesLabel→{"t[s]","F23z[N]"}];

Print["Verification "];

tc=5;

Print["using Lagrange for t = ",tc," s =>"];
T01L=Evaluate[T01xL/.t→tc];
T12L=Evaluate[T12yL/.t→tc];
F23L=Evaluate[F23zL/.t→tc];
Print["T01x = ",T01L," N m"];
Print["T12y = ",T12L," N m"];
Print["F23z = ",F23L," N"];

Print["using Kane for t = ",tc," s =>"];
T01K=Evaluate[T01xK/.t→tc];
T12K=Evaluate[T12yK/.t→tc];
F23K=Evaluate[F23zK/.t→tc];
Print["T01x = ",T01K," N m"];
Print["T12y = ",T12K," N m"];
```

```
Print["F23z = ",F23K," N"];

Print[" "];
Print[" "];

(*================*)
(*DIRECT DYNAMICS*)
(*================*)
Print["DIRECT DYNAMICS"]

(*control*)
control={
T01x→-b01 q1'[t]-g01(q1[t]-q1f),
T12y→-b12 q2'[t]-g12(q2[t]-q2f)+g (m2 L2+m3 (L2+q3[t])) Cos[q2[t]],
F23z→-b23 q3'[t]-g23(q3[t]-q3f)+g m3 Sin[q2[t]]
}/.indata;

(*numerical simulation of Lagrange's eom*)
lageq={
(Lagr1/.indata)==0,(Lagr2/.indata)==0,(Lagr3/.indata)==0,
q1'[0]==0.,q2'[0]==0.,q3'[0]==0.,
q1[0]==N[Pi/18],q2[0]==N[Pi/6],q3[0]==0.25};
Lagrange=NDSolve[lageq/.control,{q1,q2,q3},{t,0,Tp}];

Print["=== Lagrange ==="];
Plot[Evaluate[q1[t]/.Lagrange],{t,0,Tp},PlotRange→{All,All},
  AxesLabel→{"t[s]","q1[rad]"}];
Plot[Evaluate[q2[t]/.Lagrange],{t,0,Tp},PlotRange→{All,All},
  AxesLabel→{"t[s]","q2[rad]"}];
Plot[Evaluate[q3[t]/.Lagrange],{t,0,Tp},PlotRange→{All,All},
  AxesLabel→{"t[s]","q3[m]"}];

(*numerical simulation of Kane's eom*)
kaneq =
{(kane1/.indata)==0,(kane2/.indata)==0,(kane3/.indata)==0,
q1'[0]==0.,q2'[0]==0.,q3'[0]==0.,
q1[0]==N[Pi/18],q2[0]==N[Pi/6],q3[0]==0.25};
Kane=NDSolve[kaneq/.control,{q1,q2,q3},{t,0,Tp}];

Print["=== Kane ==="];
Plot[Evaluate[q1[t]/.Kane],{t,0,Tp},PlotRange→{All,All},
  AxesLabel→{"t[s]","q1[rad]"}];
Plot[Evaluate[q2[t]/.Kane],{t,0,Tp},PlotRange→{All,All},
  AxesLabel→{"t[s]","q2[rad]"}];
Plot[Evaluate[q3[t]/.Kane],{t,0,Tp},PlotRange→{All,All},
  AxesLabel→{"t[s]","q3[m]"}];

Print["Verification "];

td=1.55;

Print["using Lagrange for t = ",td," s =>"];
q110L=Evaluate[q1[t]/.Lagrange/.t->td][[1]];
q210L=Evaluate[q2[t]/.Lagrange/.t->td][[1]];
q310L=Evaluate[q3[t]/.Lagrange/.t->td][[1]];
Print["q1 = ",q110L," rad"];
Print["q2 = ",q210L," rad"];
Print["q3 = ",q310L," m"];

Print["using Kane for t = ",td," s =>"];
q110K=Evaluate[q1[t]/.Kane/.t->td][[1]];
q210K=Evaluate[q2[t]/.Kane/.t->td][[1]];
q310K=Evaluate[q3[t]/.Kane/.t->td][[1]];
Print["q1 = ",q110K," rad"];
Print["q2 = ",q210K," rad"];
```

```
Print["q3 = ",q31OK," m"];


Print[" "]
Print["================"];
tt=15;
tLagr=Timing[{q1[tt],q2[tt],q3[tt]}/.Lagrange];
tKane=Timing[{q1[tt],q2[tt],q3[tt]}/.Kane];
Print["For t = ",tt," s"];
Print["the time used to evaluate {q1[t],q2[t],q3[t]} = ",tLagr[[2]][[1]]," with
Lagrange is = ",tLagr[[1]] ];
Print["the time used to evaluate {q1[t],q2[t],q3[t]} = ",tKane[[2]][[1]]," with Kane
is = ",tKane[[1]] ];
```

transformation matrix from RF1 to RF0: R10=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[q1[t]] & \text{Sin}[q1[t]] \\ 0 & -\text{Sin}[q1[t]] & \text{Cos}[q1[t]] \end{pmatrix}$$

transformation matrix from RF2 to RF1: R21=

$$\begin{pmatrix} \text{Cos}[q2[t]] & 0 & -\text{Sin}[q2[t]] \\ 0 & 1 & 0 \\ \text{Sin}[q2[t]] & 0 & \text{Cos}[q2[t]] \end{pmatrix}$$

angular velocity of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}: w10=

{q1′[t], 0, 0}

angular velocity of link 2 in RF0 expressed in terms of RF1 {i1,j1,k1}: w201=

{q1′[t], q2′[t], 0}

angular velocity of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}: w20=

{Cos[q2[t]] q1′[t], q2′[t], Sin[q2[t]] q1′[t]}

angular acceleration of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}: α10=

{q1″[t], 0, 0}

angular acceleration of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}: α20=

{-Sin[q2[t]] q1′[t] q2′[t] + Cos[q2[t]] q1″[t],
 q2″[t], Cos[q2[t]] q1′[t] q2′[t] + Sin[q2[t]] q1″[t]}

position vector of mass center C1 of
  link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}: rC1=

{0, 0, L1}

linear velocity of mass center C1 of
  link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}: vC1=

{0, -L1 q1′[t], 0}

linear velocity of joint B in RF0 expressed in terms of RF1 {i1,j1,k1}: vB=

$$\{0, -2\,L1\,q1'[t], 0\}$$

```
position vector of mass center C2 of
  link 2  in RF0 expressed in terms of RF2 {i2,j2,k2}: rC2=
```

$$\{-2\,L1\,Sin[q2[t]], 0, L2 + 2\,L1\,Cos[q2[t]]\}$$

```
linear velocity of mass center C2 of
  link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}: vC2=
```

$$\{L2\,q2'[t], -L2\,Cos[q2[t]]\,q1'[t] - 2\,L1\,Cos[q2[t]]^2\,q1'[t] - 2\,L1\,Sin[q2[t]]^2\,q1'[t], 0\}$$

```
position vector of mass center C3 of
  link 3 in RF0 expressed in terms of RF2 {i2,j2,k2}: rC3=
```

$$\{-2\,L1\,Sin[q2[t]], 0, L2 + 2\,L1\,Cos[q2[t]] + q3[t]\}$$

```
linear velocity of mass center C3 of
  link 3 in RF0 expressed in terms of RF2 {i2,j2,k2}: vC3=
```

$$\{(L2 + q3[t])\,q2'[t], -(2\,L1 + L2\,Cos[q2[t]] + Cos[q2[t]]\,q3[t])\,q1'[t], q3'[t]\}$$

```
linear velocity of  C32 of link 2 in RF0 expressed in terms of
  RF2 {i2,j2,k2}  C32 of link 2 is superposed with C3 of link 3: vC32=
```

$$\{(L2 + q3[t])\,q2'[t], -(2\,L1 + L2\,Cos[q2[t]] + Cos[q2[t]]\,q3[t])\,q1'[t], 0\}$$

```
linear acceleration of mass center C1 of
  link 1  in RF0 expressed in terms of RF1 {i1,j1,k1}: aC1=
```

$$\{0, -L1\,q1''[t], -L1\,q1'[t]^2\}$$

```
linear acceleration of mass center C2 of
  link 2  in RF0 expressed in terms of RF2 {i2,j2,k2}: aC2=
```

$$\{L2\,Cos[q2[t]]\,Sin[q2[t]]\,q1'[t]^2 + 2\,L1\,Cos[q2[t]]^2\,Sin[q2[t]]\,q1'[t]^2 + 2\,L1\,Sin[q2[t]]^3\,q1'[t]^2 + L2\,q2''[t], 2\,L2\,Sin[q2[t]]\,q1'[t]\,q2'[t] - L2\,Cos[q2[t]]\,q1''[t] - 2\,L1\,Cos[q2[t]]^2\,q1''[t] - 2\,L1\,Sin[q2[t]]^2\,q1''[t], -L2\,Cos[q2[t]]^2\,q1'[t]^2 - 2\,L1\,Cos[q2[t]]^3\,q1'[t]^2 - 2\,L1\,Cos[q2[t]]\,Sin[q2[t]]^2\,q1'[t]^2 - L2\,q2'[t]^2\}$$

```
linear acceleration of mass center C3 of
  link 3  in RF0 expressed in terms of RF2 {i2,j2,k2}: aC3=
```

$$\{2\,L1\,Sin[q2[t]]\,q1'[t]^2 + L2\,Cos[q2[t]]\,Sin[q2[t]]\,q1'[t]^2 + Cos[q2[t]]\,q3[t]\,Sin[q2[t]]\,q1'[t]^2 + 2\,q2'[t]\,q3'[t] + (L2 + q3[t])\,q2''[t], L2\,Sin[q2[t]]\,q1'[t]\,q2'[t] + q3[t]\,Sin[q2[t]]\,q1'[t]\,q2'[t] - Cos[q2[t]]\,q1'[t]\,q3'[t] - q1'[t]\,(-L2\,Sin[q2[t]]\,q2'[t] - q3[t]\,Sin[q2[t]]\,q2'[t] + Cos[q2[t]]\,q3'[t]) - (2\,L1 + L2\,Cos[q2[t]] + Cos[q2[t]]\,q3[t])\,q1''[t], -2\,L1\,Cos[q2[t]]\,q1'[t]^2 - L2\,Cos[q2[t]]^2\,q1'[t]^2 - Cos[q2[t]]^2\,q3[t]\,q1'[t]^2 - L2\,q2'[t]^2 - q3[t]\,q2'[t]^2 + q3''[t]\}$$

```
gravitational force that acts on link
  1 at C1 in RF0 expressed in terms of RF1 {i1,j1,k1}: G1=
```

$$\{-g\,m1, 0, 0\}$$

```
gravitational force that acts on link
  2 at C2 in RF0 expressed in terms of RF2 {i2,j2,k2}: G2=
```

{-g m2 Cos[q2[t]], 0, -g m2 Sin[q2[t]]}

gravitational force that acts on link
3 at C3 in RF0 expressed in terms of RF2 {i2,j2,k2}: G3=

{-g m3 Cos[q2[t]], 0, -g m3 Sin[q2[t]]}

contact torque of 0 that acts on link
1 in RF0 expressed in terms of RF1 {i1,j1,k1}: T01=

{T01x, T01y, T01z}

contact torque of link 1 that acts on
link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}: T12=

{T12x, T12y, T12x}

contact force of link 2 that acts on link
3 at C3 in RF0 expressed in terms of RF2 {i2,j2,k2}: F23=

{F23x, F23y, F23z}

generalized active force Q1=

T01x

generalized active force Q2=

T12y - g L2 m2 Cos[q2[t]] - g m3 Cos[q2[t]] (L2 + q3[t])

generalized active force Q3=

F23z - g m3 Sin[q2[t]]

central inertia dyadic for link 1 expressed in terms of RF1 {i1,j1,k1}: I1=

$$\left\{\left\{\frac{L1^2\, m1}{3}, 0, 0\right\}, \left\{0, \frac{L1^2\, m1}{3}, 0\right\}, \{0, 0, 0\}\right\}$$

central inertia dyadic for link 2 expressed in terms of RF2 {i2,j2,k2}: I2=

$$\left\{\left\{\frac{L2^2\, m2}{3}, 0, 0\right\}, \left\{0, \frac{L2^2\, m2}{3}, 0\right\}, \{0, 0, 0\}\right\}$$

central inertia torque for link 3 expressed in terms of RF2 {i2,j2,k2}: I3=

{{I3x, 0, 0}, {0, I3y, 0}, {0, 0, I3z}}

kinetic energy of link 1: T1=

$$\frac{2}{3}\, L1^2\, m1\, q1'[t]^2$$

kinetic energy of link 2: T2=

$$\frac{1}{2}\, m2 \left((-L2\, Cos[q2[t]]\, q1'[t] - 2\, L1\, Cos[q2[t]]^2\, q1'[t] - 2\, L1\, Sin[q2[t]]^2\, q1'[t])^2 + L2^2\, q2'[t]^2 \right) + \frac{1}{2} \left(\frac{1}{3}\, L2^2\, m2\, Cos[q2[t]]^2\, q1'[t]^2 + \frac{1}{3}\, L2^2\, m2\, q2'[t]^2 \right)$$

```
kinetic energy of link 3: T3=
```

$$\frac{1}{2}\left(\text{I3x}\,\text{Cos}[q2[t]]^2\,q1'[t]^2+\text{I3z}\,\text{Sin}[q2[t]]^2\,q1'[t]^2+\text{I3y}\,q2'[t]^2\right)+\frac{1}{2}\,m3\,\left((2\,L1+L2\,\text{Cos}[q2[t]]+\text{Cos}[q2[t]]\,q3[t])^2\,q1'[t]^2+(L2+q3[t])^2\,q2'[t]^2+q3'[t]^2\right)$$

```
total kinetic energy: T=
```

$$\frac{2}{3}\,L1^2\,m1\,q1'[t]^2+\frac{1}{2}\left(\text{I3x}\,\text{Cos}[q2[t]]^2\,q1'[t]^2+\text{I3z}\,\text{Sin}[q2[t]]^2\,q1'[t]^2+\text{I3y}\,q2'[t]^2\right)+\frac{1}{2}\,m2\left((-L2\,\text{Cos}[q2[t]]\,q1'[t]-2\,L1\,\text{Cos}[q2[t]]^2\,q1'[t]-2\,L1\,\text{Sin}[q2[t]]^2\,q1'[t])^2+L2^2\,q2'[t]^2\right)+\frac{1}{2}\left(\frac{1}{3}\,L2^2\,m2\,\text{Cos}[q2[t]]^2\,q1'[t]^2+\frac{1}{3}\,L2^2\,m2\,q2'[t]^2\right)+\frac{1}{2}\,m3\,\left((2\,L1+L2\,\text{Cos}[q2[t]]+\text{Cos}[q2[t]]\,q3[t])^2\,q1'[t]^2+(L2+q3[t])^2\,q2'[t]^2+q3'[t]^2\right)$$

```
First Lagrange's equation of motion

D[D[T,q1'[t]],t] - D[T,q1[t]] = Q1
```

$$-\text{T01x}-4\,L1\,L2\,m3\,\text{Sin}[q2[t]]\,q1'[t]\,q2'[t]-2\,\text{I3x}\,\text{Cos}[q2[t]]\,\text{Sin}[q2[t]]\,q1'[t]\,q2'[t]+2\,\text{I3z}\,\text{Cos}[q2[t]]\,\text{Sin}[q2[t]]\,q1'[t]\,q2'[t]-\frac{8}{3}\,L2^2\,m2\,\text{Cos}[q2[t]]\,\text{Sin}[q2[t]]\,q1'[t]\,q2'[t]-2\,L2^2\,m3\,\text{Cos}[q2[t]]\,\text{Sin}[q2[t]]\,q1'[t]\,q2'[t]-4\,L1\,L2\,m2\,\text{Cos}[q2[t]]^2\,\text{Sin}[q2[t]]\,q1'[t]\,q2'[t]-4\,L1\,m3\,q3[t]\,\text{Sin}[q2[t]]\,q1'[t]\,q2'[t]-4\,L2\,m3\,\text{Cos}[q2[t]]\,q3[t]\,\text{Sin}[q2[t]]\,q1'[t]\,q2'[t]-2\,m3\,\text{Cos}[q2[t]]\,q3[t]^2\,\text{Sin}[q2[t]]\,q1'[t]\,q2'[t]-4\,L1\,L2\,m2\,\text{Sin}[q2[t]]^3\,q1'[t]\,q2'[t]+4\,L1\,m3\,\text{Cos}[q2[t]]\,q1'[t]\,q3'[t]+2\,L2\,m3\,\text{Cos}[q2[t]]^2\,q1'[t]\,q3'[t]+2\,m3\,\text{Cos}[q2[t]]^2\,q3[t]\,q1'[t]\,q3'[t]+\frac{4}{3}\,L1^2\,m1\,q1''[t]+4\,L1^2\,m3\,q1''[t]+4\,L1\,L2\,m3\,\text{Cos}[q2[t]]\,q1''[t]+\text{I3x}\,\text{Cos}[q2[t]]^2\,q1''[t]+\frac{4}{3}\,L2^2\,m2\,\text{Cos}[q2[t]]^2\,q1''[t]+L2^2\,m3\,\text{Cos}[q2[t]]^2\,q1''[t]+4\,L1\,L2\,m2\,\text{Cos}[q2[t]]^3\,q1''[t]+4\,L1^2\,m2\,\text{Cos}[q2[t]]^4\,q1''[t]+4\,L1\,m3\,\text{Cos}[q2[t]]\,q3[t]\,q1''[t]+2\,L2\,m3\,\text{Cos}[q2[t]]^2\,q3[t]\,q1''[t]+m3\,\text{Cos}[q2[t]]^2\,q3[t]^2\,q1''[t]+\text{I3z}\,\text{Sin}[q2[t]]^2\,q1''[t]+4\,L1\,L2\,m2\,\text{Cos}[q2[t]]\,\text{Sin}[q2[t]]^2\,q1''[t]+8\,L1^2\,m2\,\text{Cos}[q2[t]]^2\,\text{Sin}[q2[t]]^2\,q1''[t]+4\,L1^2\,m2\,\text{Sin}[q2[t]]^4\,q1''[t]$$

```
Second Lagrange's equation of motion

D[D[T,q2'[t]],t] - D[T,q2[t]] = Q2
```

$$-\text{T12y}+g\,L2\,m2\,\text{Cos}[q2[t]]+g\,m3\,\text{Cos}[q2[t]]\,(L2+q3[t])+2\,L1\,L2\,m3\,\text{Sin}[q2[t]]\,q1'[t]^2+\text{I3x}\,\text{Cos}[q2[t]]\,\text{Sin}[q2[t]]\,q1'[t]^2-\text{I3z}\,\text{Cos}[q2[t]]\,\text{Sin}[q2[t]]\,q1'[t]^2+\frac{4}{3}\,L2^2\,m2\,\text{Cos}[q2[t]]\,\text{Sin}[q2[t]]\,q1'[t]^2+L2^2\,m3\,\text{Cos}[q2[t]]\,\text{Sin}[q2[t]]\,q1'[t]^2+2\,L1\,L2\,m2\,\text{Cos}[q2[t]]^2\,\text{Sin}[q2[t]]\,q1'[t]^2+2\,L1\,m3\,q3[t]\,\text{Sin}[q2[t]]\,q1'[t]^2+2\,L2\,m3\,\text{Cos}[q2[t]]\,q3[t]\,\text{Sin}[q2[t]]\,q1'[t]^2+m3\,\text{Cos}[q2[t]]\,q3[t]^2\,\text{Sin}[q2[t]]\,q1'[t]^2+2\,L1\,L2\,m2\,\text{Sin}[q2[t]]^3\,q1'[t]^2+2\,L2\,m3\,q2'[t]\,q3'[t]+2\,m3\,q3[t]\,q2'[t]\,q3'[t]+\text{I3y}\,q2''[t]+\frac{4}{3}\,L2^2\,m2\,q2''[t]+L2^2\,m3\,q2''[t]+2\,L2\,m3\,q3[t]\,q2''[t]+m3\,q3[t]^2\,q2''[t]$$

```
Third Lagrange's equation of motion

D[D[T,q3'[t]],t] - D[T,q3[t]] = Q3
```

$$-\text{F23z}+g\,m3\,\text{Sin}[q2[t]]-2\,L1\,m3\,\text{Cos}[q2[t]]\,q1'[t]^2-L2\,m3\,\text{Cos}[q2[t]]^2\,q1'[t]^2-m3\,\text{Cos}[q2[t]]^2\,q3[t]\,q1'[t]^2-L2\,m3\,q2'[t]^2-m3\,q3[t]\,q2'[t]^2+m3\,q3''[t]$$

inertia moment for link 1 expressed in terms of RF1 {i1,j1,k1}: M1in=

$$\left\{-\frac{1}{3} L1^2\, m1\, q1''[t],\ 0,\ 0\right\}$$

inertia moment for link 2 expressed in terms of RF2 {i2,j2,k2}: M2in=

$$\left\{\frac{1}{3} L2^2\, m2\, \mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] - \frac{1}{3} L2^2\, m2\, (-\mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] + \mathrm{Cos}[q2[t]]\, q1''[t]),\right.$$
$$\left. -\frac{1}{3} L2^2\, m2\, \mathrm{Cos}[q2[t]]\, \mathrm{Sin}[q2[t]]\, q1'[t]^2 - \frac{1}{3} L2^2\, m2\, q2''[t],\ 0\right\}$$

inertia moment for link 3 expressed in terms of RF2 {i2,j2,k2}: M3in=

$$\{I3y\, \mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] - I3z\, \mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] -$$
$$I3x\, (-\mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] + \mathrm{Cos}[q2[t]]\, q1''[t]),$$
$$-I3x\, \mathrm{Cos}[q2[t]]\, \mathrm{Sin}[q2[t]]\, q1'[t]^2 + I3z\, \mathrm{Cos}[q2[t]]\, \mathrm{Sin}[q2[t]]\, q1'[t]^2 - I3y\, q2''[t],$$
$$I3x\, \mathrm{Cos}[q2[t]]\, q1'[t]\, q2'[t] - I3y\, \mathrm{Cos}[q2[t]]\, q1'[t]\, q2'[t] -$$
$$I3z\, (\mathrm{Cos}[q2[t]]\, q1'[t]\, q2'[t] + \mathrm{Sin}[q2[t]]\, q1''[t])\}$$

generalized inertia forces corresponding to q1: Kin1=

$$-\frac{4}{3} L1^2\, m1\, q1''[t] - m3\, (-2\, L1 - L2\, \mathrm{Cos}[q2[t]] - \mathrm{Cos}[q2[t]]\, q3[t])$$
$$(L2\, \mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] + q3[t]\, \mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] - \mathrm{Cos}[q2[t]]\, q1'[t]\, q3'[t] -$$
$$q1'[t]\, (-L2\, \mathrm{Sin}[q2[t]]\, q2'[t] - q3[t]\, \mathrm{Sin}[q2[t]]\, q2'[t] + \mathrm{Cos}[q2[t]]\, q3'[t]) -$$
$$(2\, L1 + L2\, \mathrm{Cos}[q2[t]] + \mathrm{Cos}[q2[t]]\, q3[t])\, q1''[t]) -$$
$$m2\, (-L2\, \mathrm{Cos}[q2[t]] - 2\, L1\, \mathrm{Cos}[q2[t]]^2 - 2\, L1\, \mathrm{Sin}[q2[t]]^2)\, (2\, L2\, \mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] -$$
$$L2\, \mathrm{Cos}[q2[t]]\, q1''[t] - 2\, L1\, \mathrm{Cos}[q2[t]]^2\, q1''[t] - 2\, L1\, \mathrm{Sin}[q2[t]]^2\, q1''[t]) +$$
$$\mathrm{Cos}[q2[t]]\, (I3y\, \mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] - I3z\, \mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] -$$
$$I3x\, (-\mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] + \mathrm{Cos}[q2[t]]\, q1''[t])) +$$
$$\mathrm{Cos}[q2[t]]\, \left(\frac{1}{3} L2^2\, m2\, \mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] -\right.$$
$$\left.\frac{1}{3} L2^2\, m2\, (-\mathrm{Sin}[q2[t]]\, q1'[t]\, q2'[t] + \mathrm{Cos}[q2[t]]\, q1''[t])\right) +$$
$$\mathrm{Sin}[q2[t]]\, (I3x\, \mathrm{Cos}[q2[t]]\, q1'[t]\, q2'[t] - I3y\, \mathrm{Cos}[q2[t]]\, q1'[t]\, q2'[t] -$$
$$I3z\, (\mathrm{Cos}[q2[t]]\, q1'[t]\, q2'[t] + \mathrm{Sin}[q2[t]]\, q1''[t]))$$

generalized inertia forces corresponding to q2: Kin2=

$$-I3x\, \mathrm{Cos}[q2[t]]\, \mathrm{Sin}[q2[t]]\, q1'[t]^2 + I3z\, \mathrm{Cos}[q2[t]]\, \mathrm{Sin}[q2[t]]\, q1'[t]^2 -$$
$$\frac{1}{3} L2^2\, m2\, \mathrm{Cos}[q2[t]]\, \mathrm{Sin}[q2[t]]\, q1'[t]^2 - I3y\, q2''[t] -$$
$$\frac{1}{3} L2^2\, m2\, q2''[t] - L2\, m2\, (L2\, \mathrm{Cos}[q2[t]]\, \mathrm{Sin}[q2[t]]\, q1'[t]^2 +$$
$$2\, L1\, \mathrm{Cos}[q2[t]]^2\, \mathrm{Sin}[q2[t]]\, q1'[t]^2 + 2\, L1\, \mathrm{Sin}[q2[t]]^3\, q1'[t]^2 + L2\, q2''[t]) -$$
$$m3\, (L2 + q3[t])\, (2\, L1\, \mathrm{Sin}[q2[t]]\, q1'[t]^2 + L2\, \mathrm{Cos}[q2[t]]\, \mathrm{Sin}[q2[t]]\, q1'[t]^2 +$$
$$\mathrm{Cos}[q2[t]]\, q3[t]\, \mathrm{Sin}[q2[t]]\, q1'[t]^2 + 2\, q2'[t]\, q3'[t] + (L2 + q3[t])\, q2''[t])$$

generalized inertia forces corresponding to q3: Kin3=

$$-m3\, (-2\, L1\, \mathrm{Cos}[q2[t]]\, q1'[t]^2 - L2\, \mathrm{Cos}[q2[t]]^2\, q1'[t]^2 -$$
$$\mathrm{Cos}[q2[t]]^2\, q3[t]\, q1'[t]^2 - L2\, q2'[t]^2 - q3[t]\, q2'[t]^2 + q3''[t])$$

First Kane's dynamical equation

$$T01x - \frac{4}{3} L1^2\, m1\, q1''[t] - m3\, (-2\, L1 - L2\, Cos[q2[t]] - Cos[q2[t]]\, q3[t])$$
$$(L2\, Sin[q2[t]]\, q1'[t]\, q2'[t] + q3[t]\, Sin[q2[t]]\, q1'[t]\, q2'[t] - Cos[q2[t]]\, q1'[t]\, q3'[t] -$$
$$q1'[t]\, (-L2\, Sin[q2[t]]\, q2'[t] - q3[t]\, Sin[q2[t]]\, q2'[t] + Cos[q2[t]]\, q3'[t]) -$$
$$(2\, L1 + L2\, Cos[q2[t]] + Cos[q2[t]]\, q3[t])\, q1''[t]) -$$
$$m2\, (-L2\, Cos[q2[t]] - 2\, L1\, Cos[q2[t]]^2 - 2\, L1\, Sin[q2[t]]^2)\, (2\, L2\, Sin[q2[t]]\, q1'[t]\, q2'[t] -$$
$$L2\, Cos[q2[t]]\, q1''[t] - 2\, L1\, Cos[q2[t]]^2\, q1''[t] - 2\, L1\, Sin[q2[t]]^2\, q1''[t]) +$$
$$Cos[q2[t]]\, (I3y\, Sin[q2[t]]\, q1'[t]\, q2'[t] - I3z\, Sin[q2[t]]\, q1'[t]\, q2'[t] -$$
$$I3x\, (-Sin[q2[t]]\, q1'[t]\, q2'[t] + Cos[q2[t]]\, q1''[t])) +$$
$$Cos[q2[t]]\, \left(\frac{1}{3} L2^2\, m2\, Sin[q2[t]]\, q1'[t]\, q2'[t] -\right.$$
$$\left.\frac{1}{3} L2^2\, m2\, (-Sin[q2[t]]\, q1'[t]\, q2'[t] + Cos[q2[t]]\, q1''[t])\right) +$$
$$Sin[q2[t]]\, (I3x\, Cos[q2[t]]\, q1'[t]\, q2'[t] - I3y\, Cos[q2[t]]\, q1'[t]\, q2'[t] -$$
$$I3z\, (Cos[q2[t]]\, q1'[t]\, q2'[t] + Sin[q2[t]]\, q1''[t]))$$

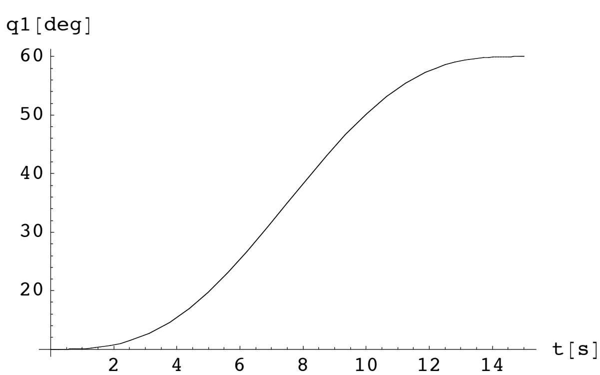
Second Kane's dynamical equation

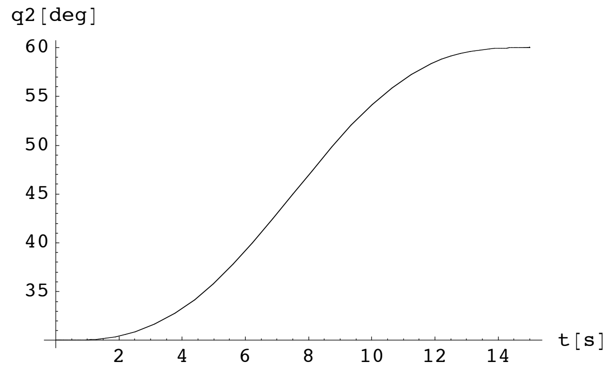
$$T12y - g\, L2\, m2\, Cos[q2[t]] - g\, m3\, Cos[q2[t]]\, (L2 + q3[t]) - I3x\, Cos[q2[t]]\, Sin[q2[t]]\, q1'[t]^2 +$$
$$I3z\, Cos[q2[t]]\, Sin[q2[t]]\, q1'[t]^2 - \frac{1}{3} L2^2\, m2\, Cos[q2[t]]\, Sin[q2[t]]\, q1'[t]^2 -$$
$$I3y\, q2''[t] - \frac{1}{3} L2^2\, m2\, q2''[t] - L2\, m2\, (L2\, Cos[q2[t]]\, Sin[q2[t]]\, q1'[t]^2 +$$
$$2\, L1\, Cos[q2[t]]^2\, Sin[q2[t]]\, q1'[t]^2 + 2\, L1\, Sin[q2[t]]^3\, q1'[t]^2 + L2\, q2''[t]) -$$
$$m3\, (L2 + q3[t])\, (2\, L1\, Sin[q2[t]]\, q1'[t]^2 + L2\, Cos[q2[t]]\, Sin[q2[t]]\, q1'[t]^2 +$$
$$Cos[q2[t]]\, q3[t]\, Sin[q2[t]]\, q1'[t]^2 + 2\, q2'[t]\, q3'[t] + (L2 + q3[t])\, q2''[t])$$

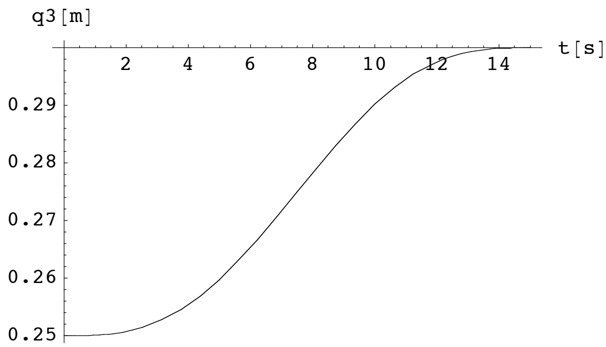
Third Kane's dynamical equation

$$F23z - g\, m3\, Sin[q2[t]] -$$
$$m3\, (-2\, L1\, Cos[q2[t]]\, q1'[t]^2 - L2\, Cos[q2[t]]^2\, q1'[t]^2 - Cos[q2[t]]^2\, q3[t]\, q1'[t]^2 -$$
$$L2\, q2'[t]^2 - q3[t]\, q2'[t]^2 + q3''[t])$$

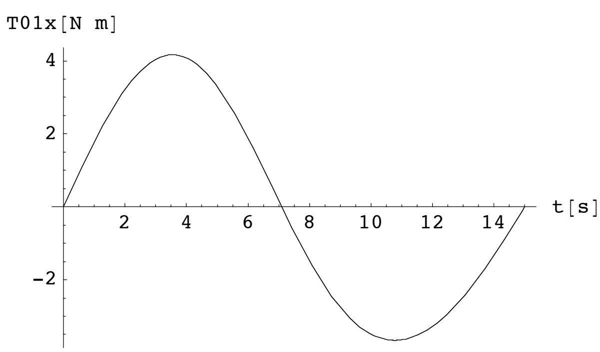
INVERSE DYNAMICS

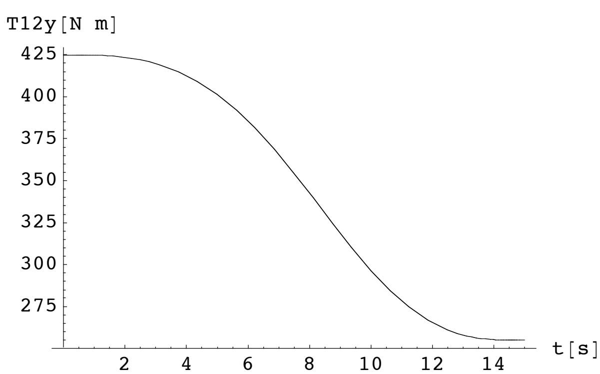


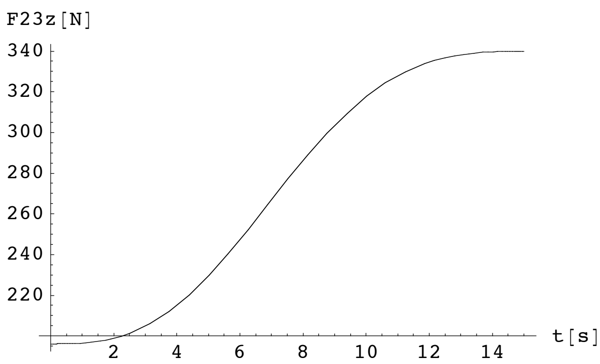

q2[deg]
60
55
50
45
40
35
t[s]
2 4 6 8 10 12 14


q3[m]
t[s]
2 4 6 8 10 12 14
0.29
0.28
0.27
0.26
0.25

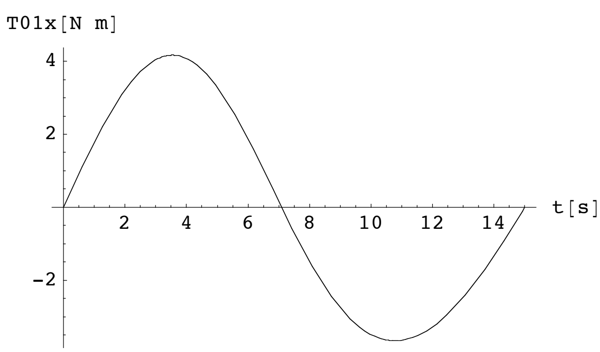
=== Lagrange ===

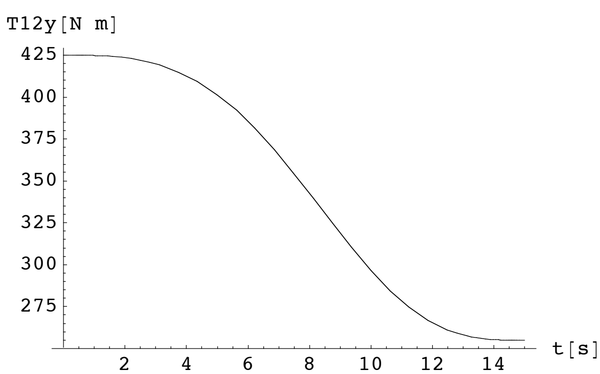

T01x[N m]
4
2
t[s]
2 4 6 8 10 12 14
-2

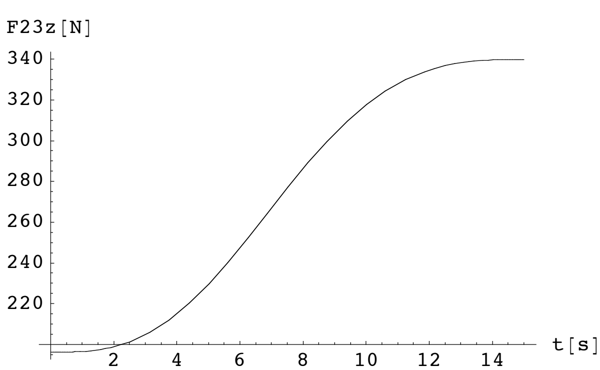

T12y[N m]
425
400
375
350
325
300
275
t[s]
2 4 6 8 10 12 14



=== Kane ===







Verification

using Lagrange for t = 5 s =>

T01x = 3.30361 N m

T12y = 401.36 N m

F23z = 229.545 N

using Kane for t = 5 s =>

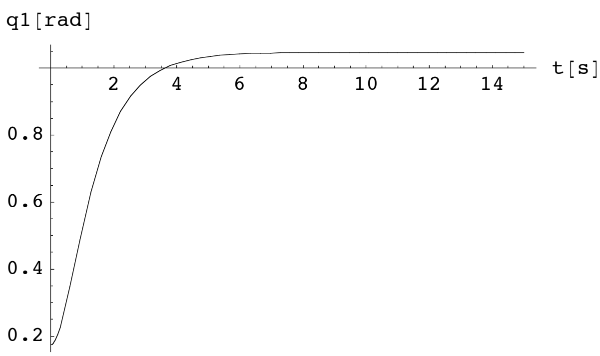
T01x = 3.30361 N m

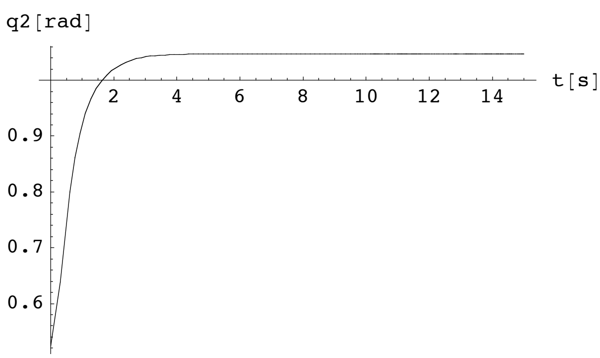
T12y = 401.36 N m

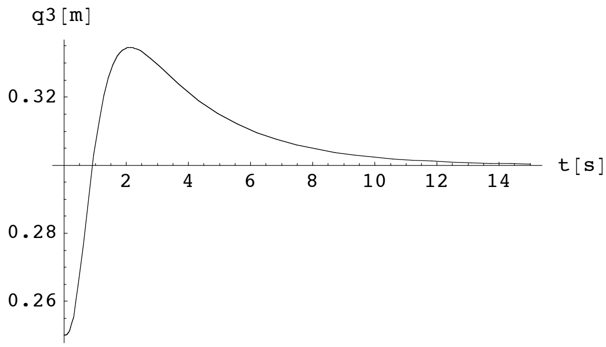
F23z = 229.545 N

DIRECT DYNAMICS

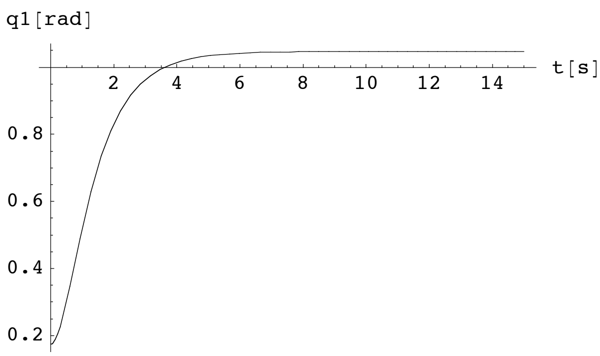
=== Lagrange ===

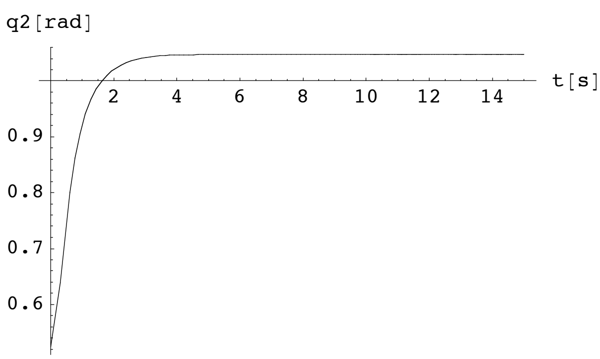


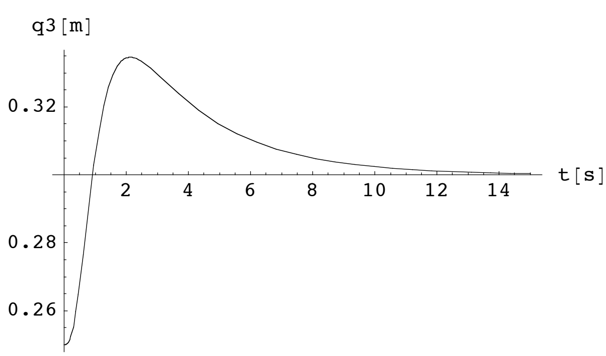




=== Kane ===







Verification

using Lagrange for t = 1.55 s =>

q1 = 0.722135 rad

q2 = 0.995257 rad

q3 = 0.329127 m

using Kane for t = 1.55 s =>

q1 = 0.722135 rad

q2 = 0.995257 rad

q3 = 0.329127 m

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For t = 15 s

the time used to evaluate {q1[t],q2[t],q3[t]} =
{1.0472, 1.0472, 0.300369} with Lagrange is = 0.000138 Second

the time used to evaluate {q1[t],q2[t],q3[t]} =
{1.0472, 1.0472, 0.300369} with Kane is = 0.000084 Second