RRT Robot

Lagrange and Kane Equations of Motion

Figure 1(a) is a schematic representation of an open kinematic chain (robot arm) consisting of three links 1, 2, and 3. Let m_1, m_2, m_3 be the masses of 1, 2, 3, respectively. Link 1 can be rotated at A in a "fixed" reference frame (0) of unit vectors $[\mathbf{\imath}_0, \mathbf{\jmath}_0, \mathbf{k}_0]$ about a vertical axis $\mathbf{\imath}_0$. The unit vector $\mathbf{\imath}_0$ is fixed in 1. The link 1 is connected to link 2 at the pin joint B. The element 2 rotates relative to 1 about an axis fixed in both 1 and 2, passing through B, and perpendicular to the axis of 1. The last link 3 is connected to 2 by means of a slider joint. The mass centers of links 1, 2, and 3 are C_1, C_2, and C_3, respectively. The distances $L_1 = AC_1$, $L_B = AB = 2\,L_1$, and $L_2 = BC_2$ are indicated in Fig. 1. The length of link 1 is $2\,L_1$ and the length of link 2 is $2\,L_2$. The reference frame (1) of the unit vectors $[\mathbf{\imath}_1, \mathbf{\jmath}_1, \mathbf{k}_1]$ is attached to link 1, and the reference frame (2) of the unit vectors $[\mathbf{\imath}_2, \mathbf{\jmath}_2, \mathbf{k}_2]$ is attached to link 2, as shown in Fig. 1(b).

The generalized coordinates (quantities associated with the the instantaneous position of the system) are $q_1(t)$, $q_2(t)$, $q_3(t)$.

The first generalized coordinate q_1 denotes the radian measure of the angle between the axes of (1) and (0). The unit vectors $\mathbf{\imath}_1$, $\mathbf{\jmath}_1$, and $\mathbf{k}_1$ can be expressed as functions of $\mathbf{\imath}_0$, $\mathbf{\jmath}_0$, and $\mathbf{k}_0$

$$\begin{aligned} \mathbf{\imath}_1 &= \mathbf{\imath}_0, \\ \mathbf{\jmath}_1 &= c_1\,\mathbf{\jmath}_0 + s_1\,\mathbf{k}_0, \\ \mathbf{k}_1 &= -s_1\,\mathbf{\jmath}_0 + c_1\,\mathbf{k}_0, \end{aligned} \tag{1}$$

or

$$\begin{bmatrix} \mathbf{\imath}_1 \\ \mathbf{\jmath}_1 \\ \mathbf{k}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix} \begin{bmatrix} \mathbf{\imath}_0 \\ \mathbf{\jmath}_0 \\ \mathbf{k}_0 \end{bmatrix},$$

where $s_1 = \sin q_1$ and $c_1 = \cos q_1$. The transformation matrix from (1) to (0) is

$$R_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix}. \tag{2}$$

The second generalized coordinate designates also a radian measure of the rotation angle between (1) and (2). The unit vectors $\boldsymbol{\imath}_2$, $\boldsymbol{\jmath}_2$ and $\mathbf{k}_2$ can be expressed as

$$\begin{aligned}
\boldsymbol{\imath}_2 &= c_2\boldsymbol{\imath}_1 - s_2\mathbf{k}_1 \\
&= c_2\boldsymbol{\imath}_0 + s_1 s_2\,\boldsymbol{\jmath}_0 - c_1 s_2\,\mathbf{k}_0, \\
\boldsymbol{\jmath}_2 &= \boldsymbol{\jmath}_1, \\
&= c_1\,\boldsymbol{\jmath}_0 + s_1\,\mathbf{k}_0, \\
\mathbf{k}_2 &= s_2\boldsymbol{\imath}_1 + c_2\mathbf{k}_1 \\
&= s_2\boldsymbol{\imath}_0 - c_2 s_1\,\boldsymbol{\jmath}_0 + c_1 c_2\,\mathbf{k}_0, \qquad (3)
\end{aligned}$$

where $s_2 = \sin q_2$ and $c_2 = \cos q_2$. The transformation matrix from (2) to (1) is

$$R_{21} = \begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix}. \qquad (4)$$

The last generalized coordinate q_3 is the distance from C_2 to C_3.

Angular velocities

Next the angular velocity of the links 1, 2, and 3 will be expressed in the fixed reference frame (0). The angular velocity of 1 in (0) is

$$\boldsymbol{\omega}_{10} = \dot{q}_1\,\boldsymbol{\imath}_1. \qquad (5)$$

The angular velocity of the link 2 with respect to (1) is

$$\boldsymbol{\omega}_{21} = \dot{q}_2\,\boldsymbol{\jmath}_2.$$

The angular velocity of the link 2 with respect to the fixed reference frame (0) is

$$\boldsymbol{\omega}_{20} = \boldsymbol{\omega}_{10} + \boldsymbol{\omega}_{21} = \dot{q}_1\,\boldsymbol{\imath}_1 + \dot{q}_2\boldsymbol{\jmath}_2.$$

With $\boldsymbol{\imath}_0 = \boldsymbol{\imath}_1 = c_2\,\boldsymbol{\imath}_2 + s_2\,\mathbf{k}_2$ the angular velocity of the link 2 in the reference frame (0) written in terms of the reference frame (2) is

$$\boldsymbol{\omega}_{20} = \dot{q}_1(c_2\,\boldsymbol{\imath}_2 + s_2\,\mathbf{k}_2) + \dot{q}_2\,\boldsymbol{\jmath}_2 = \dot{q}_1\,c_2\,\boldsymbol{\imath}_2 + \dot{q}_2\,\boldsymbol{\jmath}_2 + \dot{q}_1\,s_2\,\mathbf{k}_2. \qquad (6)$$

The link 3 has the same rotational motion as link 2, i.e., $\boldsymbol{\omega}_{30} = \boldsymbol{\omega}_{20}$.

Angular accelerations

The angular acceleration of the link 1 in the reference frame (0) is

$$\boldsymbol{\alpha}_{10} = \ddot{q}_1\,\mathbf{ı}_1. \tag{7}$$

The angular acceleration of the link 2 with respect to the reference frame (0) is

$$\boldsymbol{\alpha}_{20} \;=\; \frac{d}{dt}\boldsymbol{\omega}_{20} = \frac{{}^{(2)}d}{dt}\boldsymbol{\omega}_{20} + \boldsymbol{\omega}_{20}\times\boldsymbol{\omega}_{20} = \frac{{}^{(2)}d}{dt}\boldsymbol{\omega}_{20}.$$

where $\frac{{}^{(2)}d}{dt}$ represents the derivative with respect to time in reference frame (2), $[\mathbf{ı}_2, \mathbf{ȷ}_2, \mathbf{k}_2]$. The angular acceleration of the link 2 is

$$\begin{aligned}&\boldsymbol{\alpha}_{20} = \frac{{}^{(2)}d}{dt}(\dot{q}_1\,c_2\,\mathbf{ı}_2 + \dot{q}_2\,\mathbf{ȷ}_2 + \dot{q}_1\,s_2\,\mathbf{k}_2) = \\ &(\ddot{q}_1\,c_2 - \dot{q}_1\,\dot{q}_2\,s_2)\,\mathbf{ı}_2 + \ddot{q}_2\,\mathbf{ȷ}_2 + (\ddot{q}_1\,s_2 + \dot{q}_1\,\dot{q}_2\,c_2)\,\mathbf{k}_2.\end{aligned} \tag{8}$$

The link 3 has the same angular acceleration as link 2, i.e., $\boldsymbol{\alpha}_{30} = \boldsymbol{\alpha}_{20}$.

Linear velocities

The position vector of C_1, the mass center of link 1, is

$$\mathbf{r}_{C_1} = L_1\mathbf{k}_1,$$

and the velocity of C_1 in (0) is

$$\begin{aligned}\mathbf{v}_{C_1} &= \frac{d}{dt}\mathbf{r}_{C_1} = \frac{{}^{(1)}d}{dt}\mathbf{r}_{C_1} + \boldsymbol{\omega}_{10}\times\mathbf{r}_{C_1} \\ &= \mathbf{0} + \begin{vmatrix}\mathbf{ı}_1 & \mathbf{ȷ}_1 & \mathbf{k}_1\\ \dot{q}_1 & 0 & 0\\ 0 & 0 & L_1\end{vmatrix} = -\dot{q}_1L_1\mathbf{ȷ}_1.\end{aligned} \tag{9}$$

The position vector of C_2, the mass center of link 2, is

$$\begin{aligned}\mathbf{r}_{C_2} &= L_B\mathbf{k}_1 + L_2\mathbf{k}_2 = L_B(-s_2\mathbf{ı}_2 + c_2\mathbf{k}_2) + L_2\mathbf{k}_2\\ &= -L_Bs_2\mathbf{ı}_2 + (L_Bc_2 + L_2)\mathbf{k}_2,\end{aligned}$$

where $L_B = 2\,L_1$. The velocity of C_2 in (0) is

$$\begin{aligned}
\mathbf{v}_{C_2} &= \frac{d}{dt}\mathbf{r}_{C_2} = \frac{{}^{(2)}d}{dt}\mathbf{r}_{C_2} + \boldsymbol{\omega}_{20} \times \mathbf{r}_{C_2} \\
&= -L_B c_1 \dot{q}_2 \boldsymbol{\imath}_2 - L_B c_2 \dot{q}_2 \mathbf{k}_2 + \begin{vmatrix} \boldsymbol{\imath}_2 & \boldsymbol{\jmath}_2 & \mathbf{k}_2 \\ \dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\ -L_B s_2 & 0 & L_B c_2 + L_2 \end{vmatrix} \\
&= L_2 \dot{q}_2 \boldsymbol{\imath}_2 - (L_B + L_2 c_2)\dot{q}_1 \boldsymbol{\jmath}_2. \qquad (10)
\end{aligned}$$

The position vector of C_3 with respect to reference frame (0) is

$$\begin{aligned}
\mathbf{r}_{C_3} &= \mathbf{r}_{C_2} + q_3 \mathbf{k}_2 \\
&= -L_B s_2 \boldsymbol{\imath}_2 + (L_B c_2 + L_2 + q_3)\mathbf{k}_2,
\end{aligned}$$

and the velocity of this mass center in (0) is

$$\begin{aligned}
\mathbf{v}_{C_3} &= \frac{d}{dt}\mathbf{r}_{C_3} = \frac{{}^{(2)}d}{dt}\mathbf{r}_{C_3} + \boldsymbol{\omega}_{20} \times \mathbf{r}_{C_3} \\
&= -L_B c_2 \dot{q}_2 \boldsymbol{\imath}_2 - (L_B c_2 \dot{q}_2 + \dot{q}_3)\mathbf{k}_2 + \begin{vmatrix} \boldsymbol{\imath}_2 & \boldsymbol{\jmath}_2 & \mathbf{k}_2 \\ \dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\ -L_B s_2 & 0 & L_B c_2 + L_2 + q_2 \end{vmatrix} \\
&= (L_2 + q_3)\dot{q}_2 \boldsymbol{\imath}_2 - (L_B + L_2 c_2 + c_2 q_2)\dot{q}_1 \boldsymbol{\jmath}_2 + \dot{q}_3 \mathbf{k}_2. \qquad (11)
\end{aligned}$$

Linear accelerations

The acceleration of C_1 is

$$\begin{aligned}
\mathbf{a}_{C_1} &= \frac{d}{dt}\mathbf{v}_{C_1} = \frac{{}^{(1)}d}{d\,t}\mathbf{v}_{C_1} + \boldsymbol{\omega}_{10} \times \mathbf{v}_{C_1} \\
&= -L_1 \ddot{q}_1 \boldsymbol{\jmath} + \begin{vmatrix} \boldsymbol{\imath}_1 & \boldsymbol{\jmath}_1 & \mathbf{k}_1 \\ \dot{q}_1 & 0 & 0 \\ 0 & -L_1 \dot{q}_1 & 0 \end{vmatrix} \\
&= -L_1 \ddot{q}_1 \boldsymbol{\jmath}_1 - L_1 \dot{q}_1^2 \mathbf{k}_1. \qquad (12)
\end{aligned}$$

The linear acceleration of the mass center C_2 is

$$\mathbf{a}_{C_2} = \frac{d}{dt}\mathbf{v}_{C_2} = \frac{{}^{(2)}d}{d\,t}\mathbf{v}_{C_2} + \boldsymbol{\omega}_{20} \times \mathbf{v}_{C_2}. \qquad (13)$$

The linear acceleration of C_2 is symbolically calculated in the program *RRTrobot.nb*.

The acceleration of C_3 is

$$\mathbf{a}_{C_3} = \frac{d}{dt}\mathbf{v}_{C_3} = \frac{{}^{(2)}d}{d\,t}\mathbf{v}_{C_3} + \boldsymbol{\omega}_{20} \times \mathbf{v}_{C_3}. \tag{14}$$

The linear acceleration of C_3 is symbolically calculated in the program *RRTrobot.nb*.

Generalized forces

Remark: If a set of contact and/or body forces acting on a rigid body is equivalent to a couple of torque $\mathbf{T}$ together with force $\mathbf{R}$ applied at a point P of the rigid body, then the contribution of this set of forces to the generalized force, Q_r, is given by

$$Q_r = \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_r} \cdot \mathbf{T} + \frac{\partial \mathbf{v}_P}{\partial \dot{q}_r} \cdot \mathbf{R}, \quad r = 1, 2, ...,$$

where $\boldsymbol{\omega}$ is the angular velocity of the rigid body in (0), $\mathbf{v}_P$ is the velocity of P in (0), and r represents the generalized coordinates.

In the case of the robotic arm, there are two kinds of forces that contribute to the generalized forces Q_1, Q_2, and Q_3 namely, contact forces applied in order to drive the links 1, 2, and 3, and gravitational forces exerted on 1, 2, and 3 by the Earth.

The set of contact forces transmitted from 0 to 1 can be replaced with a couple of torque $\mathbf{T}_{01}$ applied to 1 at A.

Similarly, the set of contact forces transmitted from 1 to 2 can be replaced with a couple of torque $\mathbf{T}_{12}$ applied to 2 at B. The law of action and reaction then guarantees that the set of contact forces transmitted from 1 to 2 is equivalent to a couple of torque $-\mathbf{T}_{12}$ to 1 at B.

Next, the set of contact forces exerted by link 2 on link 3 can be replaced with a force $\mathbf{F}_{23}$ applied to 3 at C_3. The law of action and reaction guarantees that the set of contact forces transmitted from 3 to 2 is equivalent to a force $-\mathbf{F}_{23}$ applied to 2 at C_{32}.

The point C_{32} ($C_{32} \in$ link2) instantaneously coincides with C_3, ($C_3 \in$ link3).

The expressions $\mathbf{T}_{01}$, $\mathbf{T}_{12}$, and $\mathbf{F}_{23}$ are

$$\mathbf{T}_{01} = T_{01x}\mathbf{i}_1 + T_{01y}\mathbf{j}_1 + T_{01z}\mathbf{k}_1, \quad \mathbf{T}_{12} = T_{12x}\mathbf{i}_2 + T_{12y}\mathbf{j}_2 + T_{12z}\mathbf{k}_2, \quad \text{and}$$
$$\mathbf{F}_{23} = F_{23x}\mathbf{i}_2 + F_{23y}\mathbf{j}_2 + F_{23z}\mathbf{k}_2.$$

The external gravitational forces exerted on the links 1, 2, and 3 by the Earth, can be denoted by $\mathbf{G}_1$, $\mathbf{G}_2$, and $\mathbf{G}_3$ respectively, and can be expressed as

$$\mathbf{G}_1 \;=\; -m_1\, g\mathbf{i}_1,$$

$$\begin{aligned}\mathbf{G}_2 &= -m_2\, g\, \boldsymbol{\imath}_1 = -m_2\, g\, (c_2\, \boldsymbol{\imath}_2 + s_2\, \mathbf{k}_2),\\ \mathbf{G}_3 &= -m_3\, g\, \boldsymbol{\imath}_1 = -m_3\, g\, (c_2\, \boldsymbol{\imath}_2 + s_2\, \mathbf{k}_2).\end{aligned}$$

The reason for replacing $\boldsymbol{\imath}_1$ with $c_2\, \boldsymbol{\imath}_2 + s_2\, \mathbf{k}_2$ in connection with the forces $\mathbf{G}_2$ and $\mathbf{G}_3$ is that they are soon to be dot-multiplied with $\dfrac{\partial \mathbf{v}_{C_2}}{\partial \dot{q}_r}$ and $\dfrac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r}$ which have been expressed in terms of $\boldsymbol{\imath}_2, \boldsymbol{\jmath}_2,$ and $\mathbf{k}_2$.

One can express $(Q_r)_1$, the contribution to the generalized active force Q_r of all the forces and torques acting on the particles of the link 1, as

$$(Q_r)_1 = \frac{\partial \boldsymbol{\omega}_{10}}{\partial \dot{q}_r} \cdot (\mathbf{T}_{01} - \mathbf{T}_{12}) + \frac{\partial \mathbf{v}_{C_1}}{\partial \dot{q}_r} \cdot \mathbf{G}_1, \;\; r = 1, 2, 3.$$

The contribution $(Q_r)_2$ to the generalized active force of all the forces and torques acting on the link 2 is

$$(Q_r)_2 = \frac{\partial \boldsymbol{\omega}_{20}}{\partial \dot{q}_r} \cdot \mathbf{T}_{12} + \frac{\partial \mathbf{v}_{C_2}}{\partial \dot{q}_r} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{v}_{C_{32}}}{\partial \dot{q}_r} \cdot (-\mathbf{F}_{23})\,, \;\; r = 1, 2, 3,$$

where

$$\mathbf{v}_{C_{32}} \;=\; \mathbf{v}_{C_2} + \boldsymbol{\omega}_{20} \times \mathbf{r}_{C_2 C_3} = \mathbf{v}_{C_2} + \boldsymbol{\omega}_{20} \times q_3 \mathbf{k}_2.$$

The contribution $(Q_r)_3$, to the generalized active force of all the forces and torques acting on the link 3 is

$$(Q_r)_3 = \frac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r} \cdot \mathbf{G}_3 + \frac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r} \cdot \mathbf{F}_{23}, \;\; r = 1, 2, 3.$$

The generalized active force Q_r of all the forces and torques acting on the links 1, 2, and 3 are

$$Q_r = (Q_r)_1 + (Q_r)_2 + (Q_r)_3. \;\; r = 1, 2, 3,$$

The generalized forces Q_r, $r = 1, 2, 3$ are symbolically calculated in the program *RRTrobot.nb* and have the values

$$\begin{aligned}&Q_1 = T_{01x},\\ &Q_2 = T_{12y} - g\, m_2\, L_2\, c_2 - g\, m_3\, c_2\, (L_2 + q_3),\\ &Q_3 = F_{23z} - g\, m_3\, s_2.\end{aligned} \tag{15}$$

Lagrange's Equations of Motion

Kinetic energy

The total kinetic energy of the robot arm in the reference frame (0) is

$$T = \sum_{i=1}^{3} T_i .$$

The kinetic energy of the link i, $i = 1, 2, 3$, is

$$T_i = \frac{1}{2} m_i \mathbf{v}_{C_i} \cdot \mathbf{v}_{C_i} + \frac{1}{2} \boldsymbol{\omega}_{i0} \cdot (\bar{I}_i \cdot \boldsymbol{\omega}_{i0}).$$

Remark: The kinetic energy for a rigid body is

$$T_{\text{rigid body}} = \frac{1}{2} m \mathbf{v}_C \cdot \mathbf{v}_C + \frac{1}{2} \boldsymbol{\omega} \cdot (\bar{I}_C \cdot \boldsymbol{\omega}),$$

where m is the mass of the rigid body, $\mathbf{v}_C$ is the velocity of the mass center of the rigid body in (0), $\boldsymbol{\omega} = \omega_x \boldsymbol{\imath} + \omega_y \boldsymbol{\jmath} + \omega_z \mathbf{k}$ is the angular velocity of the rigid body in (0), and $\bar{I} = (I_x \boldsymbol{\imath})\boldsymbol{\imath} + (I_y \boldsymbol{\jmath})\boldsymbol{\jmath} + (I_z \mathbf{k})\mathbf{k}$ is the central inertia dyadic of the rigid body. The central principal axes of the rigid body are parallel to $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ and the associated moments of inertia have the values I_x, I_y, I_z, respectively. The inertia matrix associated with $\bar{I}$ is

$$\bar{I} \rightarrow \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} .$$

The dot product of the vector $\boldsymbol{\omega}$ with the dyadic $\bar{I}$ is

$$\boldsymbol{\omega} \cdot \bar{I} = \bar{I} \cdot \boldsymbol{\omega} \;=\; \omega_x I_x \, \boldsymbol{\imath} + \omega_y I_y \, \boldsymbol{\jmath} + \omega_z I_z \, \mathbf{k}.$$

The central moments of inertia of links 1 and 2 are calculated using Fig. 2.

The central principal axes of 1 are parallel to $\boldsymbol{\imath}_1$, $\boldsymbol{\jmath}_1$, $\mathbf{k}_1$ and the associated moments of inertia have the values I_{1x}, I_{1y}, I_{1z}, respectively. The inertia matrix associated with link 1 is

$$\bar{I}_1 \rightarrow \begin{bmatrix} I_{1x} & 0 & 0 \\ 0 & I_{1y} & 0 \\ 0 & 0 & I_{1z} \end{bmatrix} = \begin{bmatrix} \dfrac{m_1 (2\,L_1)^2}{12} & 0 & 0 \\ 0 & \dfrac{m_1 (2\,L_1)^2}{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{m_1 L_1^2}{3} & 0 & 0 \\ 0 & \dfrac{m_1 L_1^2}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The central principal axes of 2 and 3 are parallel to $\boldsymbol{\imath}_2$, $\boldsymbol{\jmath}_2$, $\mathbf{k}_2$ and the associated moments of inertia have values I_{2x}, I_{2y}, I_{2z}, and I_{3x}, I_{3y}, I_{3z} respectively. The inertia matrix associated with link 2 is

$$\bar{I}_2 \rightarrow \begin{bmatrix} I_{2x} & 0 & 0 \\ 0 & I_{2y} & 0 \\ 0 & 0 & I_{2z} \end{bmatrix} = \begin{bmatrix} \dfrac{m_2(2\,L_2)^2}{12} & 0 & 0 \\ 0 & \dfrac{m_2(2\,L_2)^2}{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{m_2 L_2^2}{3} & 0 & 0 \\ 0 & \dfrac{m_2 L_2^2}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The inertia matrix associated with the slider 3 is

$$\bar{I}_3 \rightarrow \begin{bmatrix} I_{3x} & 0 & 0 \\ 0 & I_{3y} & 0 \\ 0 & 0 & I_{3z} \end{bmatrix}$$

The kinetic energy of link 1 is

$$T_1 = \frac{1}{2} m_1 \mathbf{v}_{C_1} \cdot \mathbf{v}_{C_1} + \frac{1}{2} \boldsymbol{\omega}_{10} \cdot (\bar{I}_1 \cdot \boldsymbol{\omega}_{10}) = \frac{1}{2} m_1 L_1 \dot{q}_1^2 + \frac{1}{6} m_1 L_1 \dot{q}_1^2 = \frac{2}{3} m_1 L_1 \dot{q}_1^2.$$

The kinetic energy of bar 2 is

$$\begin{aligned} T_2 \;\; = \;\; & \frac{1}{2} m_2 \mathbf{v}_{C_2} \cdot \mathbf{v}_{C_2} + \frac{1}{2} \boldsymbol{\omega}_{20} \cdot (\bar{I}_2 \cdot \boldsymbol{\omega}_{20}) = \\ & \frac{m_2}{3} \left[\left(6\,L_1^2 + L_2^2 + 6\,L_1\,L_2\,c_2 + L_2^2\,\cos 2q_2 \right) \dot{q}_1^2 + 2\,L_2^2\,\dot{q}_2^2 \right]. \end{aligned}$$

The kinetic energy of link 3 is

$$\begin{aligned} & T_3 = \frac{1}{2} m_3 \mathbf{v}_{C_3} \cdot \mathbf{v}_{C_3} + \frac{1}{2} \boldsymbol{\omega}_{20} \cdot (\bar{I}_3 \cdot \boldsymbol{\omega}_{20}) = \\ & \frac{1}{2} \{ I_{3x}\,c_2^2\,\dot{q}_1^2 + I_{3z}\,s_2^2\,\dot{q}_1^2 + I_{3y}\,\dot{q}_2^2 + \\ & m_3 \left[(2\,L_1 + L_2\,c_2 + c_2\,q_3)^2 \; \dot{q}_1^2 + (L_2 + q_3)^2 \; \dot{q}_2^2 + \dot{q}_3^2 \right] \}. \end{aligned}$$

The total kinetic energy of the robot arm is

$$T = T_1 + T_2 + T_3,$$

and is symbolically calculated in the program *RRTrobot.nb*.

The left hand sides of Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r}, \;\; r = 1,\, 2,\, 3,$$

and are symbolically symbolically calculated in the program *RRTrobot.nb*.

To arrive at the dynamical equations governing the robot arm, all that remains to be done is to substitute into Lagrange's equations, namely,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r, \;\; r = 1,\, 2,\, 3.$$

The Lagrange's equations are symbolically calculated in the program *RRTrobot.nb*.

Kane's Dynamical Equations

Generalized inertia forces

To explain what the *generalized inertia forces* are, a system $\{S\}$ formed by ν particles $P_1, ..., P_\nu$ and having masses $m_1, ..., m_\nu$ is considered. Suppose that n generalized speeds u_r, $r = 1, ..., n$ have been introduced. (For the robotic arm $u_r = \dot{q}_r$, $r = 1, ..., n$.) Let $\mathbf{v}_{P_j}$ and $\mathbf{a}_{P_j}$ denote, respectively, the velocity of P_j and the acceleration of P_j in a reference frame (0).

Define $\mathbf{F}_{\text{in}\, j}$, called the *inertia force* for P_j, as

$$\mathbf{F}_{\text{in}\, j} = -m_j\, \mathbf{a}_{P_j}. \tag{16}$$

The quantities $K_1^*, ..., K_n^*$, defined as

$$K_r^* = \sum_{j=1}^{\nu} \frac{\partial \mathbf{v}_{P_j}}{\partial u_r} \cdot \mathbf{a}_{P_j}, \;\; r = 1, ..., n, \tag{17}$$

are called *generalized inertia forces* for $\{S\}$.
The contribution to F_r^*, made by the particles of a rigid body RB belonging to $\{S\}$, are

$$(K_r^*)_R = \frac{\partial \mathbf{v}_C}{\partial u_r} \cdot \mathbf{F}_{\text{in}} + \frac{\partial \boldsymbol{\omega}}{\partial u_r} \cdot \mathbf{M}_{\text{in}}, \;\; r = 1, ..., n, \tag{18}$$

where $\mathbf{v}_C$ is the velocity of the center of gravity of RB in (0), and $\boldsymbol{\omega} = \omega_x \boldsymbol{\imath} + \omega_y \boldsymbol{\jmath} + \omega_z \mathbf{k}$ is the angular velocity of RB in (0).
The inertia force for the rigid body RB is

$$\mathbf{F}_{\text{in}} = -m\, \mathbf{a}_C, \tag{19}$$

where m is the mass of RB, and $\mathbf{a}_C$ is the acceleration of the mass center of RB in the fixed reference frame. The inertia moment $\mathbf{M}_{\text{in}}$ for RB is

$$\mathbf{M}_{\text{in}} \;\; = \;\; -\boldsymbol{\alpha} \cdot \bar{I} - \boldsymbol{\omega} \times (\bar{I} \cdot \boldsymbol{\omega}), \tag{20}$$

where $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \alpha_x \boldsymbol{\imath} + \alpha_y \boldsymbol{\jmath} + \alpha_z \mathbf{k}$ is the angular acceleration of RB in (0), and $\bar{I} = (I_x \boldsymbol{\imath})\boldsymbol{\imath} + (I_y \boldsymbol{\jmath})\boldsymbol{\jmath} + (I_z \mathbf{k})\mathbf{k}$ is the central inertia dyadic of RB. The central principal axes of RB are parallel to $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ and the associated moments of inertia have the values I_x, I_y, I_z, respectively. The inertia matrix associated with $\bar{I}$ is

$$\bar{I} \rightarrow \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}. \tag{21}$$

The dot product of the vector $\boldsymbol{\alpha}$ with the dyadic $\bar{I}$ is

$$\boldsymbol{\alpha} \cdot \bar{I} = \bar{I} \cdot \boldsymbol{\alpha} \;\;=\;\; \alpha_x I_x \,\mathbf{\imath} + \alpha_y I_y \,\mathbf{\jmath} + \alpha_z I_z \,\mathbf{k}, \tag{22}$$

and the cross product between a vector and a dyadic is

$$\begin{aligned} \boldsymbol{\omega} \times (\bar{I} \cdot \boldsymbol{\omega}) = & \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ \omega_x I_x & \omega_y I_y & \omega_z I_z \end{vmatrix} = \\ & -\omega_y \omega_z (I_y - I_z)\mathbf{\imath} - \omega_z \omega_x (I_z - I_x)\mathbf{\jmath} - \omega_x \omega_y (I_x - I_y)\mathbf{k}. \end{aligned} \tag{23}$$

The inertia moment of 1 in (0) can be written as

$$\mathbf{M}_{\text{in }1} = -\boldsymbol{\alpha}_{10} \cdot \bar{I}_1 - \boldsymbol{\omega}_{10} \times (\bar{I}_1 \cdot \boldsymbol{\omega}_{10}) = -I_{1x} \ddot{q}_1 \mathbf{\imath}_1. \tag{24}$$

The inertia moment of 2 in (0) is

$$\mathbf{M}_{\text{in }2} = -\boldsymbol{\alpha}_{20} \cdot \bar{I}_2 - \boldsymbol{\omega}_{20} \times (\bar{I}_2 \cdot \boldsymbol{\omega}_{20}).$$

Similarly the inertia moment of 3 in (0) is

$$\mathbf{M}_{\text{in }3} = \boldsymbol{\alpha}_{20} \cdot \bar{I}_3 - \boldsymbol{\omega}_{20} \times (\bar{I}_3 \cdot \boldsymbol{\omega}_{20}). \tag{25}$$

The inertia force for link $j = 1, 2, 3$ is

$$\mathbf{F}_{\text{in }j} = -m_j \,\mathbf{a}_{C_j}, \tag{26}$$

The contribution of link $j = 1, 2, 3$ to the generalized inertia force K_r^* is

$$(K_r^*)_j = \frac{\partial \mathbf{v}_{C_j}}{\partial u_r} \cdot \mathbf{F}_{\text{in }j} + \frac{\partial \boldsymbol{\omega}_{j0}}{\partial u_r} \cdot \mathbf{M}_{\text{in }j}, \;\; r = 1, 2, 3. \tag{27}$$

The three generalized inertia forces are computed with

$$\begin{aligned} K_r^* &= \sum_{j=1}^{3} (K_r^*)_j = \\ & \sum_{j=1}^{3} \left(\frac{\partial \mathbf{v}_{C_j}}{\partial u_r} \cdot \mathbf{F}_{\text{in } j} + \frac{\partial \boldsymbol{\omega}_{j0}}{\partial u_r} \cdot \mathbf{M}_{\text{in } j} \right), \;\; r = 1,\, 2,\, 3, \end{aligned}$$

or

$$\begin{aligned} K_r^* &= \frac{\partial \mathbf{v}_{C_1}}{\partial \dot{q}_r} \cdot (-m_1\, \mathbf{a}_{C_1}) + \frac{\partial \boldsymbol{\omega}_{10}}{\partial \dot{q}_r} \cdot \mathbf{M}_{\text{in } 1} + \frac{\partial \mathbf{v}_{C_2}}{\partial \dot{q}_r} \cdot (-m_2\, \mathbf{a}_{C_2}) + \frac{\partial \boldsymbol{\omega}_{20}}{\partial \dot{q}_r} \cdot \mathbf{M}_{\text{in } 2} + \\ & \frac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r} \cdot (-m_3\, \mathbf{a}_{C_3}) + \frac{\partial \boldsymbol{\omega}_{30}}{\partial \dot{q}_r} \cdot \mathbf{M}_{\text{in } 3}, \;\; r = 1,\, 2,\, 3. \end{aligned} \tag{28}$$

To arrive at the dynamical equations governing the robot arm, all that remains to be done is to substitute into Kane's dynamical equations, namely,

$$K_r^* + Q_r = 0, \;\; r = 1,\, 2,\, 3. \tag{29}$$

Numerical Simulations

The robot arm is characterized by the following geometry: $L_1 = 0.4$ m, $L_2 = 0.7$ m, $I_{3x} = 5$ kg·m^2, $I_{3y} = 4$ kg·m^2, $I_{3z} = 1$ kg·m^2. The masses of the rigid bodies are $m_1 = 90$ kg, $m_2 = 60$ kg, $m_3 = 40$ kg, and the gravitational acceleration is $g = 9.81$ m/s^2.
The initial conditions, at $t = 0$ s, are $q_1(0) = \pi/18$ rad, $q_2(0) = \pi/6$ rad, $q_3(0) = 0.25$ m, and $\dot{q}_1(0) = \dot{q}_2(0) = \dot{q}_3(0) = 0$.
The robot arm can be brought from an initial state of rest in reference frame (0) to a final state of rest in (0) in such a way that q_1, q_2, and q_3 have specified values q_{1f}, q_{2f}, and q_{3f}, respectively ($q_{1f} = \pi/3$ rad, $q_{2f} = \pi/3$ rad, and $q_{3f} = 0.3$ m).

Inverse dynamics

A desired motion of the robot arm has been specified for a time interval $0 \le t \le T_p = 15$ s. The generalized coordinates can be established explicitly

$$q_r(t) = q_r(0) + \frac{q_r(T_p) - q_r(0)}{T_p}\left[t - \frac{T_p}{2\,\pi}\sin\left(\frac{2\,\pi\,t}{T_p}\right)\right], \quad r = 1, 2, 3, \tag{30}$$

with $q_r(T_p) = q_{rf}$.

Find $T_{01}(t)$, $T_{12}(t)$, and $F_{23}(t)$ for $0 \le t \le T_p = 15$ s.

Direct dynamics

The following feedback control laws are used

$$\begin{aligned}
T_{01x} &= -\beta_{01}\dot{q}_1 - \gamma_{01}(q_1 - q_{1f}),\\
T_{12y} &= -\beta_{12}\dot{q}_2 - \gamma_{12}(q_2 - q_{2f}) + g\,m_2\,L_2\,c_2 + g\,m_3\,c_2\,(L_2 + q_3),\\
F_{23z} &= -\beta_{23}\dot{q}_3 - \gamma_{23}(q_3 - q_{3f}) + g\,m_3\,s_2.
\end{aligned} \tag{31}$$

The constant gains are: $\beta_{01} = 450$ N·m·s/rad, $\gamma_{01} = 300$ N·m/rad, $\beta_{12} = 200$ N·m·s/rad, $\gamma_{12} = 300$ N·m/rad, $\beta_{23} = 150$ N·s/m, and $\gamma_{23} = 50$ N/m.

Find $q_1(t)$, $q_2(t)$, and $q_3(t)$ for $0 \le t \le 15$ s.