RRT Robot

Lagrange and Kane Equations of Motion

Figure 1(a) is a schematic representation of an open kinematic chain (robot arm) consisting of three links 1, 2, and 3. Let \( m_1, m_2, m_3 \) be the masses of 1, 2, 3, respectively. Link 1 can be rotated at \( A \) in a “fixed” reference frame \((0)\) of unit vectors \([\hat{i}_0, \hat{j}_0, \hat{k}_0]\) about a vertical axis \(\hat{i}_0\). The unit vector \(\hat{i}_0\) is fixed in 1. The link 1 is connected to link 2 at the pin joint \(B\). The element 2 rotates relative to 1 about an axis fixed in both 1 and 2, passing through \(B\), and perpendicular to the axis of 1. The last link 3 is connected to 2 by means of a slider joint. The mass centers of links 1, 2, and 3 are \(C_1, C_2, \) and \(C_3\), respectively. The distances \(L_1 = AC_1, L_B = AB = 2L_1, \) and \(L_2 = BC_2\) are indicated in Fig. 1. The length of link 1 is \(2L_1\) and the length of link 2 is \(2L_2\). The reference frame \((1)\) of the unit vectors \([\hat{i}_1, \hat{j}_1, \hat{k}_1]\) is attached to link 1, and the reference frame \((2)\) of the unit vectors \([\hat{i}_2, \hat{j}_2, \hat{k}_2]\) is attached to link 2, as shown in Fig. 1(b).

The generalized coordinates (quantities associated with the instantaneous position of the system) are \(q_1(t), q_2(t), q_3(t)\).

The first generalized coordinate \(q_1\) denotes the radian measure of the angle between the axes of \((1)\) and \((0)\). The unit vectors \(\hat{i}_1, \hat{j}_1, \) and \(\hat{k}_1\) can be expressed as functions of \(\hat{i}_0, \hat{j}_0, \) and \(\hat{k}_0\)

\[
\begin{align*}
\hat{i}_1 &= \hat{i}_0, \\
\hat{j}_1 &= c_1 \hat{j}_0 + s_1 \hat{k}_0, \\
\hat{k}_1 &= -s_1 \hat{j}_0 + c_1 \hat{k}_0,
\end{align*}
\]

or

\[
\begin{bmatrix}
\hat{i}_1 \\
\hat{j}_1 \\
\hat{k}_1
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & c_1 & s_1 \\
0 & -s_1 & c_1
\end{bmatrix}
\begin{bmatrix}
\hat{i}_0 \\
\hat{j}_0 \\
\hat{k}_0
\end{bmatrix},
\]

where \(s_1 = \sin q_1\) and \(c_1 = \cos q_1\). The transformation matrix from \((1)\) to \((0)\) is

\[
R_{10} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & c_1 & s_1 \\
0 & -s_1 & c_1
\end{bmatrix}.
\]
The second generalized coordinate designates also a radian measure of the rotation angle between (1) and (2). The unit vectors $\mathbf{i}_2$, $\mathbf{j}_2$ and $\mathbf{k}_2$ can be expressed as

\[
\begin{align*}
\mathbf{i}_2 &= c_2 \mathbf{i}_1 - s_2 \mathbf{k}_1 \\
&= c_2 \mathbf{i}_0 + s_1 s_2 \mathbf{j}_0 - c_1 s_2 \mathbf{k}_0, \\
\mathbf{j}_2 &= \mathbf{j}_1, \\
&= c_1 \mathbf{j}_0 + s_1 \mathbf{k}_0, \\
\mathbf{k}_2 &= s_2 \mathbf{i}_1 + c_2 \mathbf{k}_1 \\
&= s_2 \mathbf{i}_0 - c_2 s_1 \mathbf{j}_0 + c_1 c_2 \mathbf{k}_0,
\end{align*}
\]

where $s_2 = \sin q_2$ and $c_2 = \cos q_2$. The transformation matrix from (2) to (1) is

\[
R_{21} = \begin{bmatrix}
c_2 & 0 & -s_2 \\
0 & 1 & 0 \\
s_2 & 0 & c_2
\end{bmatrix}.
\]

The last generalized coordinate $q_3$ is the distance from $C_2$ to $C_3$.

**Angular velocities**

Next the angular velocity of the links 1, 2, and 3 will be expressed in the fixed reference frame (0). The angular velocity of 1 in (0) is

\[
\mathbf{\omega}_{10} = \dot{q}_1 \mathbf{i}_1.
\]

The angular velocity of the link 2 with respect to (1) is

\[
\mathbf{\omega}_{21} = \dot{q}_2 \mathbf{j}_2.
\]

The angular velocity of the link 2 with respect to the fixed reference frame (0) is

\[
\mathbf{\omega}_{20} = \mathbf{\omega}_{10} + \mathbf{\omega}_{21} = \dot{q}_1 \mathbf{i}_1 + \dot{q}_2 \mathbf{j}_2.
\]

With $\mathbf{i}_0 = \mathbf{i}_1 = c_2 \mathbf{i}_2 + s_2 \mathbf{k}_2$ the angular velocity of the link 2 in the reference frame (0) written in terms of the reference frame (2) is

\[
\mathbf{\omega}_{20} = \dot{q}_1 (c_2 \mathbf{i}_2 + s_2 \mathbf{k}_2) + \dot{q}_2 \mathbf{j}_2 = \dot{q}_1 c_2 \mathbf{i}_2 + \dot{q}_2 \mathbf{j}_2 + \dot{q}_1 s_2 \mathbf{k}_2.
\]

The link 3 has the same rotational motion as link 2, i.e., $\mathbf{\omega}_{30} = \mathbf{\omega}_{20}$. 

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**RRT robot**

The second generalized coordinate designates also a radian measure of the rotation angle between (1) and (2). The unit vectors $\mathbf{i}_2$, $\mathbf{j}_2$ and $\mathbf{k}_2$ can be expressed as
Angular accelerations
The angular acceleration of the link 1 in the reference frame (0) is
\[ \alpha_{10} = \ddot{q}_1 \mathbf{i}_1. \] (7)

The angular acceleration of the link 2 with respect to the reference frame (0) is
\[ \alpha_{20} = \frac{d}{dt} \omega_{20} = \frac{(2)}{d} \omega_{20} + \omega_{20} \times \omega_{20} = \frac{(2)}{d} \omega_{20}. \]
where \( \frac{d}{dt} \) represents the derivative with respect to time in reference frame (2), \([\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2]\). The angular acceleration of the link 2 is
\[ \alpha_{20} = \frac{(2)}{d} (\dot{q}_1 c_2 \mathbf{i}_2 + \dot{q}_2 \mathbf{j}_2 + \dot{q}_1 s_2 \mathbf{k}_2) = (\dot{q}_1 c_2 - \dot{q}_1 \dot{q}_2 s_2) \mathbf{i}_2 + \dot{q}_2 \mathbf{j}_2 + (\dot{q}_1 s_2 + \dot{q}_1 \dot{q}_2 c_2) \mathbf{k}_2. \] (8)

The link 3 has the same angular acceleration as link 2, i.e., \( \alpha_{30} = \alpha_{20} \).

Linear velocities
The position vector of \( C_1 \), the mass center of link 1, is
\[ \mathbf{r}_{C_1} = L_1 \mathbf{k}_1, \]
and the velocity of \( C_1 \) in (0) is
\[ \mathbf{v}_{C_1} = \frac{d}{dt} \mathbf{r}_{C_1} = \frac{(1)}{d} \mathbf{r}_{C_1} + \omega_{10} \times \mathbf{r}_{C_1} = 0 + \left| \begin{array}{ccc} 1_1 & \mathbf{j}_1 & \mathbf{k}_1 \\ \dot{q}_1 & 0 & 0 \\ 0 & 0 & L_1 \end{array} \right| = -\dot{q}_1 L_1 \mathbf{j}_1. \] (9)

The position vector of \( C_2 \), the mass center of link 2, is
\[ \mathbf{r}_{C_2} = L_B \mathbf{k}_1 + L_2 \mathbf{k}_2 = L_B (-s_2 \mathbf{i}_2 + c_2 \mathbf{k}_2) + L_2 \mathbf{k}_2 = -L_B s_2 \mathbf{i}_2 + (L_B c_2 + L_2) \mathbf{k}_2. \]
where $L_B = 2 L_1$. The velocity of $C_2$ in (0) is

$$v_{C_2} = \frac{d}{dt} r_{C_2} = \frac{(2)}{dt} r_{C_2} + \omega_{20} \times r_{C_2}$$

$$= -L_B c_1 \dot{q}_2 \mathbf{i}_2 - L_B c_2 \dot{q}_2 \mathbf{k}_2 + \begin{vmatrix} \mathbf{i}_2 & \mathbf{j}_2 & \mathbf{k}_2 \\ \dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\ -L_B s_2 & 0 & L_B c_2 + L_2 \end{vmatrix}$$

$$= L_2 \dot{q}_2 \mathbf{i}_2 - (L_B + L_2 c_2) \dot{q}_1 \mathbf{j}_2. \quad (10)$$

The position vector of $C_3$ with respect to reference frame (0) is

$$r_{C_3} = r_{C_2} + q_3 \mathbf{k}_2$$

$$= -L_B s_2 \mathbf{i}_2 + (L_B c_2 + L_2 + q_3) \mathbf{k}_2,$$

and the velocity of this mass center in (0) is

$$v_{C_3} = \frac{d}{dt} r_{C_3} = \frac{(2)}{dt} r_{C_3} + \omega_{20} \times r_{C_3}$$

$$= -L_B c_2 \dot{q}_2 \mathbf{i}_2 - (L_B c_2 \dot{q}_2 + \dot{q}_3) \mathbf{k}_2 + \begin{vmatrix} \mathbf{i}_2 & \mathbf{j}_2 & \mathbf{k}_2 \\ \dot{q}_1 c_2 & \dot{q}_2 & \dot{q}_1 s_2 \\ -L_B s_2 & 0 & L_B c_2 + L_2 + q_2 \end{vmatrix}$$

$$= (L_2 + q_3) \dot{q}_2 \mathbf{i}_2 - (L_B + L_2 c_2 + c_2 q_3) \dot{q}_1 \mathbf{j}_2 + \dot{q}_3 \mathbf{k}_2. \quad (11)$$

**Linear accelerations**

The acceleration of $C_1$ is

$$a_{C_1} = \frac{d}{dt} v_{C_1} = \frac{(1)}{dt} v_{C_1} + \omega_{10} \times v_{C_1}$$

$$= -L_1 \dddot{q}_1 \mathbf{i}_1 - L_1 \dddot{q}_1 \mathbf{j}_1 + \begin{vmatrix} \mathbf{i}_1 & \mathbf{j}_1 & \mathbf{k}_1 \\ \dot{q}_1 & \dot{q}_1 & \dot{q}_1 \\ 0 & 0 & -L_1 \dot{\dot{q}}_1 \end{vmatrix}$$

$$= -L_1 \dddot{q}_1 \mathbf{i}_1 - L_1 \dddot{q}_1 \mathbf{j}_1 \mathbf{k}_1. \quad (12)$$

The linear acceleration of the mass center $C_2$ is

$$a_{C_2} = \frac{d}{dt} v_{C_2} = \frac{(2)}{dt} v_{C_2} + \omega_{20} \times v_{C_2}. \quad (13)$$

The linear acceleration of $C_2$ is symbolically calculated in the program RRTrobot.nb.
The acceleration of $C_3$ is

$$a_{C_3} = \frac{d}{dt}v_{C_3} = (^{(2)}d)\frac{d}{dt}v_{C_3} + \omega_{20} \times v_{C_3}. \quad (14)$$

The linear acceleration of $C_3$ is symbolically calculated in the program $RRTrobot.nb$.\"
Generalized forces

Remark: If a set of contact and/or body forces acting on a rigid body is equivalent to a couple of torque $\mathbf{T}$ together with force $\mathbf{R}$ applied at a point $P$ of the rigid body, then the contribution of this set of forces to the generalized force, $Q_r$, is given by

$$Q_r = \frac{\partial \omega}{\partial \dot{q}_r} \cdot \mathbf{T} + \frac{\partial \mathbf{v}_P}{\partial \dot{q}_r} \cdot \mathbf{R}, \quad r = 1, 2, \ldots,$$

where $\omega$ is the angular velocity of the rigid body in (0), $\mathbf{v}_P$ is the velocity of $P$ in (0), and $r$ represents the generalized coordinates.

In the case of the robotic arm, there are two kinds of forces that contribute to the generalized forces $Q_1$, $Q_2$, and $Q_3$ namely, contact forces applied in order to drive the links 1, 2, and 3, and gravitational forces exerted on 1, 2, and 3 by the Earth.

The set of contact forces transmitted from 0 to 1 can be replaced with a couple of torque $\mathbf{T}_{01}$ applied to 1 at $A$.

Similarly, the set of contact forces transmitted from 1 to 2 can be replaced with a couple of torque $\mathbf{T}_{12}$ applied to 2 at $B$. The law of action and reaction then guarantees that the set of contact forces transmitted from 1 to 2 is equivalent to a couple of torque $-\mathbf{T}_{12}$ to 1 at $B$.

Next, the set of contact forces exerted by link 2 on link 3 can be replaced with a force $\mathbf{F}_{23}$ applied to 3 at $C_3$. The law of action and reaction guarantees that the set of contact forces transmitted from 3 to 2 is equivalent to a force $-\mathbf{F}_{23}$ applied to 2 at $C_{32}$.

The point $C_{32}$ ($C_{32} \in \text{link2}$) instantaneously coincides with $C_3$, ($C_3 \in \text{link3}$).

The expressions $\mathbf{T}_{01}$, $\mathbf{T}_{12}$, and $\mathbf{F}_{23}$ are

$$\mathbf{T}_{01} = T_{01x}\mathbf{i}_1 + T_{01y}\mathbf{j}_1 + T_{01z}\mathbf{k}_1, \quad \mathbf{T}_{12} = T_{12x}\mathbf{i}_2 + T_{12y}\mathbf{j}_2 + T_{12z}\mathbf{k}_2, \quad \text{and} \quad \mathbf{F}_{23} = F_{23x}\mathbf{i}_2 + F_{23y}\mathbf{j}_2 + F_{23z}\mathbf{k}_2.$$

The external gravitational forces exerted on the links 1, 2, and 3 by the Earth, can be denoted by $\mathbf{G}_1$, $\mathbf{G}_2$, and $\mathbf{G}_3$ respectively, and can be expressed as

$$\mathbf{G}_1 = -m_1 g \mathbf{i}_1,$$
\[ \mathbf{G}_2 = -m_2 \mathbf{g} \mathbf{1}_1 = -m_2 g (c_2 \mathbf{1}_2 + s_2 \mathbf{k}_2), \]
\[ \mathbf{G}_3 = -m_3 \mathbf{g} \mathbf{1}_1 = -m_3 g (c_2 \mathbf{1}_2 + s_2 \mathbf{k}_2). \]

The reason for replacing \( \mathbf{1}_1 \) with \( c_2 \mathbf{1}_2 + s_2 \mathbf{k}_2 \) in connection with the forces \( \mathbf{G}_2 \) and \( \mathbf{G}_3 \) is that they are soon to be dot-multiplied with \( \frac{\partial \mathbf{v}_{C_2}}{\partial \dot{q}_r} \) and \( \frac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r} \) which have been expressed in terms of \( \mathbf{1}_2, \mathbf{j}_2, \) and \( \mathbf{k}_2 \).

One can express \((Q_r)_1\), the contribution to the generalized active force \( Q_r \) of all the forces and torques acting on the particles of the link 1, as

\[ (Q_r)_1 = \frac{\partial \omega_{10}}{\partial \dot{q}_r} \cdot (\mathbf{T}_{01} - \mathbf{T}_{12}) + \frac{\partial \mathbf{v}_{C_1}}{\partial \dot{q}_r} \cdot \mathbf{G}_1, \quad r = 1, 2, 3. \]

The contribution \((Q_r)_2\) to the generalized active force of all the forces and torques acting on the link 2 is

\[ (Q_r)_2 = \frac{\partial \omega_{20}}{\partial \dot{q}_r} \cdot \mathbf{T}_{12} + \frac{\partial \mathbf{v}_{C_2}}{\partial \dot{q}_r} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{v}_{C_{32}}}{\partial \dot{q}_r} \cdot (-\mathbf{F}_{23}), \quad r = 1, 2, 3, \]

where

\[ \mathbf{v}_{C_{32}} = \mathbf{v}_{C_2} + \omega_{20} \times r_{C_2} \mathbf{c}_3 = \mathbf{v}_{C_2} + \omega_{20} \times q_3 \mathbf{k}_2. \]

The contribution \((Q_r)_3\), to the generalized active force of all the forces and torques acting on the link 3 is

\[ (Q_r)_3 = \frac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r} \cdot \mathbf{G}_3 + \frac{\partial \mathbf{v}_{C_3}}{\partial \dot{q}_r} \cdot \mathbf{F}_{23}, \quad r = 1, 2, 3. \]

The generalized active force \( Q_r \) of all the forces and torques acting on the links 1, 2, and 3 are

\[ Q_r = (Q_r)_1 + (Q_r)_2 + (Q_r)_3, \quad r = 1, 2, 3, \]

The generalized forces \( Q_r, r = 1, 2, 3 \) are symbolically calculated in the program RRTrobot.nb and have the values

\[ Q_1 = T_{01x}, \]
\[ Q_2 = T_{12y} - g m_2 l_2 c_2 - g m_3 c_2 (L_2 + q_3), \]
\[ Q_3 = F_{23z} - g m_3 s_2. \]
Lagrange’s Equations of Motion

Kinetic energy
The total kinetic energy of the robot arm in the reference frame (0) is

\[ T = \sum_{i=1}^{3} T_i. \]

The kinetic energy of the link \( i \), \( i = 1, 2, 3 \), is

\[ T_i = \frac{1}{2} m_i v_{C_i} \cdot v_{C_i} + \frac{1}{2} \omega_{i0} \cdot (\bar{I}_i \cdot \omega_{i0}). \]

Remark: The kinetic energy for a rigid body is

\[ T_{\text{rigid body}} = \frac{1}{2} m v_C \cdot v_C + \frac{1}{2} \omega \cdot (\bar{I}_C \cdot \omega), \]

where \( m \) is the mass of the rigid body, \( v_C \) is the velocity of the mass center of the rigid body in (0), \( \omega = \omega_x \hat{1} + \omega_y \hat{j} + \omega_z \hat{k} \) is the angular velocity of the rigid body in (0), and \( \bar{I} = (I_{x1}\hat{1} + (I_{y1} \hat{j})\hat{j} + (I_{z1} \hat{k})\hat{k} \) is the central inertia dyadic of the rigid body. The central principal axes of the rigid body are parallel to \( \hat{1}, \hat{j}, \hat{k} \) and the associated moments of inertia have the values \( I_x, I_y, I_z \), respectively. The inertia matrix associated with \( \bar{I} \) is

\[
\bar{I} \rightarrow \begin{bmatrix}
I_x & 0 & 0 \\
0 & I_y & 0 \\
0 & 0 & I_z \\
\end{bmatrix}.
\]

The dot product of the vector \( \omega \) with the dyadic \( \bar{I} \) is

\[ \omega \cdot \bar{I} = I \cdot \omega = \omega_x I_x \hat{1} + \omega_y I_y \hat{j} + \omega_z I_z \hat{k}. \]

The central moments of inertia of links 1 and 2 are calculated using Fig. 2. The central principal axes of 1 are parallel to \( \hat{1}_1, \hat{j}_1, \hat{k}_1 \) and the associated moments of inertia have the values \( I_{1x}, I_{1y}, I_{1z} \), respectively. The inertia matrix associated with link 1 is

\[
\bar{I}_1 \rightarrow \begin{bmatrix}
I_{1x} & 0 & 0 \\
0 & I_{1y} & 0 \\
0 & 0 & I_{1z} \\
\end{bmatrix} = \begin{bmatrix}
\frac{m_1(2L_1)^2}{12} & 0 & 0 \\
0 & \frac{m_1(2L_1)^2}{12} & 0 \\
0 & 0 & \frac{3m_1L_1^2}{3} \\
\end{bmatrix} = \begin{bmatrix}
\frac{m_1L_1^2}{3} & 0 & 0 \\
0 & \frac{m_1L_1^2}{3} & 0 \\
0 & 0 & 0 \\
\end{bmatrix}.
\]
The central principal axes of 2 and 3 are parallel to \( \mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2 \) and the associated moments of inertia have values \( I_{2x}, I_{2y}, I_{2z}, \) and \( I_{3x}, I_{3y}, I_{3z} \) respectively. The inertia matrix associated with link 2 is

\[
\bar{I}_2 \rightarrow \begin{bmatrix} I_{2x} & 0 & 0 \\ 0 & I_{2y} & 0 \\ 0 & 0 & I_{2z} \end{bmatrix} = \begin{bmatrix} \frac{m_2(2L_2)^2}{12} & 0 & 0 \\ 0 & \frac{m_2(2L_2)^2}{12} & 0 \\ 0 & 0 & \frac{m_2L_2^2}{3} \end{bmatrix} = \begin{bmatrix} \frac{m_2L_2^2}{3} & 0 & 0 \\ 0 & \frac{m_2L_2^2}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

The inertia matrix associated with the slider 3 is

\[
\bar{I}_3 \rightarrow \begin{bmatrix} I_{3x} & 0 & 0 \\ 0 & I_{3y} & 0 \\ 0 & 0 & I_{3z} \end{bmatrix}
\]

The kinetic energy of link 1 is

\[
T_1 = \frac{1}{2}m_1 \mathbf{v}_{C_1} \cdot \mathbf{v}_{C_1} + \frac{1}{2} \mathbf{\omega}_{10} \cdot (\bar{I}_1 \cdot \mathbf{\omega}_{10}) = \frac{1}{2}m_1 L_1 \dot{q}_1^2 + \frac{1}{6}m_1 L_1 q_1^2 = \frac{2}{3}m_1 L_1 \dot{q}_1^2.
\]

The kinetic energy of bar 2 is

\[
T_2 = \frac{1}{2}m_2 \mathbf{v}_{C_2} \cdot \mathbf{v}_{C_2} + \frac{1}{2} \mathbf{\omega}_{20} \cdot (\bar{I}_2 \cdot \mathbf{\omega}_{20}) = \frac{m_2}{3} \left[ (6L_1^2 + L_2^2 + 6L_1 L_2 c_2 + L_2^2 \cos 2q_2) \dot{q}_1^2 + 2L_2^2 \dot{q}_2^2 \right].
\]

The kinetic energy of link 3 is

\[
T_3 = \frac{1}{2}m_3 \mathbf{v}_{C_3} \cdot \mathbf{v}_{C_3} + \frac{1}{2} \mathbf{\omega}_{20} \cdot (\bar{I}_3 \cdot \mathbf{\omega}_{20}) = \frac{1}{2} \left( I_{3x} c_2^2 \dot{q}_1^2 + I_{3z} s_2^2 \dot{q}_2^2 + I_{3y} \dot{q}_2^2 \right) + m_3 \left[ (2L_1 + L_2 c_2 + c_2 q_3)^2 \dot{q}_1^2 + (L_2 + q_3)^2 \dot{q}_2^2 + q_3^2 \right]
\]

The total kinetic energy of the robot arm is

\[
T = T_1 + T_2 + T_3,
\]

and is symbolically calculated in the program \( RRTrobot.nb \).
The left hand sides of Lagrange’s equations are

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r}, \quad r = 1, 2, 3,
\]

and are symbolically calculated in the program \textit{RRTrobot.nb}.

To arrive at the dynamical equations governing the robot arm, all that remains to be done is to substitute into Lagrange’s equations, namely,

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = Q_r, \quad r = 1, 2, 3.
\]

The Lagrange’s equations are symbolically calculated in the program \textit{RRTrobot.nb}. 
Kane’s Dynamical Equations

Generalized inertia forces
To explain what the generalized inertia forces are, a system \( \{S\} \) formed by \( \nu \) particles \( P_1, ..., P_\nu \) and having masses \( m_1, ..., m_\nu \) is considered. Suppose that \( n \) generalized speeds \( u_r, r = 1, ..., n \) have been introduced. (For the robotic arm \( u_r = \dot{q}_r, r = 1, ..., n \).) Let \( \mathbf{v}_{P_j} \) and \( \mathbf{a}_{P_j} \) denote, respectively, the velocity of \( P_j \) and the acceleration of \( P_j \) in a reference frame \( (0) \).

Define \( \mathbf{F}_{in,j} \), called the inertia force for \( P_j \), as

\[
\mathbf{F}_{in,j} = -m_j \mathbf{a}_{P_j}, \tag{16}
\]

The quantities \( K_r^*, r = 1, ..., n \), defined as

\[
K_r^* = \sum_{j=1}^{\nu} \frac{\partial \mathbf{v}_{P_j}}{\partial u_r} \cdot \mathbf{a}_{P_j}, \quad r = 1, ..., n, \tag{17}
\]

are called generalized inertia forces for \( \{S\} \).

The contribution to \( \mathbf{F}_r^* \), made by the particles of a rigid body \( RB \) belonging to \( \{S\} \), are

\[
(K_r^*)_R = \frac{\partial \mathbf{v}_C}{\partial u_r} \cdot \mathbf{F}_{in} + \frac{\partial \omega}{\partial u_r} \cdot \mathbf{M}_{in}, \quad r = 1, ..., n, \tag{18}
\]

where \( \mathbf{v}_C \) is the velocity of the center of gravity of \( RB \) in \( (0) \), and \( \omega = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k} \) is the angular velocity of \( RB \) in \( (0) \).

The inertia force for the rigid body \( RB \) is

\[
\mathbf{F}_{in} = -m \mathbf{a}_C, \tag{19}
\]

where \( m \) is the mass of \( RB \), and \( \mathbf{a}_C \) is the acceleration of the mass center of \( RB \) in the fixed reference frame. The inertia moment \( \mathbf{M}_{in} \) for \( RB \) is

\[
\mathbf{M}_{in} = -\alpha \cdot \mathbf{I} - (\mathbf{I}_C \cdot \omega), \tag{20}
\]

where \( \alpha = \dot{\omega} = \alpha_x \mathbf{i} + \alpha_y \mathbf{j} + \alpha_z \mathbf{k} \) is the angular acceleration of \( RB \) in \( (0) \), and \( \mathbf{I} = (I_x \mathbf{i} + (I_y \mathbf{j}) + (I_z \mathbf{k}) \mathbf{k} \) is the central inertia dyadic of \( RB \). The central principal axes of \( RB \) are parallel to \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) and the associated moments of
inertia have the values $I_x$, $I_y$, $I_z$, respectively. The inertia matrix associated with $\bar{I}$ is

$$
\bar{I} \rightarrow \begin{bmatrix}
I_x & 0 & 0 \\
0 & I_y & 0 \\
0 & 0 & I_z
\end{bmatrix}.
$$

(21)

The dot product of the vector $\alpha$ with the dyadic $\bar{I}$ is

$$
\alpha \cdot \bar{I} = \bar{I} \cdot \alpha = \alpha_x I_x \mathbf{1} + \alpha_y I_y \mathbf{j} + \alpha_z I_z \mathbf{k},
$$

(22)

and the cross product between a vector and a dyadic is

$$
\omega \times (\bar{I} \cdot \omega) = \begin{vmatrix}
1 & \mathbf{j} & \mathbf{k} \\
\omega_x & \omega_y & \omega_z \\
\omega_x I_x & \omega_y I_y & \omega_z I_z
\end{vmatrix} = \omega_x \omega_z (I_y - I_z) \mathbf{i} - \omega_z \omega_x (I_z - I_x) \mathbf{j} - \omega_y \omega_x (I_x - I_y) \mathbf{k}.
$$

(23)

The inertia moment of 1 in (0) can be written as

$$
M_{in 1} = -\alpha_{10} \cdot \bar{I}_1 - \omega_{10} \times (\bar{I}_1 \cdot \omega_{10}) = -I_{1x}\ddot{q}_1 \mathbf{1}.
$$

(24)

The inertia moment of 2 in (0) is

$$
M_{in 2} = -\alpha_{20} \cdot \bar{I}_2 - \omega_{20} \times (\bar{I}_2 \cdot \omega_{20}).
$$

Similarly the inertia moment of 3 in (0) is

$$
M_{in 3} = \alpha_{20} \cdot \bar{I}_3 - \omega_{20} \times (\bar{I}_3 \cdot \omega_{20}).
$$

(25)

The inertia force for link $j = 1, 2, 3$ is

$$
F_{in j} = -m_j \mathbf{a}_{C_j},
$$

(26)

The contribution of link $j = 1, 2, 3$ to the generalized inertia force $K_r^*$ is

$$
(K_r^*)_{j} = \frac{\partial \mathbf{C}_j}{\partial u_r} \cdot F_{in j} + \frac{\partial \omega_{j0}}{\partial u_r} \cdot M_{in j}, \quad r = 1, 2, 3.
$$

(27)
The three generalized inertia forces are computed with

\[
K^*_r = \sum_{j=1}^{3} (K^*_r)_j = \sum_{j=1}^{3} \left( \frac{\partial v_{C_j}}{\partial u_r} \cdot F_{in j} + \frac{\partial \omega_{j0}}{\partial u_r} \cdot M_{in j} \right), \quad r = 1, 2, 3,
\]

or

\[
K^*_r = \frac{\partial v_{C_1}}{\partial q_r} \cdot (-m_1 a_{C_1}) + \frac{\partial \omega_{10}}{\partial q_r} \cdot M_{in 1} + \frac{\partial v_{C_2}}{\partial q_r} \cdot (-m_2 a_{C_2}) + \frac{\partial \omega_{20}}{\partial q_r} \cdot M_{in 2} + \frac{\partial v_{C_3}}{\partial q_r} \cdot (-m_3 a_{C_3}) + \frac{\partial \omega_{30}}{\partial q_r} \cdot M_{in 3}, \quad r = 1, 2, 3.
\]

(28)

To arrive at the dynamical equations governing the robot arm, all that remains to be done is to substitute into Kane’s dynamical equations, namely,

\[
K^*_r + Q_r = 0, \quad r = 1, 2, 3.
\]

(29)
Numerical Simulations

The robot arm is characterized by the following geometry: \( L_1 = 0.4 \text{ m}, \ L_2 = 0.7 \text{ m}, \ I_{3x} = 5 \text{ kg}\cdot\text{m}^2, \ I_{3y} = 4 \text{ kg}\cdot\text{m}^2, \ I_{3z} = 1 \text{ kg}\cdot\text{m}^2. \) The masses of the rigid bodies are \( m_1 = 90 \text{ kg}, \ m_2 = 60 \text{ kg}, \ m_3 = 40 \text{ kg}, \) and the gravitational acceleration is \( g = 9.81 \text{ m/s}^2. \)

The initial conditions, at \( t = 0 \text{ s}, \) are \( q_1(0) = \pi/18 \text{ rad}, \ q_2(0) = \pi/6 \text{ rad}, \ q_3(0) = 0.25 \text{ m}, \) and \( \dot{q}_1(0) = \dot{q}_2(0) = \dot{q}_3(0) = 0. \)

The robot arm can be brought from an initial state of rest in reference frame \((0)\) to a final state of rest in \((0)\) in such a way that \( q_1, \ q_2, \) and \( q_3 \) have specified values \( q_{1f}, \ q_{2f}, \) and \( q_{3f}, \) respectively \( (q_{1f} = \pi/3 \text{ rad}, \ q_{2f} = \pi/3 \text{ rad}, \) and \( q_{3f} = 0.3 \text{ m}). \)

Inverse dynamics

A desired motion of the robot arm has been specified for a time interval \( 0 \leq t \leq T_p = 15 \text{ s}. \) The generalized coordinates can be established explicitly

\[
q_r(t) = q_r(0) + \frac{q_r(T_p) - q_r(0)}{T_p} \left[ t - \frac{T_p}{2\pi} \sin \left( \frac{2\pi t}{T_p} \right) \right], \quad r = 1, 2, 3, \quad (30)
\]

with \( q_r(T_p) = q_r,f. \)

Find \( T_{01}(t), \ T_{12}(t), \) and \( F_{23}(t) \) for \( 0 \leq t \leq T_p = 15 \text{ s}. \)

Direct dynamics

The following feedback control laws are used

\[
T_{01x} = -\beta_{01}\dot{q}_1 - \gamma_{01}(q_1 - q_{1f}),
\]

\[
T_{12y} = -\beta_{12}\dot{q}_2 - \gamma_{12}(q_2 - q_{2f}) + g m_2 L_2 c_2 + g m_3 c_2 (L_2 + q_3),
\]

\[
F_{23z} = -\beta_{23}\dot{q}_3 - \gamma_{23}(q_3 - q_{3f}) + g m_3 s_2. \quad (31)
\]

The constant gains are: \( \beta_{01} = 450 \text{ N\cdotm/s/rad}, \ \gamma_{01} = 300 \text{ N\cdotm/rad}, \ \beta_{12} = 200 \text{ N\cdotm/s/rad}, \ \gamma_{12} = 300 \text{ N\cdotm/rad}, \ \beta_{23} = 150 \text{ N\cdots/m}, \) and \( \gamma_{23} = 50 \text{ N/m}. \)

Find \( q_1(t), \ q_2(t), \) and \( q_3(t) \) for \( 0 \leq t \leq 15 \text{ s}. \)