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"Lagrange's equations of motion  - example 3 "
Apply [Clear,Names["Global`*"]];
Off[General::spell]
Off[General::spell1]

"kinematics"
"transformation matrix from RF1 to RF0"
R10 = {{1,0,0},
      {0,Cos[q1[t]],Sin[q1[t]]},
      {0,-Sin[q1[t]],Cos[q1[t]]}};
MatrixForm[R10]
"transformation matrix from RF2 to RF1"
R21={{Cos[q2[t]],0,-Sin[q2[t]]},
     {0,1,0},
     {Sin[q2[t]],0,Cos[q2[t]]}};
MatrixForm[R21]
"angular velocity of link 1 in RF0
expressed in terms of RF1 {i1,j1,k1}"
w10 = {D[q1[t],t],0,0}

"angular velocity of link 2 in RF0
expressed  in terms of RF1 {i1,j1,k1}"
w201 = {D[q1[t],t],D[q2[t],t],0}

"angular velocity of link 2 in RF0
expressed in terms of RF2 {i2,j2,k2}"
w202=w201.Transpose[R21]

"angular velocity of link 2 in RF0
expressed in terms of RF0 {i0,j0,k0}"
w200=w201.R10

"position vector of mass center C1 of link 1
in RF0 expressed in terms of RF1 {i1,j1,k1}"
rC1={L1,0,0}

"linear velocity of mass center C1 of link 1
in RF0 expressed in terms of RF1 {i1,j1,k1}"
vC1 =D[rC1,t]+Cross[w10,rC1]

"position vector of mass center C2 of link 2
in RF0 expressed in terms of RF2 {i2,j2,k2}"
rC2={L2,0,0}.Transpose[R21]

"position vector of mass center C2 of link 2
in RF0 expressed in terms of RF0 {i0,j0,k0}"
rC20={L2,0,0}.R10

"linear velocity of mass center C2 of link 2 in RF0
expressed in terms of RF2 {i2,j2,k2}"
vC2 =D[rC2,t]+Cross[w202,rC2]

"position vector of mass center C3 of link 3 in RF0
expressed in terms of RF2 {i2,j2,k2}"
rC3=rC2+{0,0,q3[t]}

"linear velocity of mass center C3 of link 3 in RF0
expressed in terms of RF2 {i2,j2,k2}"
vC3 =Expand[D[rC3,t]+Cross[w202,rC3]];
Simplify[vC3]

"position vector of mass center C3 of link 3 in RF0
expressed in terms of RF0 {i0,j0,k0}"
rC30=((rC2+{0,0,q3[t]}).R21).R10
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vC30=D[rC30,t];
Expand[vC3.vC3,Trig->True]==Simplify[vC30.vC30];

"position vector of mass center of rigid body RB
in RF0 expressed in terms of RF2 {i2,j2,k2}"
rCR=rC3+{0,0,L/2}


"linear velocity of mass center of rigid body RB
in RF0 expressed in terms of RF2 {i2,j2,k2}"
vCR =Expand[D[rCR,t]+Cross[w202,rCR]];

Simplify[rCR];
Simplify[vCR]

" position vector of mass center CR of rigid body RB in RF0
expressed in terms of RF0 {i0,j0,k0} "
rCR0=(rCR.R21).R10
vCR0=D[rCR0,t];
Expand[vCR.vCR,Trig->True]==Simplify[vCR0.vCR0];

"inertia matrix associated to link 1
expressed in terms of RF1 {i1,j1,k1} "
I1={{I1x,0,0},{0,I1y,0},{0,0,I1z}};
MatrixForm[I1]

"inertia matrix associated to bar 2
expressed in terms of RF2 {i2,j2,k2} "
I2x=I2z=m2 l^2/12;
I2y=0;
I2={{I2x,0,0},{0,I2y,0},{0,0,I2z}};
MatrixForm[I2]

"inertia matrix associated to slider 2'
expressed in terms of RF2 {i2,j2,k2}"
I2S={{I2Sx,0,0},{0,I2Sy,0},{0,0,I2Sz}};
MatrixForm[I2S]

"inertia matrix associated to bar 3
expressed in terms of RF2 {i2,j2,k2} "
I3x=I3y=m3 L^2/12;
I3z=0;
I3={{I3x,0,0},{0,I3y,0},{0,0,I3z}};
MatrixForm[I3]


" kinetic energy "

" kinetic energy of link 1 "
T1=m1 vC1.vC1/2+ w10.I1.w10/2

" kinetic energy of link 2 "
T2=m2 vC2.vC2/2+ w202.I2.w202/2;
Simplify[T2]

" kinetic energy of slider 2' "
T2S=m2 vC2.vC2/2+ w202.I2S.w202/2

" kinetic energy of link 3 "
T3=m3 vC3.vC3/2+ w202.I3.w202/2;
Simplify[T3]

" kinetic energy of RB "
TR=Simplify[mR vCR.vCR/2,Trig->True]
```

```
" total kinetic energy "
Simplify[T1+T2+T2S+T3+TR]
T=Expand[T1+T2+T2S+T3+TR];

" gravitational force that acts on link 1 at C1
in RF0 expressed in terms of RF0 {i0,j0,k0} or in
in terms of RF1 {i1,j1,k1} "
G1={ -m1 g, 0, 0 }

" gravitational force that acts on bar 2 and
slider 2' at C2 in RF0 expressed in terms of
RF0 {i0,j0,k0} or RF1 {i1,j1,k1} "
G2={ -(m2+m2S) g, 0, 0 }

(* gravitational force that acts on bar 2 and
slider 2' at C2 in RF0 expressed in terms of
RF2 {i2,j2,k2} *)
G22={ -(m2+m2S) g, 0, 0 }.Transpose[R21];


" gravitational force that acts on link 3 at C3
in RF0 expressed in terms of RF0 {i0,j0,k0} "
G3={ -m3 g, 0 ,0 }

(* gravitational force that acts on link 3 at C3
in RF0 expressed in terms of RF2 {i2,j2,k2} *)
G32={ -m3 g, 0 ,0 }.Transpose[R21];

" gravitational force that acts on link 3 at CR
in RF0 expressed in terms of RF0 {i0,j0,k0} "
GR={ -mR g, 0, 0 }

(* gravitational force that acts on link 3 at C3
in RF0 expressed in terms of RF2 {i2,j2,k2} *)
GR2={ -mR g, 0, 0 }.Transpose[R21];

"generalized active force Qi=∑Fj.∂(rj)/∂(qi): Q1, Q2, Q3"

Q1=D[rC1,q1[t]].G1+D[rC20,q1[t]].G2+
   D[rC30,q1[t]].G3+D[rCR0,q1[t]].GR;

Q2=D[rC1,q2[t]].G1+D[rC20,q2[t]].G2+
   D[rC30,q2[t]].G3+D[rCR0,q2[t]].GR;

Q3=D[rC1,q3[t]].G1+D[rC20,q3[t]].G2+
   D[rC30,q3[t]].G3+D[rCR0,q3[t]].GR;


Simplify[Q1]
Simplify[Q2]
Simplify[Q3]


(*generalized active force*)


"generalized active force Qi=∑Fj.∂(vj)/∂(qi'): Q1, Q2, Q3"

F1=D[vC1,q1'[t]].G1+
   D[vC2,q1'[t]].G22+
   D[vC3,q1'[t]].G32+
   D[vCR,q1'[t]].GR2;

F2=D[vC1,q2'[t]].G1 +
   D[vC2,q2'[t]].G22+
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   D[vC3,q2'[t]].G32+
   D[vCR,q2'[t]].GR2;

F3=D[vC1,q3'[t]].G1 +
   D[vC2,q3'[t]].G22+
   D[vC3,q3'[t]].G32+
   D[vCR,q3'[t]].GR2;

Expand[Q1]==Expand[F1];
Expand[Q2]==Expand[F2];
Expand[Q3]==Expand[F3];

Simplify[F1]
Simplify[F2]
Simplify[F3]



" Lagrange's eom "

Leq1=D[D[T,q1'[t]],t]-D[T,q1[t]]-Q1==0;
Leq2=D[D[T,q2'[t]],t]-D[T,q2[t]]-Q2==0;
Leq3=D[D[T,q3'[t]],t]-D[T,q3[t]]-Q2==0;

Simplify[Leq1]
Simplify[Leq2]
Simplify[Leq3]
```

Lagrange's equations of motion - example 3

kinematics

transformation matrix from RF1 to RF0

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[q1[t]] & \text{Sin}[q1[t]] \\ 0 & -\text{Sin}[q1[t]] & \text{Cos}[q1[t]] \end{pmatrix}$$

transformation matrix from RF2 to RF1

$$\begin{pmatrix} \text{Cos}[q2[t]] & 0 & -\text{Sin}[q2[t]] \\ 0 & 1 & 0 \\ \text{Sin}[q2[t]] & 0 & \text{Cos}[q2[t]] \end{pmatrix}$$

angular velocity of link 1 in RF0 expressed in terms of RF1 {i1,j1,k1}

{q1′[t], 0, 0}

angular velocity of link 2 in RF0 expressed in terms of RF1 {i1,j1,k1}

{q1′[t], q2′[t], 0}

angular velocity of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}

{Cos[q2[t]] q1′[t], q2′[t], Sin[q2[t]] q1′[t]}

angular velocity of link 2 in RF0 expressed in terms of RF0 {i0,j0,k0}

{q1′[t], Cos[q1[t]] q2′[t], Sin[q1[t]] q2′[t]}

```
position vector of mass center C1
  of link 1  in RF0 expressed in terms of RF1 {i1,j1,k1}
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{L1, 0, 0}

```
linear velocity of mass center C1
  of link 1  in RF0 expressed in terms of RF1 {i1,j1,k1}
```

{0, 0, 0}

```
position vector of mass center C2
  of link 2  in RF0 expressed in terms of RF2 {i2,j2,k2}
```

{L2 Cos[q2[t]], 0, L2 Sin[q2[t]]}

```
position vector of mass center C2
  of link 2  in RF0 expressed in terms of RF0 {i0,j0,k0}
```

{L2, 0, 0}

```
linear velocity of mass center C2 of link 2 in RF0 expressed in terms of RF2 {i2,j2,k2}
```

{0, 0, 0}

```
position vector of mass center C3 of link 3 in RF0 expressed in terms of RF2 {i2,j2,k2}
```

{L2 Cos[q2[t]], 0, q3[t] + L2 Sin[q2[t]]}

```
linear velocity of mass center C3 of link 3 in RF0 expressed in terms of RF2 {i2,j2,k2}
```

{q3[t] q2′[t], -Cos[q2[t]] q3[t] q1′[t], q3′[t]}

```
position vector of mass center C3 of link 3 in RF0 expressed in terms of RF0 {i0,j0,k0}
```

{L2 Cos[q2[t]]^2 + Sin[q2[t]] (q3[t] + L2 Sin[q2[t]]),
 -Sin[q1[t]] (-L2 Cos[q2[t]] Sin[q2[t]] + Cos[q2[t]] (q3[t] + L2 Sin[q2[t]])),
 Cos[q1[t]] (-L2 Cos[q2[t]] Sin[q2[t]] + Cos[q2[t]] (q3[t] + L2 Sin[q2[t]]))}

```
position vector of mass center of
  rigid body RB in RF0 expressed in terms of RF2 {i2,j2,k2}
```

{L2 Cos[q2[t]], 0, $\frac{L}{2}$ + q3[t] + L2 Sin[q2[t]]}

```
linear velocity of mass center of
  rigid body RB in RF0 expressed in terms of RF2 {i2,j2,k2}
```

{$\frac{1}{2}$ (L + 2 q3[t]) q2′[t], -$\frac{1}{2}$ Cos[q2[t]] (L + 2 q3[t]) q1′[t], q3′[t]}

```
 position vector of mass center CR of
  rigid body RB in RF0 expressed in terms of RF0 {i0,j0,k0}
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{L2 Cos[q2[t]]^2 + Sin[q2[t]] ($\frac{L}{2}$ + q3[t] + L2 Sin[q2[t]]),
 -Sin[q1[t]] (-L2 Cos[q2[t]] Sin[q2[t]] + Cos[q2[t]] ($\frac{L}{2}$ + q3[t] + L2 Sin[q2[t]])),
 Cos[q1[t]] (-L2 Cos[q2[t]] Sin[q2[t]] + Cos[q2[t]] ($\frac{L}{2}$ + q3[t] + L2 Sin[q2[t]]))}

inertia matrix associated to link 1 expressed in terms of RF1 {i1,j1,k1}

$$\begin{pmatrix} I1x & 0 & 0 \\ 0 & I1y & 0 \\ 0 & 0 & I1z \end{pmatrix}$$

inertia matrix associated to bar 2 expressed in terms of RF2 {i2,j2,k2}

$$\begin{pmatrix} \frac{l^2\, m2}{12} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{l^2\, m2}{12} \end{pmatrix}$$

inertia matrix associated to slider 2' expressed in terms of RF2 {i2,j2,k2}

$$\begin{pmatrix} I2Sx & 0 & 0 \\ 0 & I2Sy & 0 \\ 0 & 0 & I2Sz \end{pmatrix}$$

inertia matrix associated to bar 3 expressed in terms of RF2 {i2,j2,k2}

$$\begin{pmatrix} \frac{L^2\, m3}{12} & 0 & 0 \\ 0 & \frac{L^2\, m3}{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

kinetic energy

kinetic energy of link 1

$$\frac{1}{2}\, I1x\, q1'[t]^2$$

kinetic energy of link 2

$$\frac{1}{24}\, l^2\, m2\, q1'[t]^2$$

kinetic energy of slider 2'

$$\frac{1}{2}\left(I2Sx\, Cos[q2[t]]^2\, q1'[t]^2 + I2Sz\, Sin[q2[t]]^2\, q1'[t]^2 + I2Sy\, q2'[t]^2\right)$$

kinetic energy of link 3

$$\frac{1}{24}\, m3\left(Cos[q2[t]]^2\left(L^2 + 12\, q3[t]^2\right) q1'[t]^2 + \left(L^2 + 12\, q3[t]^2\right) q2'[t]^2 + 12\, q3'[t]^2\right)$$

kinetic energy of RB

$$\frac{1}{8}\, mR\left(Cos[q2[t]]^2\left(L + 2\, q3[t]\right)^2 q1'[t]^2 + \left(L + 2\, q3[t]\right)^2 q2'[t]^2 + 4\, q3'[t]^2\right)$$

total kinetic energy

$$\frac{1}{24}\Big(\big(12\, I1x + \left(12\, I2Sx + l^2\, m2 + L^2\left(m3 + 3\, mR\right)\right) Cos[q2[t]]^2 + 12\, L\, mR\, Cos[q2[t]]^2\, q3[t] + 12\left(m3 + mR\right) Cos[q2[t]]^2\, q3[t]^2 + 12\, I2Sz\, Sin[q2[t]]^2 + l^2\, m2\, Sin[q2[t]]^2\big)\, q1'[t]^2 + \left(12\, I2Sy + L^2\left(m3 + 3\, mR\right) + 12\, L\, mR\, q3[t] + 12\left(m3 + mR\right) q3[t]^2\right) q2'[t]^2 + 12\left(m3 + mR\right) q3'[t]^2\Big)$$

gravitational force that acts on link 1 at C1 in RF0
expressed in terms of RF0 {i0,j0,k0} or in in terms of RF1 {i1,j1,k1}

{-g m1, 0, 0}

gravitational force that acts on bar 2 and slider 2' at
C2 in RF0 expressed in terms of RF0 {i0,j0,k0} or RF1 {i1,j1,k1}

{-g (m2 + m2S), 0, 0}

gravitational force that acts on
link 3 at C3 in RF0 expressed in terms of RF0 {i0,j0,k0}

{-g m3, 0, 0}

gravitational force that acts on
link 3 at CR in RF0 expressed in terms of RF0 {i0,j0,k0}

{-g mR, 0, 0}

generalized active force Qi=∑Fj.∂(rj)/∂(qi): Q1, Q2, Q3

0

$-\frac{1}{2}$ g Cos[q2[t]] (L mR + 2 (m3 + mR) q3[t])

-g (m3 + mR) Sin[q2[t]]

generalized active force Qi=∑Fj.∂(vj)/∂(qi'): Q1, Q2, Q3

0

$-\frac{1}{2}$ g Cos[q2[t]] (L mR + 2 (m3 + mR) q3[t])

-g (m3 + mR) Sin[q2[t]]

Lagrange's eom

$\frac{1}{12}$ (-2 Cos[q2[t]] q1′[t] ((12 I2Sx - 12 I2Sz + L^2 m3 + 3 L^2 mR + 12 L mR q3[t] + 12 (m3 + mR) q3[t]^2)
Sin[q2[t]] q2′[t] - 6 Cos[q2[t]] (L mR + 2 (m3 + mR) q3[t]) q3′[t]) +
(12 I1x + (12 I2Sx + l^2 m2 + L^2 (m3 + 3 mR)) Cos[q2[t]]^2 + 12 L mR Cos[q2[t]]^2 q3[t] +
12 (m3 + mR) Cos[q2[t]]^2 q3[t]^2 + 12 I2Sz Sin[q2[t]]^2 + l^2 m2 Sin[q2[t]]^2) q1″[t]) == 0

(m3 + mR) q3[t]^2 (Cos[q2[t]] Sin[q2[t]] q1′[t]^2 + q2″[t]) +
q3[t] (g m3 Cos[q2[t]] + g mR Cos[q2[t]] +
L mR Cos[q2[t]] Sin[q2[t]] q1′[t]^2 + 2 (m3 + mR) q2′[t] q3′[t] + L mR q2″[t]) +
$\frac{1}{24}$ ((12 I2Sx - 12 I2Sz + L^2 (m3 + 3 mR)) Sin[2 q2[t]] q1′[t]^2 +
2 (6 g L mR Cos[q2[t]] + 12 L mR q2′[t] q3′[t] + (12 I2Sy + L^2 (m3 + 3 mR)) q2″[t])) == 0

$\frac{1}{2}$ (g L mR Cos[q2[t]] - L mR Cos[q2[t]]^2 q1′[t]^2 - L mR q2′[t]^2 - 2 (m3 + mR) q3[t]
(-g Cos[q2[t]] + Cos[q2[t]]^2 q1′[t]^2 + q2′[t]^2) + 2 m3 q3″[t] + 2 mR q3″[t]) == 0