Contents

5 Contour Equations

This chapter provides an algebraic method to compute the velocities and accelerations of any closed kinematic chain. The classical method for obtaining the velocities and accelerations involves the computation of the derivative with respect to time of the position vectors. The method of contour equations avoids this task and uses only algebraic equations [4, 56]. Using this approach, a numerical implementation is much more efficient. The method described here can be applied to planar and spatial mechanisms.

Two rigid links (j) and (k) are connected by a joint (kinematic pair) at A (Fig. 5.1). The point A_j of the rigid body (j) is guided along a path prescribed in the body (k). The points A_j belonging to body (j) and the A_k belonging to body (k) are coincident at the instant of motion under consideration. The following relation exists between the velocity $\mathbf{v}_{A_j}$ of the point A_j and the velocity $\mathbf{v}_{A_k}$ of the point A_k:

$$\mathbf{v}_{A_j} = \mathbf{v}_{A_k} + \mathbf{v}^r_{A_{jk}}, \tag{5.1}$$

where $\mathbf{v}^r_{A_{jk}} = \mathbf{v}^r_{A_j A_k}$ indicates the velocity of A_j as seen by an observer at A_k attached to body k or the relative velocity of A_j with respect to A_k, allowed at the joint A. The direction of $\mathbf{v}^r_{A_{jk}}$ is obviously tangent to the path prescribed in the body (k).

From Eq. (5.1) the accelerations of A_j and A_k are expressed as

$$\mathbf{a}_{A_j} = \mathbf{a}_{A_k} + \mathbf{a}^r_{A_{jk}} + \mathbf{a}^c_{A_{jk}}, \tag{5.2}$$

where $\mathbf{a}^c_{A_{jk}} = \mathbf{a}^c_{A_j A_k}$ is known as the *Coriolis acceleration* and is given by

$$\mathbf{a}^c_{A_{jk}} = 2\,\boldsymbol{\omega}_k \times \mathbf{v}^r_{A_{jk}}, \tag{5.3}$$

where $\boldsymbol{\omega}_k$ is the angular velocity of the body (k).

Equations (5.1) and (5.2) are useful even for coincident points belonging to two links that may not be directly connected. A graphical representation of Eq. (5.1) is shown in Fig. 5.1(b) for a rotating slider joint.

Figure 5.2 shows a monocontour closed kinematic chain with n rigid links. The joint A_i, $i = 0, 1, 2, ..., n$ is the connection between the links (i) and $(i-1)$. The last link n is connected with the first link 0 of the chain. For the closed kinematic chain, a path is chosen from link 0 to link n. At the joint A_i there are two instantaneously coincident points: 1) the point $A_{i,i}$

belonging to link (i), $A_{i,i} \in (i)$, and 2) the point $A_{i,i-1}$ belonging to body $(i-1)$, $A_{i,i-1} \in (i-1)$.

5.1 Contour Velocity Equations

The absolute angular velocity, $\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i,0}$, of the rigid body (i), or the angular velocity of the rigid body (i) with respect to the "fixed" reference frame $Oxyz$ is

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_{i,i-1}, \tag{5.4}$$

where $\boldsymbol{\omega}_{i-1} = \boldsymbol{\omega}_{i-1,0}$ is the absolute angular velocity of the rigid body $(i-1)$ (or the angular velocity of the rigid body $(i-1)$ with respect to the "fixed" reference frame $Oxyz$) and $\boldsymbol{\omega}_{i,i-1}$ is the relative angular velocity of the rigid body (i) with respect to the rigid body $(i-1)$.

For the n link closed kinematic chain the following expressions are obtained for the angular velocities:

$$\begin{aligned}
&\boldsymbol{\omega}_1 = \boldsymbol{\omega}_0 + \boldsymbol{\omega}_{1,0} \\
&\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_{2,1} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_{i,i-1} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\boldsymbol{\omega}_0 = \boldsymbol{\omega}_n + \boldsymbol{\omega}_{0,n}\ .
\end{aligned} \tag{5.5}$$

Summing the expressions given in Eq. (5.5), the following relation is obtained:

$$\boldsymbol{\omega}_{1,0} + \boldsymbol{\omega}_{2,1} + \ldots + \boldsymbol{\omega}_{0,n} = \mathbf{0}, \tag{5.6}$$

which may be rewritten as

$$\sum_{(i)} \boldsymbol{\omega}_{i,i-1} = \mathbf{0}. \tag{5.7}$$

Equation (5.7) represents the first vectorial equation for the angular velocities of a simple closed kinematic chain.

The following relation exists between the velocity $\mathbf{v}_{A_{i,i}}$ of the point $A_{i,i}$ and the velocity $\mathbf{v}_{Ai,i-1}$ of the point $A_{i,i-1}$

$$\mathbf{v}_{A_{i,i}} = \mathbf{v}_{A_{i,i-1}} + \mathbf{v}^r_{A_{i,i-1}}\ , \tag{5.8}$$

where $\mathbf{v}^r_{A_{i,i-1}} = \mathbf{v}^r_{A_{i,i}A_{i,i-1}}$ is the relative velocity of $A_{i,i}$ on link (i) with respect to $A_{i,i-1}$ on link $(i-1)$. Using the velocity relation for two particles on the rigid body (i), the following relation exists:

$$\mathbf{v}_{A_{i+1,i}} = \mathbf{v}_{A_{i,i}} + \boldsymbol{\omega}_i \times \mathbf{r}_{A_iA_{i+1}} \ , \tag{5.9}$$

where $\boldsymbol{\omega}_i$ is the absolute angular velocity of the link (i) in the reference frame $Oxyz$, and $\mathbf{r}_{A_iA_{i+1}}$ is the distance vector from A_i to A_{i+1}. Using Eqs. (5.8) and (5.9), the velocity of the point $A_{i+1,i} \in (i+1)$ is written as

$$\mathbf{v}_{A_{i+1,i}} = \mathbf{v}_{A_{i,i-1}} + \boldsymbol{\omega}_i \times \mathbf{r}_{A_iA_{i+1}} + \mathbf{v}^r_{A_{i,i-1}} \ . \tag{5.10}$$

For the n link closed kinematic chain the following expressions are obtained

$$\begin{aligned}
&\mathbf{v}_{A_{3,2}} = \mathbf{v}_{A_{2,1}} + \boldsymbol{\omega}_2 \times \mathbf{r}_{A_2A_3} + \mathbf{v}^r_{A_{2,1}}\\
&\mathbf{v}_{A_{4,3}} = \mathbf{v}_{A_{3,2}} + \boldsymbol{\omega}_3 \times \mathbf{r}_{A_3A_4} + \mathbf{v}^r_{A_{3,2}}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\mathbf{v}_{A_{i+1,i}} = \mathbf{v}_{A_{i,i-1}} + \boldsymbol{\omega}_i \times \mathbf{r}_{A_iA_{i+1}} + \mathbf{v}^r_{A_{i,i-1}}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\mathbf{v}_{A_{1,0}} = \mathbf{v}_{A_{0,n}} + \boldsymbol{\omega}_0 \times \mathbf{r}_{A_0A_1} + \mathbf{v}^r_{A_{0,n}}\\
&\mathbf{v}_{A_{2,1}} = \mathbf{v}_{A_{1,0}} + \boldsymbol{\omega}_1 \times \mathbf{r}_{A_1A_2} + \mathbf{v}^r_{A_{1,0}} \ .
\end{aligned} \tag{5.11}$$

Summing the relations in Eq. (5.11):

$$\begin{aligned}
&\left[\boldsymbol{\omega}_1 \times \mathbf{r}_{A_1A_2} + \boldsymbol{\omega}_2 \times \mathbf{r}_{A_2A_3} + \dots + \boldsymbol{\omega}_i \times \mathbf{r}_{A_iA_{i+1}} + \dots + \boldsymbol{\omega}_0 \times \mathbf{r}_{A_0A_1}\right] +\\
&\qquad \left[\mathbf{v}^r_{A_{2,1}} + \mathbf{v}^r_{A_{3,2}} + \dots + \mathbf{v}^r_{A_{i,i-1}} + \dots + \mathbf{v}^r_{A_{0,n}} + \mathbf{v}^r_{A_{1,0}}\right] = \mathbf{0}.
\end{aligned} \tag{5.12}$$

Because the reference system $Oxyz$ is considered “fixed”, the vector $\mathbf{r}_{A_{i-1}A_i}$ is written in terms of the position vectors of the points A_{i-1} and A_i:

$$\mathbf{r}_{A_{i-1}A_i} = \mathbf{r}_{A_i} - \mathbf{r}_{A_{i-1}}, \tag{5.13}$$

where $\mathbf{r}_{A_i} = \mathbf{r}_{OA_i}$ and $\mathbf{r}_{A_{i-1}} = \mathbf{r}_{OA_{i-1}}$. Equation (5.12) becomes

$$\begin{aligned}
&[\mathbf{r}_{A_1} \times (\boldsymbol{\omega}_1 - \boldsymbol{\omega}_0) + \mathbf{r}_{A_2} \times (\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1) + \dots + \mathbf{r}_{A_0} \times (\boldsymbol{\omega}_0 - \boldsymbol{\omega}_n)] +\\
&\qquad \left[\mathbf{v}^r_{A_{1,0}} + \mathbf{v}^r_{A_{2,1}} + \dots + \mathbf{v}^r_{A_{i,i-1}} + \dots + \mathbf{v}^r_{A_{0,n}}\right] = \mathbf{0}.
\end{aligned} \tag{5.14}$$

Using Eq. (5.5), Eq. (5.14) becomes

$$\begin{aligned}
&[\mathbf{r}_{A_1} \times \boldsymbol{\omega}_{1,0} + \mathbf{r}_{A_2} \times \boldsymbol{\omega}_{2,1} + \dots + \mathbf{r}_{A_0} \times \boldsymbol{\omega}_{0,n}] +\\
&\qquad \left[\mathbf{v}^r_{A_{1,0}} + \mathbf{v}^r_{A_{2,1}} + \dots + \mathbf{v}^r_{A_{0,n}}\right] = \mathbf{0}.
\end{aligned} \tag{5.15}$$

The previous equation is written as

$$\sum_{(i)} \mathbf{r}_{A_i} \times \boldsymbol{\omega}_{i,i-1} + \sum_{(i)} \mathbf{v}^r_{A_{i,i-1}} = \mathbf{0}. \tag{5.16}$$

Equation (5.16) represents the second vectorial equation for the angular velocities of a simple closed kinematic chain.

Equations such as

$$\sum_{(i)} \boldsymbol{\omega}_{i,i-1} = \mathbf{0} \text{ and } \sum_{(i)} \mathbf{r}_{A_i} \times \boldsymbol{\omega}_{i,i-1} + \sum_{(i)} \mathbf{v}^r_{A_{i,i-1}} = \mathbf{0}, \tag{5.17}$$

represent the velocity equations for a simple closed kinematic chain.

5.2 Contour Acceleration Equations

The absolute angular acceleration, $\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_{i,0}$, of the rigid body (i) (or the angular acceleration of the rigid body (i) with respect to the ‘fixed” reference frame $Oxyz$) is

$$\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_{i-1} + \boldsymbol{\alpha}_{i,i-1} + \boldsymbol{\omega}_i \times \boldsymbol{\omega}_{i,i-1}, \tag{5.18}$$

where $\boldsymbol{\alpha}_{i-1} = \boldsymbol{\alpha}_{i-1,0}$ is the absolute angular acceleration of the rigid body $(i-1)$ [or the angular acceleration of the rigid body $(i-1)$ with respect to the "fixed" reference frame $Oxyz$] and $\boldsymbol{\alpha}_{i,i-1}$ is the relative angular acceleration of the rigid body (i) with respect to the rigid body $(i-1)$.

For the n link closed kinematic chain the following expressions are obtained for the angular accelerations:

$$\begin{aligned}
&\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_{2,1} + \boldsymbol{\omega}_2 \times \boldsymbol{\omega}_{2,1} \\
&\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_{3,2} + \boldsymbol{\omega}_3 \times \boldsymbol{\omega}_{3,2} \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_{i-1} + \boldsymbol{\alpha}_{i,i-1} + \boldsymbol{\omega}_i \times \boldsymbol{\omega}_{i,i-1} \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_{1,0} + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_{1,0}\ .
\end{aligned} \tag{5.19}$$

Summing all the expressions in Eq. (5.19):

$$\boldsymbol{\alpha}_{2,1} + \boldsymbol{\alpha}_{3,2} + ... + \boldsymbol{\alpha}_{1,0} + \boldsymbol{\omega}_2 \times \boldsymbol{\omega}_{2,1} + ... + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_{1,0} = \mathbf{0}. \tag{5.20}$$

Equation (5.20) is rewritten as

$$\sum_{(i)} \boldsymbol{\alpha}_{i,i-1} + \sum_{(i)} \boldsymbol{\omega}_i \times \boldsymbol{\omega}_{i,i-1} = \mathbf{0}. \tag{5.21}$$

Equation (5.21) represents the first vectorial equation for the angular accelerations of a simple closed kinematic chain.

Using the acceleration distributions of the relative motion of two rigid bodies (i) and $(i-1)$:

$$\mathbf{a}_{A_{i,i}} = \mathbf{a}_{A_{i,i-1}} + \mathbf{a}^r_{A_{i,i-1}} + \mathbf{a}^c_{A_{i,i-1}} \ , \tag{5.22}$$

where $\mathbf{a}_{A_{i,i}}$ and $\mathbf{a}_{A_{i,i-1}}$ are the linear accelerations of the points $A_{i,i}$ and $A_{i,i-1}$, and $\mathbf{a}^r_{A_{i,i-1}} = \mathbf{a}^r_{A_{i,i}A_{i,i-1}}$ is the relative acceleration between $A_{i,i}$ on link (i) and $A_{i,i-1}$ on link $(i-1)$. Finally, $\mathbf{a}^c_{A_{i,i-1}}$ is the Coriolis acceleration defined as

$$\mathbf{a}^c_{A_{i,i-1}} = 2\,\boldsymbol{\omega}_{i-1} \times \mathbf{v}^r_{A_{i,i-1}} \ . \tag{5.23}$$

Using the acceleration distribution relations for two particles on a rigid body:

$$\mathbf{a}_{A_{i+1,i}} = \mathbf{a}_{A_{i,i}} + \boldsymbol{\alpha}_i \times \mathbf{r}_{A_iA_{i+1}} + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}_{A_iA_{i+1}}), \tag{5.24}$$

where $\boldsymbol{\alpha}_i$ is the angular acceleration of the link (i). From Eqs. (5.22) and (5.24):

$$\mathbf{a}_{A_{i+1,i}} = \mathbf{a}_{A_{i,i-1}} + \mathbf{a}^r_{A_{i,i-1}} + \mathbf{a}^c_{A_{i,i-1}} + \boldsymbol{\alpha}_i \times \mathbf{r}_{A_iA_{i+1}} + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}_{A_iA_{i+1}}). \tag{5.25}$$

Writing similar equations for all the links of the kinematic chain, the following relations are obtained:

$$\begin{aligned}
\mathbf{a}_{A_{3,2}} &= \mathbf{a}_{A_{2,1}} + \mathbf{a}^r_{A_{2,1}} + \mathbf{a}^c_{A_{2,1}} + \boldsymbol{\alpha}_2 \times \mathbf{r}_{A_2A_3} + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{r}_{A_2A_3}),\\
\mathbf{a}_{A_{4,3}} &= \mathbf{a}_{A_{3,2}} + \mathbf{a}^r_{A_{3,2}} + \mathbf{a}^c_{A_{3,2}} + \boldsymbol{\alpha}_3 \times \mathbf{r}_{A_3A_4} + \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{r}_{A_3A_4}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\mathbf{a}_{A_{1,0}} &= \mathbf{a}_{A_{0,n}} + \mathbf{a}^r_{A_{0,n}} + \mathbf{a}^c_{A_{0,n}} + \boldsymbol{\alpha}_0 \times \mathbf{r}_{A_0A_1} + \boldsymbol{\omega}_0 \times (\boldsymbol{\omega}_0 \times \mathbf{r}_{A_0A_1}),\\
\mathbf{a}_{A_{2,1}} &= \mathbf{a}_{A_{1,0}} + \mathbf{a}^r_{A_{1,0}} + \mathbf{a}^c_{A_{1,0}} + \boldsymbol{\alpha}_1 \times \mathbf{r}_{A_1A_2} + \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{r}_{A_1A_2}).
\end{aligned} \tag{5.26}$$

Summing the expressions in Eq. (5.26):

$$\begin{gathered}
\left[\mathbf{a}^r_{A_{1,0}} + \mathbf{a}^r_{A_{2,1}} + \ldots + \mathbf{a}^r_{A_{0,n}}\right] + \left[\mathbf{a}^c_{A_{1,0}} + \mathbf{a}^c_{A_{2,1}} + \ldots + \mathbf{a}^c_{A_{0,n}}\right] +\\
\left[\boldsymbol{\alpha}_1 \times \mathbf{r}_{A_1A_2} + \boldsymbol{\alpha}_2 \times \mathbf{r}_{A_2A_3} + \ldots + \boldsymbol{\alpha}_0 \times \mathbf{r}_{A_0A_1}\right] +\\
\boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{r}_{A_1A_2}) + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{r}_{A_2A_3}) + \ldots +\\
\boldsymbol{\omega}_0 \times (\boldsymbol{\omega}_0 \times \mathbf{r}_{A_0A1}) = \mathbf{0}.
\end{gathered} \tag{5.27}$$

Using the relation $\mathbf{r}_{A_{i-1}A_i} = \mathbf{r}_{A_i} - \mathbf{r}_{A_{i-1}}$ in Eq. (5.27):

$$\left[\mathbf{a}^r_{A_{1,0}} + \mathbf{a}^r_{A_{2,1}} + ... + \mathbf{a}^r_{A_{0,n}}\right] + \left[\mathbf{a}^c_{A_{1,0}} + \mathbf{a}^c_{A_{2,1}} + ... + \mathbf{a}^c_{A_{0,n}}\right] +$$
$$[\mathbf{r}_{A_1} \times (\boldsymbol{\alpha}_{1,0} + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_{1,0}) + ... + \mathbf{r}_{A_0} \times (\boldsymbol{\alpha}_{0,n} + \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_{0,n})] +$$
$$\boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{r}_{A_1A_2}) + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{r}_{A_2A_3}) + ...+$$
$$\boldsymbol{\omega}_0 \times (\boldsymbol{\omega}_0 \times \mathbf{r}_{A_0A_1}) = \mathbf{0}. \tag{5.28}$$

Equation (5.28) is rewritten as

$$\sum_{(i)} \mathbf{a}^r_{A_{i,i-1}} + \sum_{(i)} \mathbf{a}^c_{A_{i,i-1}} + \sum_{(i)} \mathbf{r}_{A_i} \times (\boldsymbol{\alpha}_{i,i-1} + \boldsymbol{\omega}_i \times \boldsymbol{\omega}_{i,i-1}) +$$
$$\sum_{(i)} \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}_{A_iA_{i+1}}) = \mathbf{0}. \tag{5.29}$$

Equation (5.29) represents the second vectorial equation for the angular accelerations of a simple closed kinematic chain. Thus,

$$\sum_{(i)} \boldsymbol{\alpha}_{i,i-1} + \sum_{(i)} \boldsymbol{\omega}_i \times \boldsymbol{\omega}_{i,i-1} = \mathbf{0} \text{ and}$$
$$\sum_{(i)} \mathbf{r}_{A_i} \times (\boldsymbol{\alpha}_{i,i-1} + \boldsymbol{\omega}_i \times \boldsymbol{\omega}_{i,i-1}) + \sum_{(i)} \mathbf{a}^r_{A_{i,i-1}} + \sum_{(i)} \mathbf{a}^c_{A_{i,i-1}} +$$
$$\sum_{(i)} \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}_{A_iA_{i+1}}) = \mathbf{0}. \tag{5.30}$$

are the acceleration equations for the case of a simple closed kinematic chain.
Remarks
1. For a closed kinematic chain in planar motion, simplified relations are obtained because

$$\boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}_{A_iA_{i+1}}) = -\omega_i^2 \mathbf{r}_{A_iA_{i+1}} \text{ and } \boldsymbol{\omega}_i \times \boldsymbol{\omega}_{i,i-1} = \mathbf{0}. \tag{5.31}$$

Equations

$$\sum_{(i)} \boldsymbol{\alpha}_{i,i-1} = \mathbf{0} \text{ and}$$
$$\sum_{(i)} \mathbf{r}_{A_i} \times \boldsymbol{\alpha}_{i,i-1} + \sum_{(i)} \mathbf{a}^r_{A_{i,i-1}} + \sum_{(i)} \mathbf{a}^c_{A_{i,i-1}} - \omega_i^2 \mathbf{r}_{A_iA_{i+1}} = \mathbf{0}. \tag{5.32}$$

represent the acceleration equations for a simple closed kinematic chain in planar motion.
2. The Coriolis acceleration, given by the expression

$$\mathbf{a}^c_{A_{i,i-1}} = 2\boldsymbol{\omega}_{i-1} \times \mathbf{v}^r_{A_{i,i-1}} \tag{5.33}$$

vanishes when $\boldsymbol{\omega}_{i-1} = \mathbf{0}$, or $\mathbf{v}^r_{A_{i,i-1}} = \mathbf{0}$, or when $\boldsymbol{\omega}_{i-1}$ is parallel to $\mathbf{v}^r_{A_{i,i-1}}$.

5.3 Independent Contour Equations

A diagram is used to represent to a mechanism in the following way: the numbered links are the nodes of the diagram and are represented by circles, and the joints are represented by lines which connect the nodes.

Figure 5.3 shows the diagram that represents a planar mechanism. The maximum number of independent contours is given by

$$N = c - n \quad \text{or} \quad n_c = N = c - p + 1, \tag{5.34}$$

where c is the number of joints, n is the number of moving links, and p is the number of links.

The equations for velocities and accelerations are written for any closed contour of the mechanism. However, it is best to write the contour equations only for the independent loops of the diagram representing the mechanism.

Step 1. Determine the position analysis of the mechanism.

Step 2. Draw a diagram representing the mechanism and select the independent contours. Determine a path for each contour.

Step 3. For each closed loop write the contour velocity relations [Eq. (5.17)] and contour acceleration relations [Eq. (5.30)]. For a closed kinematic chain in planar motion the following equations will be used:

$$\sum_{(i)} \boldsymbol{\omega}_{i,i-1} = \mathbf{0},$$
$$\sum_{(i)} \mathbf{r}_{A_i} \times \boldsymbol{\omega}_{i,i-1} + \sum_{(i)} \mathbf{v}^r_{A_{i,i-1}} = \mathbf{0}. \tag{5.35}$$

$$\sum_{(i)} \boldsymbol{\alpha}_{i,i-1} = \mathbf{0},$$
$$\sum_{(i)} \mathbf{r}_{A_i} \times \boldsymbol{\alpha}_{i,i-1} + \sum_{(i)} \mathbf{a}^r_{A_{i,i-1}} + \sum_{(i)} \mathbf{a}^c_{A_{i,i-1}} - \omega_i^2 \mathbf{r}_{A_i A_{i+1}} = \mathbf{0}. \tag{5.36}$$

Step 4. Project on a cartesian reference system the velocity and acceleration equations. Linear algebraic equations are obtained where the unknowns are

- the components of the relative angular velocities $\boldsymbol{\omega}_{j,j-1}$;
- the components of the relative angular accelerations $\boldsymbol{\alpha}_{j,j-1}$;
- the components of the relative linear velocities $\mathbf{v}^r_{Aj,j-1}$;
- the components of the relative linear accelerations $\mathbf{a}^r_{Aj,j-1}$.

Solve the algebraic system of equations and determine the unknown kinematic parameters.

Step 5. Determine the absolute angular velocities $\boldsymbol{\omega}_j$ and the absolute angular accelerations $\boldsymbol{\alpha}_j$. Compute the velocities and accelerations of the characteristic points and joints.

In the following examples, the contour method is applied to determine the velocities and accelerations distribution for several planar mechanisms. The following notation will be used:

$\boldsymbol{\omega}_{ij}$ is the relative angular velocity vector of the link i with respect to the link j. When the link j is the ground (denoted as link 0), then $\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i0}$ also denotes the absolute angular velocity vector of the link i. The magnitude of $\boldsymbol{\omega}_{ij}$ is ω_{ij}, i.e., $|\boldsymbol{\omega}_{ij}| = \omega_{ij}$.

$\mathbf{v}^r_{A_{ij}}$ is the relative linear velocity of the point A_i on link i with respect to the point A_j on link j. The point A_i belonging to link j, and the point A_j belonging to link j, are coincident at the instant of motion under consideration.

$\boldsymbol{\alpha}_{ij}$ is the relative angular acceleration vector of the link i with respect to the rigid body j. When the link j is the ground, then $\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_{i0}$ also denotes the absolute angular acceleration vector of the rigid body i.

$\mathbf{a}^r_{A_{ij}}$ is the relative linear acceleration vector of A_i on link i with respect to A_j on link j.

$\mathbf{a}^c_{A_{ij}}$ is the Coriolis acceleration of A_i with respect to A_j.

$\mathbf{r}_{BC}$ denotes a vector from the joint B to the joint C.

x_B, y_B, z_B denote the coordinates of the point B with respect to the fixed reference frame.

$\mathbf{v}_B$ denotes the linear velocity vector of the point B with respect to the fixed reference frame.

$\mathbf{a}_B$ denotes the linear acceleration vector of the point B with respect to the fixed reference frame.

5.4 Example

The planar mechanism considered in this example is depicted in Fig. 5.4(a). The following data are given: $AC = 0.100$ m, $BC = 0.300$ m, $BD = 0.900$ m, and $L_a = 0.100$ m. The angle of the driver element (link AB) with the horizontal axis is $\phi = 45°$. A Cartesian reference frame with the origin at A ($x_A = y_A = 0$) is selected. The coordinates of joint C are $x_C = AC$, $y_C = 0$. The coordinates of joint B are $x_B = 0.256\,\text{m}$, $y_B = 0.256\,\text{m}$. The coordinates of joint D are $x_D = 1.142$ m, $y_D = 0.100$ m. The position vectors $\mathbf{r}_{AB}$, $\mathbf{r}_{AC}$, and $\mathbf{r}_{AD}$ are defined as follows:

$$\begin{aligned}
\mathbf{r}_{AB} &= x_B\ \imath + y_B\ \jmath = 0.256\ \imath + 0.256\ \jmath,\\
\mathbf{r}_{AC} &= x_C\ \imath + y_C\ \jmath = 0.100\ \imath,\\
\mathbf{r}_{AD} &= x_D\ \imath + y_D\ \jmath = 1.142\ \imath + 0.100\ \jmath.
\end{aligned}$$

The angular velocity of the driver link is $n_1 = 100$ rpm, or

$$\omega_{10} = \omega_1 = n\frac{\pi}{30} = 100\frac{\pi}{30}\,\text{rad/s} = 10.472\,\text{rad/s}.$$

The mechanism has six links and seven full joints. Using Eq. (5.34), the number of independent loops is given by

$$n_c = l - p + 1 = 7 - 6 + 1 = 2.$$

This mechanism has two independent contours. The first contour I contains the links 0, 1, 2, and 3, while the second contour II contains the links 0, 3, 4, and 5. The diagram representing the mechanism is given in Fig.5.4(b). Clockwise paths are chosen for each closed loop I and II.

First contour

According to Fig. 5.5, the first contour has

- rotational joint R between links 0 and 1 (joint A);
- translational joint T between links 1 and 2 (joint B_{T});
- rotational joint R between links 2 and 3 (joint B_{R});

• rotational joint R between links 3 and 0 (joint C).
For the velocity analysis, the following equations are written using Eq. (5.35):

$$\boldsymbol{\omega}_{10} + \boldsymbol{\omega}_{32} + \boldsymbol{\omega}_{03} = \mathbf{0},$$
$$\mathbf{r}_{AB} \times \boldsymbol{\omega}_{32} + \mathbf{r}_{AC} \times \boldsymbol{\omega}_{03} + \mathbf{v}^r_{B_{21}} = \mathbf{0}, \tag{5.37}$$

where $\boldsymbol{\omega}_{10} = \omega_{10}\,\mathbf{k} = 10.47\,\mathbf{k}$ rad/s, $\boldsymbol{\omega}_{32} = \omega_{32}\,\mathbf{k}$, and $\boldsymbol{\omega}_{03} = \omega_{03}\,\mathbf{k}$.
The relative velocity of B_2 on link 2 with respect to B_1 on link 1, $\mathbf{v}^r_{B_{21}}$, has $\imath$ and $\jmath$ components:

$$\mathbf{v}^r_{B_{21}} = v^r_{B_{21x}}\ \imath + v^r_{B_{21y}}\ \jmath = v^r_{B_{21}} \cos\phi\ \imath + v^r_{B_{21}} \sin\phi\ \jmath,$$

where $v^r_{B_{21}}$ is the magnitude of the vector $\mathbf{v}^r_{B_{21}}$ i.e. $|\mathbf{v}^r_{B+21}| = v^r_{B_{21}}$. The sign of the unknown relative velocities is selected as positive as shown in Figs. 5.4(a) and 5.5(b). The numerical computation will give the correct orientation of the unknown vectors. The unknowns in Eq. (5.37) are ω_{32}, ω_{03}, and $v^r_{B_{21}}$. Equation (5.37) becomes

$$\omega_{10}\,\mathbf{k} + \omega_{32}\,\mathbf{k} + \omega_{03}\,\mathbf{k} = \mathbf{0},$$
$$\begin{vmatrix} \imath & \jmath & \mathbf{k} \\ x_B & y_B & 0 \\ 0 & 0 & \omega_{32} \end{vmatrix} + \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ x_C & y_C & 0 \\ 0 & 0 & \omega_{03} \end{vmatrix} +$$
$$v^r_{B_{21}} \cos\phi\ \imath + v^r_{B_{21}} \sin\phi\ \jmath = \mathbf{0}. \tag{5.38}$$

Equation (5.38) is projected onto the “fixed” reference frame $Oxyz$:

$$\omega_{10} + \omega_{32} + \omega_{03} = 0,$$
$$y_B\,\omega_{32} + y_C\,\omega_{03} + v^r_{B_{21}} \cos\phi = 0,$$
$$-x_B\,\omega_{32} - x_C\,\omega_{03} + v^r_{B_{21}} \sin\phi = 0, \tag{5.39}$$

or numerically as

$$10.472 + \omega_{32} + \omega_{03} = 0,$$
$$0.256\,\omega_{32} + v^r_{B_{21}} \cos 45^\circ = 0,$$
$$-0.256\,\omega_{32} - 0.100\,\omega_{03} + v^r_{B_{21}} \sin 45^\circ = 0. \tag{5.40}$$

Equation (5.40) represents a system of three equations with three unknowns: ω_{32}, ω_{03}, and $v^r_{B_{21}}$. Solving the algebraic equations, the following numerical values are obtained: $\omega_{32} = 2.539$ rad/s, $\omega_{03} = -13.011$ rad/s, and $v^r_{B_{21}} =$

-0.920 m/s.
The absolute angular velocity of link 3 is

$$\boldsymbol{\omega}_{30} = -\boldsymbol{\omega}_{03} = 13.011\,\mathbf{k}\,\text{rad/s}. \tag{5.41}$$

The velocity of point $B_2 = B_3$ is computed with the expression of velocity field of two points (B_3 and C) on the same rigid body (link 3):

$$\mathbf{v}_{B_2} = \mathbf{v}_{B_3} = \mathbf{v}_C + \boldsymbol{\omega}_{30} \times \mathbf{r}_{CB} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_{30} \\ x_B - x_C & y_B - y_C & 0 \end{vmatrix} =$$

$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & 13.011 \\ 0.256 - 0.100 & 0.256 & 0 \end{vmatrix} = -3.333\ \mathbf{\imath} + 2.032\ \mathbf{\jmath}\,\text{m/s},$$

where $\mathbf{v}_C = \mathbf{0}$ because joint C is grounded.
Link 2 and the driver link 1 have the same angular velocity:

$$\boldsymbol{\omega}_{10} = \boldsymbol{\omega}_{20} = \boldsymbol{\omega}_{30} + \boldsymbol{\omega}_{23} = 13.011\mathbf{k} - 2.539\mathbf{k} = 10.472\mathbf{k}\,\text{rad/s}.$$

The velocity of point B_1 on link 1 is calculated with the expression of velocity field of two points (B_1 and A) on the same rigid body (link 1):

$$\mathbf{v}_{B_1} = \mathbf{v}_A + \boldsymbol{\omega}_{10} \times \mathbf{r}_{AB} = \boldsymbol{\omega}_{10} \times \mathbf{r}_{AB} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_{10} \\ x_B & y_B & 0 \end{vmatrix} =$$

$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & 10.472 \\ 0.256 & 0.256 & 0 \end{vmatrix} = -2.682\ \mathbf{\imath} + 2.682\ \mathbf{\jmath}\,\text{m/s}.$$

Another way of calculating the velocity of the point $B_2 = B_3$ is with the help of velocity field of two points (B_1 and B_2) not situated on the same rigid body (B_1 is on link 1 and B_2 is on link 2):

$$\mathbf{v}_{B_2} = \mathbf{v}_{B_1} + \mathbf{v}^r_{B_{21}},$$

where $\mathbf{v}^r_{B_{21}} = v^r_{B_{21}} \cos\phi\ \mathbf{\imath} + v^r_{B_{21}} \sin\phi\ \mathbf{\jmath} = -0.651\ \mathbf{\imath} - 0.651\ \mathbf{\jmath}\ \text{m/s}$.

For the acceleration analysis, the following equations are written using Eq. (5.36):

$$\begin{aligned} &\boldsymbol{\alpha}_{10} + \boldsymbol{\alpha}_{32} + \boldsymbol{\alpha}_{03} = \mathbf{0}, \\ &\mathbf{r}_{AB} \times \boldsymbol{\alpha}_{32} + \mathbf{r}_{AC} \times \boldsymbol{\alpha}_{03} + \mathbf{a}_{B_{21}} + \mathbf{a}^c_{B_{21}} - \omega_{10}^2\mathbf{r}_{AB} - \omega_{30}^2\mathbf{r}_{BC} = \mathbf{0}, \end{aligned} \tag{5.42}$$

where $\boldsymbol{\alpha}_{10} = \dot{\omega}_{10}\,\mathbf{k} = \mathbf{0}$, $\boldsymbol{\alpha}_{32} = \alpha_{32}\,\mathbf{k}$, and $\boldsymbol{\alpha}_{03} = \alpha_{03}\,\mathbf{k}$.
The relative acceleration of B_2 on link 2 with respect to B_1 on link 1, $\mathbf{a}^r_{B_{21}}$, has $\imath$ and $\jmath$ components:

$$\mathbf{a}^r_{B_{21}} = a^r_{B_{21x}}\,\imath + a^r_{B_{21y}}\,\jmath = a^r_{B_{21}}\cos\phi\,\imath + a^r_{B_{21}}\sin\phi\,\jmath.$$

The sign of the unknown relative accelerations is selected positive and then the numerical computation will give the correct orientation of the unknown acceleration vectors. The expression for the Coriolis acceleration is

$$\mathbf{a}^c_{B_{21}} = 2\,\boldsymbol{\omega}_{10} \times \mathbf{v}^r_{B_{21}} = 2\,\boldsymbol{\omega}_{20} \times \mathbf{v}^r_{B_{21}} =$$
$$2\begin{vmatrix} \imath & \jmath & \mathbf{k} \\ 0 & 0 & \omega_{10} \\ v^r_{B_{21}}\cos\phi & v^r_{B_{21}}\sin\phi & 0 \end{vmatrix} = -2v^r_{B_{21}}\omega_{10}\sin\phi\,\imath + 2v^r_{B_{21}}\omega_{10}\cos\phi\,\jmath =$$
$$-2(-0.920)(10.472)\sin 45^\circ\,\imath + 2(-0.920)(10.472)\cos 45^\circ\,\jmath =$$
$$13.629\,\imath - 13.629\,\jmath \ \text{m/s}^2.$$

The unknowns in Eq. (5.42) are α_{32}, α_{03}, and $a^r_{B_{21}}$. Equation (5.42) becomes

$$\alpha_{32}\,\mathbf{k} + \alpha_{03}\,\mathbf{k} = \mathbf{0},$$
$$\begin{vmatrix} \imath & \jmath & \mathbf{k} \\ x_B & y_B & 0 \\ 0 & 0 & \alpha_{32} \end{vmatrix} + \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ x_C & y_C & 0 \\ 0 & 0 & \alpha_{03} \end{vmatrix} + a^r_{B_{21}}\cos\phi\,\imath + a^r_{B_{21}}\sin\phi\,\jmath +$$
$$\mathbf{a}^c_{B_{21}} - \omega^2_{10}(x_B\,\imath + y_B\,\jmath) - \omega^2_{30}[(x_C - x_B)\,\imath + (y_C - y_B)\,\jmath] = \mathbf{0}.$$

The previous equations are projected onto the “fixed” reference frame $Oxyz$:

$$\alpha_{32} + \alpha_{03} = 0,$$
$$y_B\,\alpha_{32} + y_C\,\alpha_{03} + a^r_{B_{21}}\cos\phi - 2v^r_{B_{21}}\omega_{10}\sin\phi - \omega^2_{10}x_B - \omega^2_{30}(x_C - x_B) = 0,$$
$$-x_B\,\alpha_{32} - x_C\,\alpha_{03} + a^r_{B_{21}}\sin\phi + 2v^r_{B_{21}}\omega_{10}\cos\phi - \omega^2_{10}y_B - \omega^2_{30}(y_C - y_B) = 0,$$

or numerically as

$$\alpha_{32} + \alpha_{03} = 0,$$
$$0.256\alpha_{32} + a^r_{B_{21}}\cos 45^\circ + 13.626 -$$
$$(10.472)^2(0.256) - (13.011)^2(0.100 - 0.256) = 0,$$
$$-0.256\alpha_{32} - 0.100\alpha_{03} + a^r_{B_{21}}\sin 45^\circ - 13.626 -$$
$$(10.472)^2(0.256) - (13.011)^2(0 - 0.256) = 0. \qquad (5.43)$$

Equation (5.43) represents a system of three equations with three unknowns: α_{32}, α_{03}, and $a^r_{B_{21}}$. Solving the algebraic equations, the following numerical values are obtained: $\alpha_{32} = -25.032\,\text{rad/s}^2$, $\alpha_{03} = 25.032\,\text{rad/s}^2$, and $a^r_{B_{21}} = -7.865\,\text{m/s}^2$.
The absolute angular acceleration of link 3 is

$$\alpha_{30} = -\alpha_{03} = -25.032\,\mathbf{k}\ \text{rad/s}^2.$$

The velocity of the point $B_2 = B_3$ is computed with the expression of velocity field of two points (B_3 and C) on the same rigid body (link 3):

$$\mathbf{v}_{B_2} = \mathbf{v}_{B_3} = \mathbf{v}_C + \boldsymbol{\omega}_{30} \times \mathbf{r}_{CB} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_{30} \\ x_B - x_C & y_B - y_C & 0 \end{vmatrix} =$$
$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & 13.011 \\ 0.256 - 0.100 & 0.256 & 0 \end{vmatrix} = -3.333\ \mathbf{\imath} + 2.032\ \mathbf{\jmath}\,\text{m/s},$$

where $\mathbf{v}_C = \mathbf{0}$ because joint C is grounded.
Link 2 and driver link 1 have the same angular velocity:

$$\boldsymbol{\omega}_{10} = \boldsymbol{\omega}_{20} = \boldsymbol{\omega}_{30} + \boldsymbol{\omega}_{23} = 13.011\mathbf{k} - 2.539\mathbf{k} = 10.472\mathbf{k}\,\text{rad/s}.$$

The velocity of point B_1 on link 1 is calculated with the expression of velocity field of two points (B_1 and A) on the same rigid body (link 1):

$$\mathbf{v}_{B_1} = \mathbf{v}_A + \boldsymbol{\omega}_{10} \times \mathbf{r}_{AB} = \boldsymbol{\omega}_{10} \times \mathbf{r}_{AB} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & \omega_{10} \\ x_B & y_B & 0 \end{vmatrix} =$$
$$\begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ 0 & 0 & 10.472 \\ 0.256 & 0.256 & 0 \end{vmatrix} = -2.682\ \mathbf{\imath} + 2.682\ \mathbf{\jmath}\,\text{m/s}.$$

Another way of calculating the velocity of the point $B_2 = B_3$ is with the help of velocity field of two points (B_1 and B_2) not situated on the same rigid body (B_1 is on link 1 and B_2 is on link 2):

$$\mathbf{v}_{B_2} = \mathbf{v}_{B_1} + \mathbf{v}^r_{B_{21}},$$

where $\mathbf{v}^r_{B_{21}} = v^r_{B_{21}} \cos\phi\ \mathbf{\imath} + v^r_{B_{21}} \sin\phi\ \mathbf{\jmath} = -0.651\ \mathbf{\imath} - 0.651\ \mathbf{\jmath}\ \ \text{m/s}$.
The angular acceleration of link 3 is:

$$\boldsymbol{\alpha}_{30} = -\boldsymbol{\alpha}_{03} = \boldsymbol{\alpha}_{32} = -25.032\,\mathbf{k}\,\text{rad/s}^2.$$

The absolute linear acceleration of point B_3 is computed as follows:

$$\mathbf{a}_{B_3} = \mathbf{a}_C + \boldsymbol{\alpha}_{30} \times \mathbf{r}_{CB} - \omega_{30}^2 \mathbf{r}_{CB} = -20.026\ \boldsymbol{\imath} - 47.277\ \boldsymbol{\jmath}\ \mathrm{m/s}^2.$$

Second contour analysis

According to Fig. 5.6, the second contour is described as

- rotational joint R between links 0 and 3 (joint C);
- rotational joint R between links 3 and 4 (joint B) ;
- rotational joint R between links 4 and 5 (joint D_{R});
- translational joint T between links 5 and 0 (joint D_{T}).

For the velocity analysis, the following equations are written:

$$\begin{aligned} &\boldsymbol{\omega}_{30} + \boldsymbol{\omega}_{43} + \boldsymbol{\omega}_{54} = \mathbf{0}, \\ &\mathbf{r}_{AC} \times \boldsymbol{\omega}_{30} + \mathbf{r}_{AB} \times \boldsymbol{\omega}_{43} + \mathbf{r}_{AD} \times \boldsymbol{\omega}_{54} + \mathbf{v}^r_{D_{05}} = \mathbf{0}. \end{aligned} \tag{5.44}$$

The relative linear velocity $\mathbf{v}^r_{D_{05}}$ has only one component, along the x axis:

$$\mathbf{v}^r_{D_{05}} = v^r_{D_{05}}\ \boldsymbol{\imath}.$$

The unknown parameters in Eq. (5.44) are ω_{43}, ω_{54}, and $v^r_{D_{05}}$. The following numerical values are obtained $\omega_{43} = -15.304\,\mathrm{rad/s}$, $\omega_{54} = 2.292\,\mathrm{rad/s}$, and $v^r_{D_{05}} = 3.691\,\mathrm{m/s}$.

The angular velocity of the link BD is

$$\boldsymbol{\omega}_{40} = \boldsymbol{\omega}_{30} + \boldsymbol{\omega}_{43} = -\boldsymbol{\omega}_{54} = -2.292\,\mathbf{k}\,\mathrm{rad/s}.$$

The absolute linear velocity of the point $D_4 = D_5$ is computed as follows:

$$\mathbf{v}_{D_4} = \mathbf{v}_{D_5} = \mathbf{v}_{B_4} + \boldsymbol{\omega}_{40} \times \mathbf{r}_{BD} = -\mathbf{v}^r_{D_{05}} = -3.691\ \boldsymbol{\imath}\,\mathrm{m/s},$$

where $\mathbf{v}_{B_4} = \mathbf{v}_{B_3}$.

For the acceleration analysis the following equations exist:

$$\begin{aligned} &\boldsymbol{\alpha}_{30} + \boldsymbol{\alpha}_{43} + \boldsymbol{\alpha}_{54} = \mathbf{0}, \\ &\mathbf{a}^r_{D_{05}} + \mathbf{a}^c_{D_{05}} + \mathbf{r}_{AC} \times \boldsymbol{\alpha}_{30} + \mathbf{r}_{AB} \times \boldsymbol{\alpha}_{43} + \mathbf{r}_{AD} \times \boldsymbol{\alpha}_{54} - \\ &\omega_{30}^2 \mathbf{r}_{CB} - \omega_{40}^2 \mathbf{r}_{BD} = \mathbf{0}. \end{aligned} \tag{5.45}$$

Because the slider 5 does not rotate ($\boldsymbol{\omega}_{50} = \mathbf{0}$), the Coriolis acceleration is

$$\mathbf{a}^c_{D_{05}} = 2\boldsymbol{\omega}_{50} \times \mathbf{v}^r_{D_{05}} = \mathbf{0}.$$

The unknowns in Eq. (5.45) are α_{43}, α_{54}, and a^r_{D05}. The following numerical results are obtained: $\alpha_{43} = 77.446\,\text{rad/s}^2$, $\alpha_{54} = -52.414\,\text{rad/s}^2$, and $a^r_{D05} = 16.499\,\text{m/s}^2$.
The absolute angular acceleration of the link BD is

$$\boldsymbol{\alpha}_{40} = \boldsymbol{\alpha}_{30} + \boldsymbol{\alpha}_{43} = \boldsymbol{\alpha}_{45} = 52.414\,\mathbf{k}\,\text{rad/s}^2,$$

and the linear acceleration of the point $D_4 = D_5$ is

$$\mathbf{a}_{D_4} = \mathbf{a}_{D_5} = \mathbf{a}_{B_4} + \boldsymbol{\alpha}_{40} \times \mathbf{r}_{BD} - \omega_{40}^2 \mathbf{r}_{BD} = -16.499\ \boldsymbol{\imath}\,\text{m/s}^2,$$

where $\mathbf{a}_{B_4} = \mathbf{a}_{B_3}$.

5.5 Problems

5.1 The four-bar mechanism shown in Fig. 3.10(a) has the dimensions: $AB = 80$ mm, $BC = 210$ mm, $CD = 120$ mm, and $AD = 190$ mm. The driver link AB rotates with a constant angular speed of 200 rpm. Find the velocities and the accelerations of the four-bar mechanism using the contour equations method for the case when the angle of the driver link AB with the horizontal axis is $\phi = 60°$.

5.2 The angular speed of the driver link 1, of the mechanism shown in Fig. 4.9, is $\omega = \omega_1 = 20$ rad/s. The distance from link 3 to the horizontal axis Ax is $a = 55$ mm. Using the contour equations find the velocity and the acceleration of point C on link 3 for $\phi = 30°$.

5.3 The slider crank mechanism shown in Fig. 4.10 has the dimensions $AB = 0.4$ m and $BC = 1$ m. The driver link 1 rotates with a constant angular speed of $n = 160$ rpm. Find the velocity and acceleration of the slider 3 using the contour equations when the angle of the driver link with the horizontal axis is $\phi = 30°$.

5.4 The planar mechanism considered is shown in Fig. 3.19. The following data are given: $AB = 0.150$ m, $BC = 0.400$ m, $CD = 0.370$ m, $CE = 0.230$ m, $EF = CE$, $L_a = 0.300$ m, $L_b = 0.450$ m, and $L_c = CD$. The angular speed of the driver link 1 is constant and has the value 180 rpm. Using the contour equations method, find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 30°$.

5.5 The R-RRR-RTT mechanism is shown in Fig. 3.20. The following data are given: $AB = 0.080$ m, $BC = 0.350$ m, $CE = 0.200$ m, $CD = 0.150$ m, $L_a = 0.200$ m, $L_b = 0.350$ m, and $L_c 0.040$ m. The driver link 1 rotates with a constant angular speed of $n = 1200$ rpm. For $\phi = 145°$ find the velocities and the accelerations of the mechanism with the contour equations.

5.6 The mechanism shown in Fig. 3.21 has the following dimensions: $AB =$ 80 mm, $AD = 250$ mm, $BC = 180$ mm, $CE = 60$ mm, $EF = 200$ mm, and $a = 170$ mm. The constant angular speed of the driver link 1 is $n =$ 400 rpm. Find the velocities and the accelerations of the mechanism using the contour equations when the angle of the driver link 1 with the horizontal axis is $\phi = \phi_1 = 300°$.

5.7 The dimensions for the mechanism shown in Fig. 3.22 are: $AB =$ 150 mm, $BD =$ 400 mm, $BC =$ 140 mm, $CD =$ 400 mm, $DE =$ 250 mm, $CF =$ 500 mm, $AE =$ 380 mm, and $b =$ 100 mm. The constant angular speed of the driver link 1 is $n = 40$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 210°$. Use the contour equations method.

5.8 The mechanism in Fig. 3.23 has the dimensions: $AB =$ 200 mm, $AC =$ 100 mm, $BD =$ 400 mm, $DE =$ 550 mm, $EF =$ 300 mm, $L_a =$ 500 mm, and $L_b =$ 100 mm. Using the contour equations method find the velocities and the accelerations of the mechanism if the constant angular speed of the driver link 1 is $n = 70$ rpm, and for $\phi = \phi_1 = 210°$.

5.9 The dimensions for the mechanism shown in Fig. 3.24 are: $AB =$ 150 mm, $BC =$ 400 mm, $AD =$ 360 mm, $CD =$ 210 mm, $DE =$ 130 mm, $EF =$ 400 mm, and $L_a =$ 40 mm. The constant angular speed of the driver link 1 is $n = 250$ rpm. Using the contour equations find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 30°$.

5.10 The mechanism in Fig. 3.25 has the dimensions: $AB =$ 250 mm, $AC =$ 800 mm, $BD =$ 1200 mm, $L_a =$ 180 mm, and $L_b =$ 300 mm. The driver link 1 rotates with a constant angular speed of $n = 50$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 210°$. Use the contour equations method.

5.11 Figure 3.26 shows a mechanism with the following dimensions: $AB =$ 120 mm, $BD =$ 400 mm, and $L_a =$ 150 mm. The constant angular speed of the driver link 1 is $n = 600$ rpm. Find the velocities and the accelerations of the mechanism, using the contour equations method, when the angle of the driver link 1 with the horizontal axis is $\phi = 210°$.

5.12 The mechanism in Fig. 3.27 has the dimensions: $AB =$ 200 mm, $AC =$ 500 mm, $BD =$ 800 mm, $DE =$ 400 mm, $EF =$ 270 mm, $L_a =$ 70 mm, and $L_b =$ 300 mm. The constant angular speed of the driver link 1 is $n = 40$ rpm. Using the contour equations, find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 300°$.

5.13 Figure 3.28 shows a mechanism with the following dimensions: $AB =$ 200 mm, $BC =$ 750 mm, $CD = DE =$ 300 mm, $EF =$ 500 mm, $L_a =$ 750 mm, and $L_b = L_c =$ 250 mm. Find the velocities and the accelerations of the mechanism, using the contour equations method, if the constant angular speed of the driver link 1 is $n = 1100$ rpm and for $\phi = \phi_1 = 120°$.

5.14 Figure 3.29 shows a mechanism with the following dimensions: $AB =$ 120 mm, $BC = 550$ mm, $CE = 180$ mm, $CD = 350$ mm, $EF =$ 300 mm, $L_a = 320$ mm, $L_b = 480$ mm, and $L_c = 600$ mm. The constant angular speed of the driver link 1 is $n = 100$ rpm. Find the velocities and the accelerations of the mechanism, using the contour equations, for $\phi = \phi_1 = 30°$.

5.15 Figure 3.30 shows a mechanism with the following dimensions: $AB =$ 180 mm, $BC = 520$ mm, $CF = 470$ mm, $CD = 165$ mm, $DE =$ 540 mm, $L_a = 630$ mm, $L_b = 360$ mm, and $L_c = 210$ mm. The constant angular speed of the driver link 1 is $n = 70$ rpm. Use the contour equations to calculate the velocities and the accelerations of the mechanism when the angle of the driver link 1 with the horizontal axis is $\phi = 210°$.

5.16 Figure 3.31 shows a mechanism with the following dimensions: $AB =$ 60 mm, $BC = 150$ mm, $AD = 70$ mm, and $BE = 170$ mm. The constant angular speed of the driver link 1 is $n = 300$ rpm. Find the velocities and the accelerations of the mechanism, using the contour equations, if the angle of the driver link 1 with the horizontal axis is $\phi = 210°$.

5.17 The dimensions of the mechanism shown in Fig. 3.32 are: $AB = 90$ mm, $BC = 240$ mm, $BE = 400$ mm, $CE = 600$ mm, $CD = 220$ mm, $EF = 900$ mm, $L_a = 250$ mm, $L_b = 150$ mm, and $L_c = 100$ mm. The constant angular speed of the driver link 1 is $n = 50$ rpm. Employing the contour equations find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 210°$.

5.18 The dimensions of the mechanism shown in Fig. 3.33 are: $AB =$ 180 mm, $AC = 300$ mm, $CD = 400$ mm, $DE = 200$ mm, and $L_a = 360$ mm. The constant angular speed of the driver link 1 is $n = 90$ rpm. Use the contour equations to calculate the velocities and the accelerations of the mechanism when the angle of the driver link 1 with the horizontal axis is $\phi = 30°$.

??.19 The dimensions of the mechanism shown in Fig. 3.34 are: $AB = 80$ mm, $AC = 40$ mm, $CD = 100$ mm, and $DE = 300$ mm. The constant angular speed of the driver link 1 is $n = 60$ rpm. Use the contour equations to calculate the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 210°$.

5.20 The dimensions of the mechanism shown in Fig. 3.35 are: $AB = 200$ mm, $AC = 350$ mm, and $CD = 600$ mm. For the distance b select a suitable value. The constant angular speed of the driver link 1 is $n = 90$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 120°$.

5.21 The dimensions of the mechanism shown in Fig. 3.36 are: $AB = 80$ mm, $AC = 60$ mm, and $CD = 70$ mm. The constant angular speed of the driver link 1 is $n = 220$ rpm. Find the velocities and the accelerations of the mechanism, using the contour equations, for $\phi = \phi_1 = 240°$.

??.22 The dimensions of the mechanism shown in Fig. 3.37 are: $AB = 150$ mm, $AC = 420$ mm, $BD = L_a = 650$ mm, and $DE = 350$ mm. The constant angular speed of the driver link 1 is $n = n_1 = 650$ rpm. Find the velocities and the accelerations of the mechanism, using the contour equations, for $\phi = \phi_1 = 240°$.

5.23 The dimensions of the mechanism shown in Fig. 3.38 are: $AB = 200$ mm, $AD = 500$ mm, and $BC = 250$ mm. The constant angular speed of the driver link 1 is $n = 160$ rpm. Use the contour equations to calculate the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 240°$. Select a suitable value for the distance a.

5.24 The mechanism in Fig. 3.11(a) has the dimensions: $AB = 0.20$ m, $AD = 0.40$ m, $CD = 0.70$ m, $CE = 0.30$ m, and $y_E = 0.35$ m. The constant angular speed of the driver link 1 is $n = 2600$ rpm. Using the contour equations find the velocities and the accelerations of the mechanism for the given input angle $\phi = \phi_1$=210°.

5.25 The mechanism in Fig. 3.12 has the dimensions: $AB = 0.03$ m, $BC = 0.05$ m, $CD = 0.08$ m, $AE = 0.07$ m, and $L_a = 0.025$ m. The constant angular speed of the driver link 1 is $n = 90$ rpm. Employing the contour equations, find the velocities and the accelerations of the mechanism for the given input angle $\phi = \phi_1 = \pi/3$.

5.26 The mechanism in Fig. 3.15 has the dimensions: $AC = 0.200$ m, $BC = 0.300$ m, $BD = 1.000$ m, and $L_a = 0.050$ m. The constant angular speed of the driver link 1 is $n = 1500$ rpm. Use the contour equations to calculate the velocities and the accelerations of the mechanism for $\phi = 330°$.

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Figure captions

Figure 5.1. Two rigid links (j) and (k) connected by a joint at A: (a) general case, (b) slider joint in general motion.

Figure 5.2. Monocontour closed kinematic chain with n rigid links.

Figure 5.3. Planar mechanism and the diagram that represents the mechanism

Figure 5.4. (a) Example mechanism (R-TRR-RRT), and (b) diagram which represents the mechanism.

Figure 5.5. First contour RTRR: (a) diagram, and (b) mechanism.

Figure 5.6. Second contour RRRT: (a) diagram, and (b) mechanism.