

1 Kinematics

1.1 Ordinary gear trains

A system of two fixed gears is shown in Figure 1.a. The angular speed of the motor connected to the center of the gear 1 is $\omega_{10} = 10$ rpm. The pinion is fixed to the ground with a pin joint. Given the radii $r_1 = 1$ m and $r_2 = 0.5$ m of the gears 1 and 2, find the angular speed ω_{20} of the gear 2.

The following relation can be written between the speeds and the radii of the two gears

$$\frac{\omega_{10}}{\omega_{20}} = -\frac{r_1}{r_2}. \quad (1)$$

Thus, the angular speed n_2 of the planet gear is

$$\omega_{20} = -\frac{r_1\omega_{10}}{r_2} = -20\text{rpm}. \quad (2)$$

Remark:

In general, if the angular speed is expressed in rpm, then the symbol n is preferred instead of ω . In this section the symbol ω is also used for the angular speed expressed in rpm.

1.2 Planetary gear trains

1.2.1 Two DOF gear train

A planetary gear train is shown in Figure 2.a. The planet gear rotates around the sun gear. The arm 3 is connected to the planet gear at the point C (pin joint) and the ground at the point D (pin joint). The sun gear is connected to the ground at the point A (pin joint). The motor connected to the sun gear has the angular speed $\omega_{10} = 20$ rpm. The motor connected to the arm has the angular speed $\omega_{30} = -10$ rpm. Given the radii $r_1 = 1$ m and $r_2 = 0.5$ m of the sun gear 1 and the planet gear 2, find the angular speed ω_{20} of the planet gear 2.

The contour of the gear train is $0 - A - 1 - B - 2 - C - 3 - D - 0$.

One can write the following relations between the relative angular velocities of the gears

$$\omega_{10} + \omega_{21} + \omega_{32} + \omega_{03} = \mathbf{0}, \quad (3)$$

$$\mathbf{AB} \times \boldsymbol{\omega}_{21} + \mathbf{AC} \times \boldsymbol{\omega}_{32} = \mathbf{0}, \quad (4)$$

where $\boldsymbol{\omega}_{10} = 20\mathbf{i}$ rpm and $\boldsymbol{\omega}_{03} = -\boldsymbol{\omega}_{30} = 10\mathbf{i}$ rpm.

Eqs. (3) and (4) can be solved simultaneously with respect to the two unknowns $\boldsymbol{\omega}_{21}$ and $\boldsymbol{\omega}_{32}$. The solutions are

$$\boldsymbol{\omega}_{21} = -90\mathbf{i} \text{ rpm}, \quad \boldsymbol{\omega}_{32} = 60\mathbf{i} \text{ rpm}. \quad (5)$$

The angular speed $\boldsymbol{\omega}_{20}$ of the planet gear 2 can be computed as

$$\boldsymbol{\omega}_{20} = \boldsymbol{\omega}_{21} + \boldsymbol{\omega}_{10} = -70\mathbf{i} \text{ rpm}. \quad (6)$$

1.2.2 One DOF gear train

A system of three gears is shown in Figure 3.a. The sun gear 1 is connected to the ground with a pin joint at point A . The arm 3 is connected with pin joints to the planet gear 2 at point C and to the ground at point D . The planet gear 2 is also in contact to the gear 4 (as internal gear) which is fixed to the ground. The angular speed of the motor connected to the sun gear is $\omega_{10} = 20$ rpm. Given the radii $r_1 = 1$ m and $r_2 = 0.5$ m of the sun gear 1 and the planet gear 2, find the angular speed ω_{20} of the planet gear 2. The arm 3 does not have a motor attached to it and the planet gear 2 is connected to the sun gear 1 and the fixed gear 4.

Contour $0 - A - 1 - B - 2 - E - 0$

One can write the following relations between the relative angular velocities of the gears

$$\boldsymbol{\omega}_{10} + \boldsymbol{\omega}_{21} + \boldsymbol{\omega}_{02} = \mathbf{0}, \quad (7)$$

$$\mathbf{AB} \times \boldsymbol{\omega}_{21} + \mathbf{AE} \times \boldsymbol{\omega}_{02} = \mathbf{0}, \quad (8)$$

where $\boldsymbol{\omega}_{10} = -20\mathbf{i}$ rpm.

Eqs. (7) and (8) can be solved simultaneously with respect to the two unknowns $\boldsymbol{\omega}_{21}$ and $\boldsymbol{\omega}_{02}$. The solutions are

$$\boldsymbol{\omega}_{21} = -40\mathbf{i} \text{ rpm}, \quad \boldsymbol{\omega}_{02} = 20\mathbf{i} \text{ rpm}. \quad (9)$$

The angular speed $\boldsymbol{\omega}_{20}$ of the planet gear 2 can be computed as

$$\boldsymbol{\omega}_{20} = -\boldsymbol{\omega}_{02} = -20\mathbf{i} \text{ rpm}. \quad (10)$$

Contour $0 - E - 2 - C - 3 - D - 0$

One can write the following relations between the relative angular velocities of the gears

$$\boldsymbol{\omega}_{20} + \boldsymbol{\omega}_{32} + \boldsymbol{\omega}_{03} = \mathbf{0}, \quad (11)$$

$$\mathbf{AE} \times \boldsymbol{\omega}_{21} + \mathbf{AC} \times \boldsymbol{\omega}_{32} = \mathbf{0}. \quad (12)$$

The relative angular speed $\boldsymbol{\omega}_{32}$ can be computed from Eq. (12) as

$$\boldsymbol{\omega}_{32} = 26.61 \text{ rpm}. \quad (13)$$

The angular speed $\boldsymbol{\omega}_{30}$ of the arm 3 can be computed from Eq. (11) as

$$\boldsymbol{\omega}_{30} = -\boldsymbol{\omega}_{03} = \boldsymbol{\omega}_{20} + \boldsymbol{\omega}_{32} = 6.61 \text{ rpm}. \quad (14)$$

2 Force analysis

Assumptions: forces of inertia, moments of inertia, and gravity forces are neglected.

2.1 Ordinary gear trains

The mechanism from section 1.1 is considered. The free body diagram is shown in Figure 1.b. An external torque $\mathbf{M}_{ext} = -400 \frac{\boldsymbol{\omega}_{20}}{|\boldsymbol{\omega}_{20}|} \mathbf{1} = 400 \mathbf{1}$ Nm acts on the driven gear 2. Find the torque of the motor M_{mot} and the reaction force between the gears 1 and 2.

Gear 2 (driven)

For the gear 2 one can write the moment equation with respect to its center C

$$\sum \mathbf{M}_C^{(2)} = \mathbf{CB} \times \mathbf{F}_{12} + \mathbf{M}_{ext} = \mathbf{0}, \quad (15)$$

where $\mathbf{F}_{12} = F_{r12} \mathbf{j} + F_{t12} \mathbf{k}$ is the reaction force of the gear 1 on the gear 2, and $\mathbf{CB} = r_2 \mathbf{j}$.

One can write Eq. (15) as

$$\begin{vmatrix} \mathbf{1} & \mathbf{j} & \mathbf{k} \\ 0 & -r_2 & 0 \\ 0 & F_{r12} & F_{t12} \end{vmatrix} + M_{ext} \mathbf{1} = \mathbf{0}. \quad (16)$$

From Eq. (16) results

$$F_{t12} = \frac{M_{ext}}{r_2} = 800 \text{ N.} \quad (17)$$

The radial reaction force F_{r12} is

$$F_{r12} = F_{t12} \tan 20^\circ = 291.176 \text{ N,} \quad (18)$$

For the gear 2 one can write the force equation

$$\sum \mathbf{F}^{(2)} = \mathbf{F}_{12} + \mathbf{F}_{02} = \mathbf{0}. \quad (19)$$

Thus, the reaction force of the ground on the gear 2 is

$$\mathbf{F}_{02} = -\mathbf{F}_{12} = -291.176\mathbf{j} - 800\mathbf{k} \text{ N.} \quad (20)$$

Gear 1 (driver)

For the gear 1 one can write the moment equation with respect to its center A

$$\sum \mathbf{M}_A^{(1)} = \mathbf{AB} \times \mathbf{F}_{21} + \mathbf{M}_{mot} = \mathbf{0}, \quad (21)$$

where $\mathbf{F}_{21} = -\mathbf{F}_{12}$, and $\mathbf{AB} = r_1\mathbf{j}$.

The motor torque M_{mot} can be computed as

$$M_{mot} = F_{t21}r_1 = -F_{t12}r_1 = \frac{r_1}{r_2}M_{ext} = 800 \text{ Nm.} \quad (22)$$

For the gear 1 one can write the force equation

$$\sum \mathbf{F}^{(1)} = \mathbf{F}_{21} + \mathbf{F}_{01} = \mathbf{0}. \quad (23)$$

Thus, the reaction force of the ground on the gear 1 is

$$\mathbf{F}_{01} = -\mathbf{F}_{21} = 291.176\mathbf{j} + 800\mathbf{k} \text{ N.} \quad (24)$$

2.2 Planetary gear trains

2.2.1 Two DOF gear train

The gear train from section 1.2.1 is considered. The force and moment diagram is shown in Figure 2.b. An external torque $\mathbf{M}_{ext} = -400 \frac{\boldsymbol{\omega}_{20}}{|\boldsymbol{\omega}_{20}|} \mathbf{1} = 400\mathbf{i}$

Nm acts on the planet gear 2. Find the torques M_{1mot} and M_{3mot} of the motors and the reaction forces.

Gear 2 (driven)

For the planet gear 2 one can write the moment equation with respect to its center C

$$\sum \mathbf{M}_C^{(2)} = \mathbf{CB} \times \mathbf{F}_{12} + \mathbf{M}_{ext} = \mathbf{0}, \quad (25)$$

where $\mathbf{F}_{12} = F_{r12}\mathbf{j} + F_{t12}\mathbf{k}$.

One can write Eq. (25) as

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -r_2 & 0 \\ 0 & F_{r12} & F_{t12} \end{vmatrix} + M_{ext}\mathbf{i} = \mathbf{0}. \quad (26)$$

From Eq. (26) results

$$F_{t12} = \frac{M_{ext}}{r_2} = 800 \text{ N}. \quad (27)$$

The radial reaction force F_{r12} is

$$F_{r12} = F_{t12} \tan 20^\circ = 291.176 \text{ N}, \quad (28)$$

For the planet gear 2 one can write the force equation

$$\sum \mathbf{F}^{(2)} = \mathbf{F}_{12} + \mathbf{F}_{32} = \mathbf{0}. \quad (29)$$

Thus, the reaction force of the arm 3 on the gear 2 is

$$\mathbf{F}_{32} = -\mathbf{F}_{12} = -291.176\mathbf{j} - 800\mathbf{k} \text{ N}. \quad (30)$$

Gear 1

For the gear 1 one can write the moment equation with respect to its center A

$$\sum \mathbf{M}_A^{(1)} = \mathbf{AB} \times \mathbf{F}_{21} + \mathbf{M}_{1mot} = \mathbf{0}, \quad (31)$$

where $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

The motor torque M_{1mot} can be computed as

$$M_{1mot} = F_{t21}r_1 = F_{t12}r_1 = 800 \text{ Nm}. \quad (32)$$

For the gear 1 one can write the force equation

$$\sum \mathbf{F}^{(1)} = \mathbf{F}_{21} + \mathbf{F}_{01} = \mathbf{0}. \quad (33)$$

Thus, the reaction force of the ground on the gear 1 is

$$\mathbf{F}_{01} = -\mathbf{F}_{21} = 291.176\mathbf{j} + 800\mathbf{k} \text{ N}. \quad (34)$$

Arm 3

For the arm 3 one can write the moment equation with respect to the point D

$$\sum \mathbf{M}_D^{(3)} = \mathbf{DC} \times \mathbf{F}_{23} + \mathbf{M}_{3mot} = \mathbf{0}, \quad (35)$$

where $\mathbf{F}_{23} = -\mathbf{F}_{32}$, and $\mathbf{DC} = (r_1 + r_2)\mathbf{j}$.

The motor torque M_3 can be computed as

$$M_{3mot} = -F_{t23}(r_1 + r_2) = -F_{t12}(r_1 + r_2) = -1200 \text{ Nm}. \quad (36)$$

For the arm 3 one can write the force equation

$$\sum \mathbf{F}^{(3)} = \mathbf{F}_{23} + \mathbf{F}_{03} = \mathbf{0}. \quad (37)$$

Thus, the reaction force of the ground on the arm 3 is

$$\mathbf{F}_{03} = -\mathbf{F}_{23} = -291.176\mathbf{j} - 800\mathbf{k} \text{ N}. \quad (38)$$

2.2.2 One DOF gear train

The gear train from section 1.2.2 is considered. The force and momentum diagram is shown in Figure 3.b. An external torque $\mathbf{M}_{ext} = -400 \frac{\boldsymbol{\omega}_{30}}{|\boldsymbol{\omega}_{30}|} \mathbf{1} = -400\mathbf{1}$ Nm acts on the driven arm 3. Find the torque M_{mot} of the motor and the reaction forces.

Arm 3 (driven)

For the arm 3 one can write the moment equation with respect to the point D

$$\sum \mathbf{M}_D^{(3)} = \mathbf{DC} \times \mathbf{F}_{23} + \mathbf{M}_{ext} = \mathbf{0}, \quad (39)$$

where $\mathbf{F}_{23} = F_{23y}\mathbf{j} + F_{23z}\mathbf{k}$.

One can write Eq. (39) as

$$\begin{vmatrix} \mathbf{1} & \mathbf{j} & \mathbf{k} \\ 0 & r_1 + r_2 & 0 \\ 0 & F_{32y} & F_{32z} \end{vmatrix} + M_{ext}\mathbf{1} = \mathbf{0}. \quad (40)$$

From Eq. (40) results

$$\begin{aligned} F_{23y} &= 0, \\ F_{23z} &= -\frac{M_{ext}}{r_1 + r_2} = 266.666 \text{ N}. \end{aligned} \quad (41)$$

For the arm 3 one can write the force equation

$$\sum \mathbf{F}^{(3)} = \mathbf{F}_{23} + \mathbf{F}_{03} = \mathbf{0}. \quad (42)$$

Thus, the reaction force $\mathbf{F}_{03} = -\mathbf{F}_{23} = -266.666\mathbf{k}$ N.

Gear 2

For the planet gear 2 one can write the force equation

$$\sum \mathbf{F}^{(2)} = \mathbf{F}_{12} + \mathbf{F}_{02} + \mathbf{F}_{32} = \mathbf{0}, \quad (43)$$

where $\mathbf{F}_{12} = F_{r12}\mathbf{j} + F_{t12}\mathbf{k}$, $\mathbf{F}_{02} = F_{r02}\mathbf{j} + F_{t02}\mathbf{k}$, and $\mathbf{F}_{32} = -\mathbf{F}_{23}$.

Projecting Eq. (43) on the Oy axis one can write

$$F_{r12} = F_{r02}. \quad (44)$$

Projecting Eq. (43) on the Oz axis one can write

$$F_{t12} + F_{t02} + F_{32z} = 0. \quad (45)$$

Thus, the tangential reaction forces F_{t12} and F_{t02} are

$$F_{t12} = F_{t02} = -\frac{F_{t32}}{2} = 133.333 \text{ N}. \quad (46)$$

The radial reaction forces F_{r12} and F_{r02} are

$$F_{r12} = F_{r02} = -\frac{F_{t32}}{2} \tan 20^\circ = 48.529 \text{ N}. \quad (47)$$

Gear 1 (driver)

For the gear 1 one can write the moment equation with respect to its center A

$$\sum \mathbf{M}_A^{(1)} = \mathbf{AB} \times \mathbf{F}_{21} + \mathbf{M}_{mot} = \mathbf{0}, \quad (48)$$

where $\mathbf{F}_{21} = -\mathbf{F}_{12}$.

The motor torque M_{mot} can be computed as

$$M_{mot} = -F_{t21}r_1 = F_{t12}r_1 = 133.333 \text{ Nm.} \quad (49)$$

For the gear 1 one can write the force equation

$$\sum \mathbf{F}^{(1)} = \mathbf{F}_{21} + \mathbf{F}_{01} = \mathbf{0}. \quad (50)$$

Thus, the reaction force $\mathbf{F}_{01} = -\mathbf{F}_{21} = 48.529\mathbf{j} + 133.333\mathbf{k}$ N.

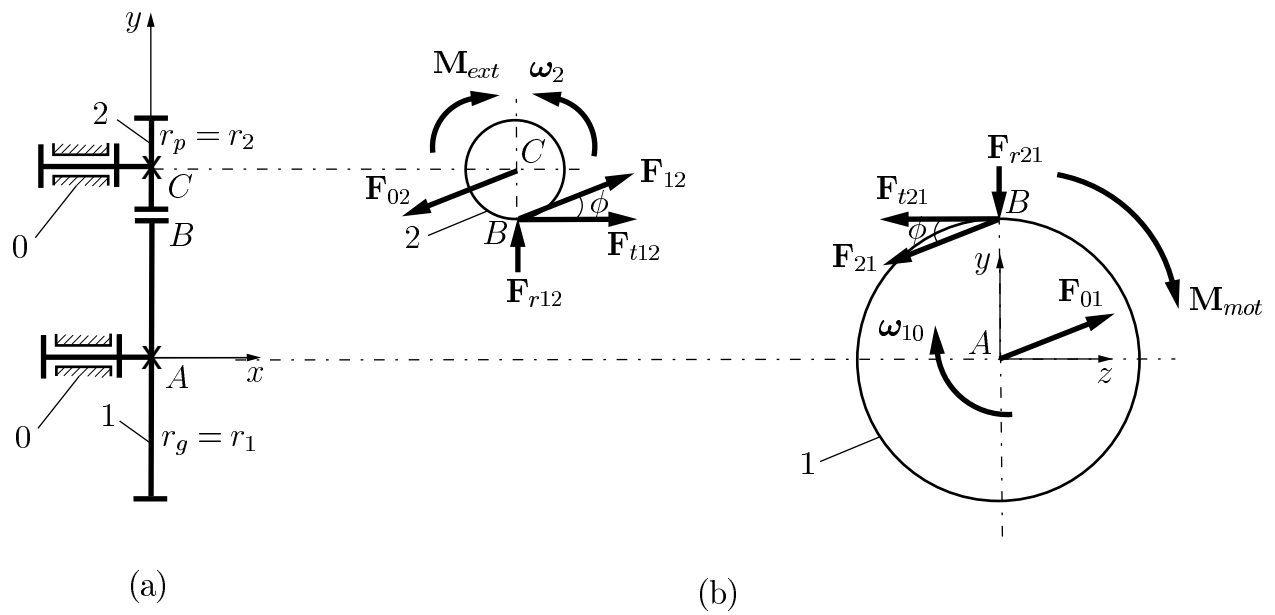
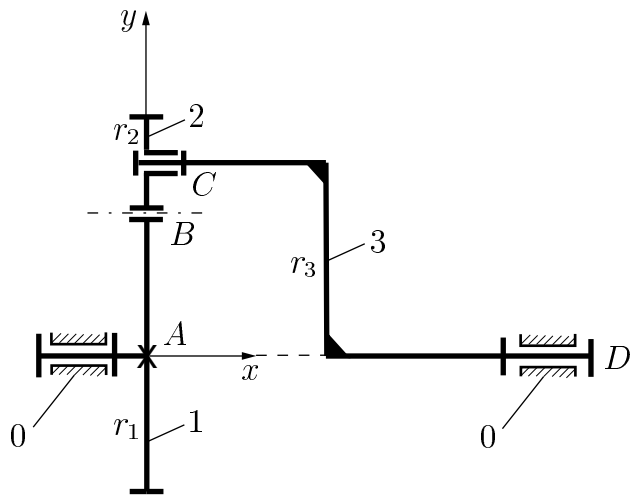
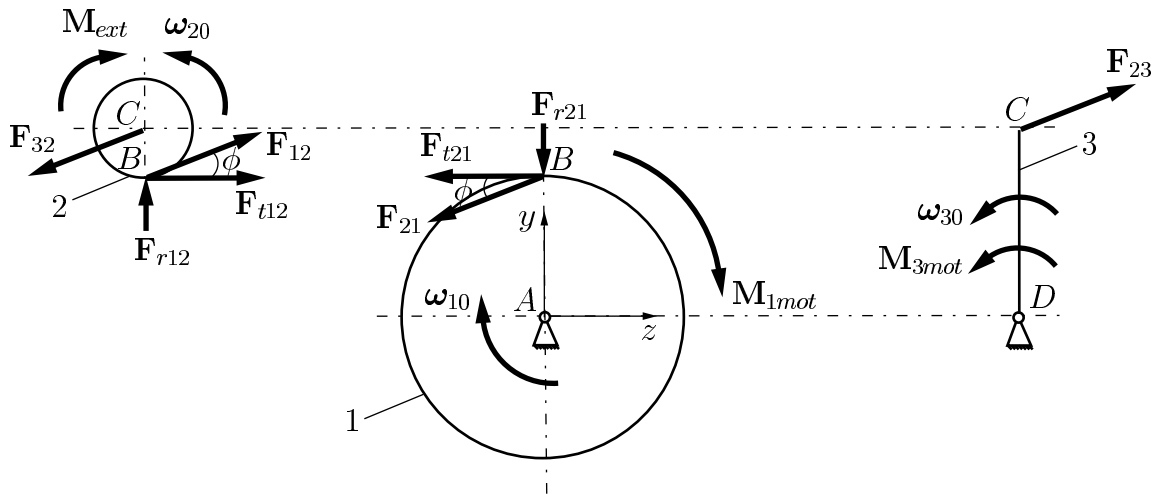


Figure 1

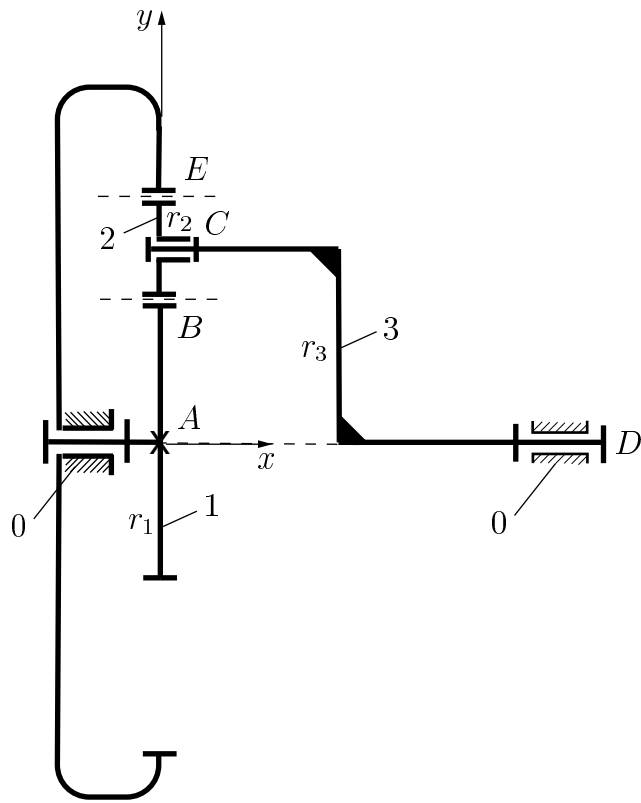


(a)

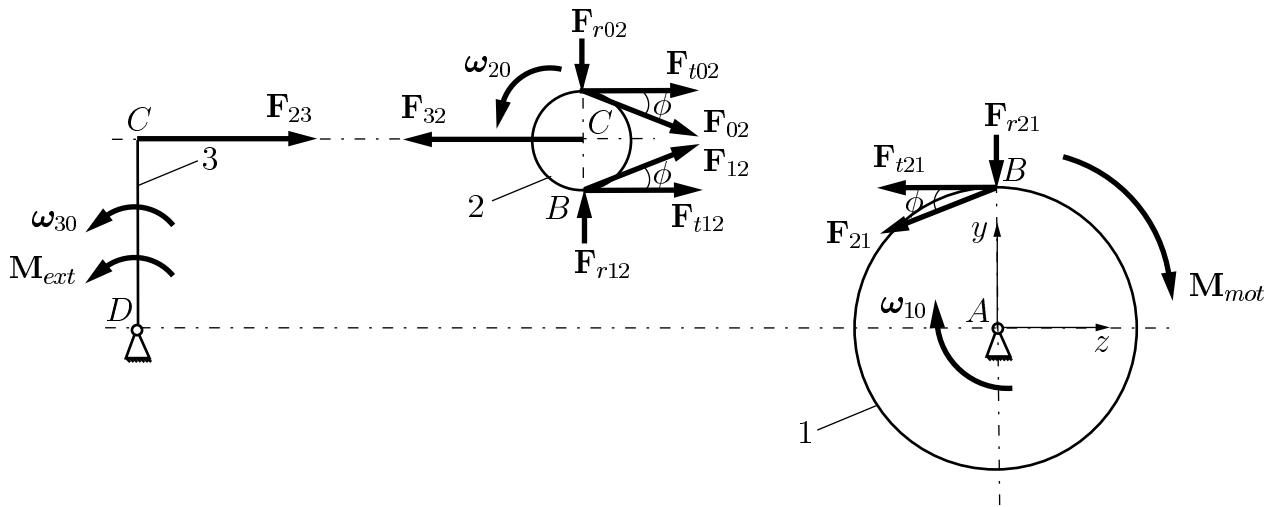


(b)

Figure 2



(a)



(b)

Figure 3