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4 Velocity and Acceleration Analysis

4.1 Kinematics of the Rigid Body

The motion of a rigid body (RB) is defined when the position vector, velocity and acceleration of all points of the rigid body are defined as functions of time with respect to a fixed reference frame with the origin at O_0 .

Let $\mathbf{i}_0, \mathbf{j}_0$ and \mathbf{k}_0 , be the constant unit vectors of a fixed orthogonal Cartesian reference frame $O_0x_0y_0z_0$ (primary reference frame). The unit vectors $\mathbf{i}_0, \mathbf{j}_0$, and \mathbf{k}_0 of the primary reference frame are constant with respect to time. Let \mathbf{i}, \mathbf{j} and \mathbf{k} be the unit vectors of a mobile orthogonal Cartesian reference frame $Oxyz$ (Fig. 4.1). A reference frame that moves with the rigid body is a *body fixed* (or mobile) reference frame. The unit vectors \mathbf{i}, \mathbf{j} , and \mathbf{k} of the body fixed reference frame are not constant, because they rotate with the body fixed reference frame. The location of the point O is arbitrary.

The position vector of a point M ($M \in (RB)$), with respect to the fixed reference frame $O_0x_0y_0z_0$ is denoted by $\mathbf{r}_1 = \mathbf{r}_{O_0M}$ and with respect to the mobile reference frame $Oxyz$ is denoted by $\mathbf{r} = \mathbf{r}_{OM}$. The location of the origin O of the mobile reference frame with respect to the fixed point O_0 is defined by the position vector $\mathbf{r}_O = \mathbf{r}_{O_0O}$. Then the relation between the vectors \mathbf{r}_1, \mathbf{r} and \mathbf{r}_O is given by

$$\mathbf{r}_1 = \mathbf{r}_O + \mathbf{r} = \mathbf{r}_O + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad (4.1)$$

where x, y and z represent the projections of the vector \mathbf{r} on the mobile reference frame. The magnitude of the vector $\mathbf{r} = \mathbf{r}_{OM}$ is a constant as the distance between the points O and M is constant ($O \in (RB)$ and $M \in (RB)$). Thus, the x, y and z components of the vector \mathbf{r} with respect to the mobile reference frame are constant. The unit vectors \mathbf{i}, \mathbf{j} and \mathbf{k} are time-dependent vector functions. The vectors \mathbf{i}, \mathbf{j} and \mathbf{k} are the unit vector of an orthogonal Cartesian reference frame, the following relations can be written

$$\mathbf{i} \cdot \mathbf{i} = 1, \quad \mathbf{j} \cdot \mathbf{j} = 1, \quad \mathbf{k} \cdot \mathbf{k} = 1, \quad (4.2)$$

$$\mathbf{i} \cdot \mathbf{j} = 0, \quad \mathbf{j} \cdot \mathbf{k} = 0, \quad \mathbf{k} \cdot \mathbf{i} = 0. \quad (4.3)$$

Velocity of a point on the rigid body

The velocity of an arbitrary point M of the rigid body with respect to the fixed reference frame $Ox_0y_0z_0$, is the derivative with respect to time of the position vector \mathbf{r}_1

$$\mathbf{v} = \frac{d\mathbf{r}_1}{dt} = \dot{\mathbf{r}}_1 = \dot{\mathbf{r}}_O + \dot{\mathbf{r}} = \mathbf{v}_O + x\dot{\mathbf{i}} + y\dot{\mathbf{j}} + z\dot{\mathbf{k}} + \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}, \quad (4.4)$$

where $\mathbf{v}_O = \dot{\mathbf{r}}_O$ represent the velocity of the origin of the mobile reference frame $O_1x_1y_1z_1$ with respect to the fixed reference frame $Oxyz$. Because all the points in the rigid body maintain their relative position, their velocity relative to the mobile reference frame $Oxyz$ is zero: $\dot{x} = \dot{y} = \dot{z} = 0$.

The velocity of the point M is

$$\mathbf{v} = \mathbf{v}_O + x\dot{\mathbf{i}} + y\dot{\mathbf{j}} + z\dot{\mathbf{k}}.$$

The derivative of the Eqs. (4.2) and (4.3) with respect to time gives

$$\dot{\mathbf{i}} \cdot \mathbf{i} = 0, \quad \dot{\mathbf{j}} \cdot \mathbf{j} = 0, \quad \dot{\mathbf{k}} \cdot \mathbf{k} = 0, \quad (4.5)$$

and

$$\dot{\mathbf{i}} \cdot \mathbf{j} + \mathbf{j} \cdot \dot{\mathbf{i}} = 0, \quad \dot{\mathbf{j}} \cdot \mathbf{k} + \dot{\mathbf{k}} \cdot \mathbf{j} = 0, \quad \dot{\mathbf{k}} \cdot \mathbf{i} + \dot{\mathbf{i}} \cdot \mathbf{k} = 0. \quad (4.6)$$

For Eq. (4.6) the following convention is introduced

$$\begin{aligned} \dot{\mathbf{i}} \cdot \mathbf{j} &= -\mathbf{j} \cdot \dot{\mathbf{i}} = \omega_z, \\ \dot{\mathbf{j}} \cdot \mathbf{k} &= -\dot{\mathbf{k}} \cdot \mathbf{j} = \omega_x, \\ \dot{\mathbf{k}} \cdot \mathbf{i} &= -\dot{\mathbf{i}} \cdot \mathbf{k} = \omega_y, \end{aligned} \quad (4.7)$$

where ω_x , ω_y and ω_z can be considered as the projections of a vector $\boldsymbol{\omega}$

$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}.$$

To calculate $\dot{\mathbf{i}}$, $\dot{\mathbf{j}}$, $\dot{\mathbf{k}}$ the following formula is introduced for an arbitrary vector, \mathbf{d} ,

$$\mathbf{d} = d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k} = (\mathbf{d} \cdot \mathbf{i})\mathbf{i} + (\mathbf{d} \cdot \mathbf{j})\mathbf{j} + (\mathbf{d} \cdot \mathbf{k})\mathbf{k}. \quad (4.8)$$

Using Eq. (4.8) and the results from Eqs. (4.5) and (4.6) it results

$$\begin{aligned} \dot{\mathbf{i}} &= (\dot{\mathbf{i}} \cdot \mathbf{i})\mathbf{i} + (\dot{\mathbf{i}} \cdot \mathbf{j})\mathbf{j} + (\dot{\mathbf{i}} \cdot \mathbf{k})\mathbf{k} \\ &= (0)\mathbf{i} + (\omega_z)\mathbf{j} - (\omega_y)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 1 & 0 & 0 \end{vmatrix} = \boldsymbol{\omega} \times \mathbf{i}, \end{aligned}$$

$$\begin{aligned}
\mathbf{j} &= (\mathbf{j} \cdot \mathbf{i}) \mathbf{i} + (\mathbf{j} \cdot \mathbf{J}) \mathbf{J} + (\mathbf{j} \cdot \mathbf{k}) \mathbf{k} \\
&= (-\omega_z) \mathbf{i} + (0) \mathbf{J} + (\omega_x) \mathbf{k} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{J} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 0 & 1 & 0 \end{vmatrix} = \boldsymbol{\omega} \times \mathbf{J},
\end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{k}} &= (\dot{\mathbf{k}} \cdot \mathbf{i}) \mathbf{i} + (\dot{\mathbf{k}} \cdot \mathbf{J}) \mathbf{J} + (\dot{\mathbf{k}} \cdot \mathbf{k}) \mathbf{k} \\
&= (\omega_y) \mathbf{i} - (\omega_x) \mathbf{J} + (0) \mathbf{k} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{J} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 0 & 0 & 1 \end{vmatrix} = \boldsymbol{\omega} \times \mathbf{k}.
\end{aligned} \tag{4.9}$$

The relations

$$\mathbf{i} = \boldsymbol{\omega} \times \mathbf{i}, \quad \mathbf{j} = \boldsymbol{\omega} \times \mathbf{J}, \quad \dot{\mathbf{k}} = \boldsymbol{\omega} \times \mathbf{k}. \tag{4.10}$$

are known as *Poisson formulas*.

Using Eqs. (4.4) and (4.10) the velocity of M is

$$\mathbf{v} = \mathbf{v}_O + x\boldsymbol{\omega} \times \mathbf{i} + y\boldsymbol{\omega} \times \mathbf{J} + z\boldsymbol{\omega} \times \mathbf{k} = \mathbf{v}_O + \boldsymbol{\omega} \times (x\mathbf{i} + y\mathbf{J} + z\mathbf{k}),$$

or

$$\mathbf{v} = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}. \tag{4.11}$$

Combining Eqs. (4.4) and (4.11) it results

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}. \tag{4.12}$$

Using Eq. (4.11) the components of the velocity are

$$\begin{aligned}
v_x &= v_{Ox} + z\omega_y - y\omega_z, \\
v_y &= v_{Oy} + x\omega_z - z\omega_x, \\
v_z &= v_{Oz} + y\omega_x - x\omega_y.
\end{aligned}$$

Acceleration of a point on the rigid body

The acceleration of an arbitrary point $M \in (RB)$ with respect to a fixed reference frame $O_0x_0y_0z_0$, represents the double derivative with respect to time of the position vector \mathbf{r}_1

$$\mathbf{a} = \ddot{\mathbf{r}}_1 = \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}) = \frac{d}{dt}\mathbf{v}_O + \frac{d}{dt}\boldsymbol{\omega} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d}{dt}\mathbf{r} = \dot{\mathbf{v}}_O + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}}. \quad (4.13)$$

The acceleration of the point O with respect to the fixed reference frame $O_0x_0y_0z_0$ is

$$\mathbf{a}_O = \dot{\mathbf{v}}_O = \ddot{\mathbf{r}}_O. \quad (4.14)$$

The derivative of the vector $\boldsymbol{\omega}$ with respect to the time is the vector $\boldsymbol{\alpha}$ given by

$$\begin{aligned} \boldsymbol{\alpha} &= \dot{\boldsymbol{\omega}} = \dot{\omega}_x \mathbf{i} + \dot{\omega}_y \mathbf{j} + \dot{\omega}_z \mathbf{k} + \omega_x \dot{\mathbf{i}} + \omega_y \dot{\mathbf{j}} + \omega_z \dot{\mathbf{k}} \\ &= \alpha_x \mathbf{i} + \alpha_y \mathbf{j} + \alpha_z \mathbf{k} + \omega_x \boldsymbol{\omega} \times \mathbf{i} + \omega_y \boldsymbol{\omega} \times \mathbf{j} + \omega_z \boldsymbol{\omega} \times \mathbf{k} \\ &= \alpha_x \mathbf{i} + \alpha_y \mathbf{j} + \alpha_z \mathbf{k} + \boldsymbol{\omega} \times \boldsymbol{\omega} = \alpha_x \mathbf{i} + \alpha_y \mathbf{j} + \alpha_z \mathbf{k}. \end{aligned} \quad (4.15)$$

where $\alpha_x = \dot{\omega}_x$, $\alpha_y = \dot{\omega}_y$, and $\alpha_z = \dot{\omega}_z$. In the previous expression the Poisson formulas have been used.

Using Eqs. (4.13), (4.14) and (4.15) the acceleration of the point M is

$$\mathbf{a} = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \quad (4.16)$$

The components of the acceleration are

$$\begin{aligned} a_x &= a_{Ox} + (z\alpha_y - y\alpha_z) + \omega_y(y\omega_x - x\omega_y) + \omega_z(x\omega_x - x\omega_z), \\ a_y &= a_{Oy} + (x\alpha_z - z\alpha_x) + \omega_z(z\omega_y - y\omega_z) + \omega_x(x\omega_y - y\omega_z), \\ a_z &= a_{Oz} + (y\alpha_x - x\alpha_y) + \omega_x(x\omega_z - z\omega_x) + \omega_y(y\omega_z - z\omega_y). \end{aligned}$$

The vector $\boldsymbol{\omega}$ characterizes the rotational motion of the rigid body and is called the *angular velocity*. The vector $\boldsymbol{\alpha}$ is called the *angular acceleration*.

The angular velocity can be introduced also in another way. If the orientation of a rigid body RB in a reference frame RF_0 depends on only a single scalar variable ζ , there exists for each value of ζ a vector $\boldsymbol{\omega}$ such that the derivative with respect to ζ in RF_0 of every vector \mathbf{c} fixed in rigid body RB is given by

$$\frac{d\mathbf{c}}{d\zeta} = \boldsymbol{\omega} \times \mathbf{c}, \quad (4.17)$$

where the vector $\boldsymbol{\omega}$ is the rate of change of orientation of the rigid body RB in the reference frame RF_0 with respect to ζ . The vector $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \frac{\frac{d\mathbf{a}}{d\zeta} \times \frac{d\mathbf{b}}{d\zeta}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}}, \quad (4.18)$$

where \mathbf{a} and \mathbf{b} are any two nonparallel vectors fixed in the rigid body RB .

Proof

The vectors \mathbf{a} and \mathbf{b} are fixed in the rigid body. The magnitudes $\mathbf{a} \cdot \mathbf{a}$, $\mathbf{b} \cdot \mathbf{b}$ and the angle between \mathbf{a} and \mathbf{b} are independent of ζ

$$\frac{d(\mathbf{a} \cdot \mathbf{a})}{d\zeta} = 0, \quad \frac{d(\mathbf{b} \cdot \mathbf{b})}{d\zeta} = 0, \quad \frac{d(\mathbf{a} \cdot \mathbf{b})}{d\zeta} = 0,$$

or

$$\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{a} = 0, \quad \frac{d\mathbf{b}}{d\zeta} \cdot \mathbf{b} = 0, \quad \frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{d\zeta} = 0.$$

Using the vector triple product of three vectors \mathbf{p} , \mathbf{q} , \mathbf{t} it results

$$\mathbf{p} \times (\mathbf{q} \times \mathbf{t}) = \mathbf{p} \cdot \mathbf{t} \mathbf{q} - \mathbf{p} \cdot \mathbf{q} \mathbf{t}, \quad (\mathbf{p} \times \mathbf{q}) \times \mathbf{t} = \mathbf{t} \cdot \mathbf{p} \mathbf{q} - \mathbf{t} \cdot \mathbf{q} \mathbf{t}.$$

From these expression it follows that

$$\begin{aligned} \frac{\frac{d\mathbf{a}}{d\zeta} \times \frac{d\mathbf{b}}{d\zeta}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}} \times \mathbf{a} &= \frac{\left(\frac{d\mathbf{a}}{d\zeta} \times \frac{d\mathbf{b}}{d\zeta}\right) \times \mathbf{a}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}} = \frac{\mathbf{a} \cdot \frac{d\mathbf{a}}{d\zeta} \frac{d\mathbf{b}}{d\zeta} - \mathbf{a} \cdot \frac{d\mathbf{b}}{d\zeta} \frac{d\mathbf{a}}{d\zeta}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}} \\ &= \frac{-\mathbf{a} \cdot \frac{d\mathbf{b}}{d\zeta} \frac{d\mathbf{a}}{d\zeta}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}} = \frac{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b} \frac{d\mathbf{a}}{d\zeta}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}} = \frac{d\mathbf{a}}{d\zeta}, \end{aligned} \quad (4.19)$$

and

$$\begin{aligned}
 \frac{\frac{d\mathbf{a}}{d\zeta} \times \frac{d\mathbf{b}}{d\zeta}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}} \times \mathbf{b} &= \frac{\left(\frac{d\mathbf{a}}{d\zeta} \times \frac{d\mathbf{b}}{d\zeta}\right) \times \mathbf{b}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}} = \frac{\mathbf{b} \cdot \frac{d\mathbf{a}}{d\zeta} \frac{d\mathbf{b}}{d\zeta} - \mathbf{b} \cdot \frac{d\mathbf{b}}{d\zeta} \frac{d\mathbf{a}}{d\zeta}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}} \\
 &= \frac{\mathbf{b} \cdot \frac{d\mathbf{a}}{d\zeta} \frac{d\mathbf{b}}{d\zeta}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}} = \frac{d\mathbf{b}}{d\zeta}.
 \end{aligned} \tag{4.20}$$

The following vector is defined

$$\boldsymbol{\omega} = \frac{\frac{d\mathbf{a}}{d\zeta} \times \frac{d\mathbf{b}}{d\zeta}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}},$$

and the Eqs. (4.19), (4.20) can be written as

$$\frac{d\mathbf{a}}{d\zeta} = \boldsymbol{\omega} \times \mathbf{a}, \quad \frac{d\mathbf{b}}{d\zeta} = \boldsymbol{\omega} \times \mathbf{b}.$$

In general a given vector \mathbf{v} can be expressed as

$$\mathbf{d} = d_1 \mathbf{n}_1 + d_2 \mathbf{n}_2 + d_3 \mathbf{n}_3,$$

where $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are three units vectors not parallel to the same plane, and d_1, d_2, d_3 are three scalars.

Any vector \mathbf{c} fixed in the rigid body RB can be expressed as

$$\mathbf{c} = c_1 \mathbf{a} + c_2 \mathbf{b} + c_3 \mathbf{a} \times \mathbf{b}, \tag{4.21}$$

where c_1, c_2 and c_3 are constant and independent of ζ . Differentiating Eq. (4.21) with respect to ζ the following expression is obtained

$$\begin{aligned}
 \frac{d\mathbf{c}}{d\zeta} &= c_1 \frac{d\mathbf{a}}{d\zeta} + c_2 \frac{d\mathbf{b}}{d\zeta} + c_3 \frac{d\mathbf{a}}{d\zeta} \times \mathbf{b} + c_3 \mathbf{a} \times \frac{d\mathbf{b}}{d\zeta} \\
 &= c_1 \boldsymbol{\omega} \times \mathbf{a} + c_2 \boldsymbol{\omega} \times \mathbf{b} + c_3 [(\boldsymbol{\omega} \times \mathbf{a}) \times \mathbf{b} + \mathbf{a} \times (\boldsymbol{\omega} \times \mathbf{b})] \\
 &= c_1 \boldsymbol{\omega} \times \mathbf{a} + c_2 \boldsymbol{\omega} \times \mathbf{b} + c_3 [\mathbf{b} \cdot \boldsymbol{\omega} \mathbf{a} - \mathbf{b} \cdot \mathbf{a} \boldsymbol{\omega} + \mathbf{a} \cdot \mathbf{b} \boldsymbol{\omega} - \mathbf{a} \cdot \boldsymbol{\omega} \mathbf{b}]
 \end{aligned}$$

$$\begin{aligned}
&= c_1 \boldsymbol{\omega} \times \mathbf{a} + c_2 \boldsymbol{\omega} \times \mathbf{b} + c_3 [\boldsymbol{\omega} \cdot \mathbf{b} \mathbf{a} - \mathbf{a} \cdot \mathbf{b} \boldsymbol{\omega} + \mathbf{a} \cdot \mathbf{b} \boldsymbol{\omega} - \mathbf{a} \cdot \boldsymbol{\omega} \mathbf{b}] \\
&= c_1 \boldsymbol{\omega} \times \mathbf{a} + c_2 \boldsymbol{\omega} \times \mathbf{b} + c_3 [\boldsymbol{\omega} \cdot \mathbf{b} \mathbf{a} - \boldsymbol{\omega} \cdot \mathbf{a} \mathbf{b}] \\
&= c_1 \boldsymbol{\omega} \times \mathbf{a} + c_2 \boldsymbol{\omega} \times \mathbf{b} + c_3 \boldsymbol{\omega} \times (\mathbf{a} \times \mathbf{b}) \\
&= \boldsymbol{\omega} \times (c_1 \mathbf{a} + c_2 \mathbf{b} + c_3 \mathbf{a} \times \mathbf{b}) \\
&= \boldsymbol{\omega} \times \mathbf{c}.
\end{aligned} \tag{4.22}$$

The vector $\boldsymbol{\omega}$ is not associated with any particular point. With the help of $\boldsymbol{\omega}$ the process of differentiation is replaced with that of cross multiplication.

The vector $\boldsymbol{\omega}$ can be expressed in a symmetrical relation in \mathbf{a} and \mathbf{b}

$$\boldsymbol{\omega} = \frac{1}{2} \left(\frac{\frac{d\mathbf{a}}{d\zeta} \times \frac{d\mathbf{b}}{d\zeta}}{\frac{d\mathbf{a}}{d\zeta} \cdot \mathbf{b}} + \frac{\frac{d\mathbf{b}}{d\zeta} \times \frac{d\mathbf{a}}{d\zeta}}{\frac{d\mathbf{b}}{d\zeta} \cdot \mathbf{a}} \right). \tag{4.23}$$

The first derivatives of a vector \mathbf{p} with respect to a scalar variable ζ in two reference frames RF_i and RF_j are related as follows

$$\frac{{}^{(j)}d\mathbf{p}}{d\zeta} = \frac{{}^{(i)}d\mathbf{p}}{d\zeta} + \boldsymbol{\omega}_{ij} \times \mathbf{p}, \tag{4.24}$$

where $\boldsymbol{\omega}_{ij}$ is the rate of change of orientation of RF_i in RF_j with respect to ζ and $\frac{{}^{(j)}d\mathbf{p}}{d\zeta}$ is the total derivative of \mathbf{p} with respect to ζ in RF_j .

Proof

The vector \mathbf{p} can be expressed as

$$\mathbf{p} = p_1 \mathbf{1}_1 + p_2 \mathbf{1}_2 + p_3 \mathbf{1}_3,$$

where $\mathbf{1}_1, \mathbf{1}_2, \mathbf{1}_3$ are three units vectors not parallel to the same plane fixed in RF_i , and p_x, p_y, p_z are the scalar measure numbers of \mathbf{p} . Differentiating in RF_j

$$\begin{aligned}
\frac{{}^{(j)}d\mathbf{p}}{d\zeta} &= \frac{{}^{(j)}d}{d\zeta} (p_1 \mathbf{1}_1 + p_2 \mathbf{1}_2 + p_3 \mathbf{1}_3) \\
&= \frac{{}^{(j)}d p_1}{d\zeta} \mathbf{1}_1 + \frac{{}^{(j)}d p_2}{d\zeta} \mathbf{1}_2 + \frac{{}^{(j)}d p_3}{d\zeta} \mathbf{1}_3 + p_1 \frac{{}^{(j)}d \mathbf{1}_1}{d\zeta} + p_2 \frac{{}^{(j)}d \mathbf{1}_2}{d\zeta} + p_3 \frac{{}^{(j)}d \mathbf{1}_3}{d\zeta} \\
&= \frac{d p_1}{d\zeta} \mathbf{1}_1 + \frac{d p_2}{d\zeta} \mathbf{1}_2 + \frac{d p_3}{d\zeta} \mathbf{1}_3 + p_1 \boldsymbol{\omega}_{ij} \times \mathbf{1}_1 + p_2 \boldsymbol{\omega}_{ij} \times \mathbf{1}_2 + p_3 \boldsymbol{\omega}_{ij} \times \mathbf{1}_3
\end{aligned}$$

$$\begin{aligned}
&= \frac{{}^{(i)}d p_2}{d\zeta} \mathbf{1}_1 + \frac{{}^{(i)}d p_2}{d\zeta} \mathbf{1}_2 + \frac{{}^{(i)}d p_3}{d\zeta} \mathbf{1}_3 + \boldsymbol{\omega}_{ij} \times (p_1 \mathbf{1}_1 + p_2 \mathbf{1}_2 + p_3 \mathbf{1}_3) \\
&= \frac{{}^{(i)}d \mathbf{p}}{d\zeta} + \boldsymbol{\omega}_{ij} \times \mathbf{p}.
\end{aligned}$$

The *angular velocity* of a rigid body RB in a reference frame RF_0 is the rate of change of orientation with respect to the time t

$$\boldsymbol{\omega} = \frac{1}{2} \left(\frac{\frac{d\mathbf{a}}{dt} \times \frac{d\mathbf{b}}{dt}}{\frac{d\mathbf{a}}{dt} \cdot \mathbf{b}} + \frac{\frac{d\mathbf{b}}{dt} \times \frac{d\mathbf{a}}{dt}}{\frac{d\mathbf{b}}{dt} \cdot \mathbf{a}} \right) = \frac{1}{2} \left(\frac{\dot{\mathbf{a}} \times \dot{\mathbf{b}}}{\dot{\mathbf{a}} \cdot \mathbf{b}} + \frac{\dot{\mathbf{b}} \times \dot{\mathbf{a}}}{\dot{\mathbf{b}} \cdot \mathbf{a}} \right). \quad (4.25)$$

The direction of $\boldsymbol{\omega}$ is related to the direction of the rotation of the rigid body through a right-hand rule.

Let RF_i , $i = 1, 2, \dots, n$ be n reference frames. The angular velocity of a rigid body r in the reference frame RF_n , can be expressed as

$$\boldsymbol{\omega}_{rn} = \boldsymbol{\omega}_{r1} + \boldsymbol{\omega}_{12} + \boldsymbol{\omega}_{23} + \dots + \boldsymbol{\omega}_{r,n-1}. \quad (4.26)$$

Proof

Let \mathbf{p} be any vector fixed in the rigid body. Then

$$\begin{aligned}
\frac{{}^{(i)}d \mathbf{p}}{dt} &= \boldsymbol{\omega}_{ri} \times \mathbf{p} \\
\frac{{}^{(i-1)}d \mathbf{p}}{dt} &= \boldsymbol{\omega}_{r,i-1} \times \mathbf{p}.
\end{aligned}$$

On the other hand

$$\frac{{}^{(i)}d \mathbf{p}}{dt} = \frac{{}^{(i-1)}d \mathbf{p}}{dt} + \boldsymbol{\omega}_{i,i-1} \times \mathbf{p}.$$

Hence

$$\boldsymbol{\omega}_{ri} \times \mathbf{p} = \boldsymbol{\omega}_{r,i-1} \times \mathbf{p} + \boldsymbol{\omega}_{i,i-1} \times \mathbf{p},$$

as this equation is satisfied for all \mathbf{p} fixed in the rigid body

$$\boldsymbol{\omega}_{ri} = \boldsymbol{\omega}_{r,i-1} + \boldsymbol{\omega}_{i,i-1}. \quad (4.27)$$

With $i = n$, Eq. (4.27) gives

$$\boldsymbol{\omega}_{rn} = \boldsymbol{\omega}_{r,n-1} + \boldsymbol{\omega}_{n,n-1}. \quad (4.28)$$

With $i = n - 1$, Eq. (4.27) gives

$$\boldsymbol{\omega}_{r,n-1} = \boldsymbol{\omega}_{r,n-2} + \boldsymbol{\omega}_{n-1,n-2}. \quad (4.29)$$

Substitute Eq. (4.29) into Eq. (4.28)

$$\boldsymbol{\omega}_{rn} = \boldsymbol{\omega}_{r,n-2} + \boldsymbol{\omega}_{n-1,n-2} + \boldsymbol{\omega}_{n,n-1}.$$

Next use Eq. (4.27) with $i = n - 2$, then with $i = n - 3$, and so forth.

Motion of a point that moves relative to a rigid body

A reference frame that moves with the rigid body is a body fixed reference frame. Figure 4.2 shows a rigid body (RB), in motion relative to a primary reference frame with its origin at point O_0 , $O_0x_0y_0z_0$. The primary reference frame is a fixed reference frame or an earth fixed reference frame. The unit vectors $\mathbf{i}_0, \mathbf{j}_0$, and \mathbf{k}_0 of the primary reference reference frame are constant.

The body fixed reference frame (mobile reference frame), $Oxyz$, has its origin at a point O of the rigid body ($O \in (RB)$), and is a moving reference frame relative to the primary reference. The unit vectors \mathbf{i}, \mathbf{j} , and \mathbf{k} of the body fixed reference frame are not constant, because they rotate with the body fixed reference frame.

The position vector of a point P of the rigid body ($P \in (RB)$) relative to the origin, O , of the body fixed reference frame is the vector \mathbf{r}_{OP} . The velocity of P relative to O is

$$\frac{d\mathbf{r}_{OP}}{dt} = \boldsymbol{\omega} \times \mathbf{r}_{OP},$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rigid body.

The position vector of a point A (the point A is not assumed to be a point of the rigid body, as shown in Fig. 4.2) relative to the origin O_0 of the primary reference frame is

$$\mathbf{r}_A = \mathbf{r}_O + \mathbf{r},$$

where

$$\mathbf{r} = \mathbf{r}_{OA} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

is the position vector of A relative to the origin O , of the body fixed reference frame, and x, y , and z are the coordinates of A in terms of the body fixed

reference frame. The velocity of the point A is the time derivative of the position vector \mathbf{r}_A

$$\begin{aligned}\mathbf{v}_A &= \frac{d\mathbf{r}_O}{dt} + \frac{d\mathbf{r}}{dt} = \mathbf{v}_O + \mathbf{v}_{AO} = \\ &\mathbf{v}_O + \frac{dx}{dt}\mathbf{i} + x\frac{d\mathbf{i}}{dt} + \frac{dy}{dt}\mathbf{j} + y\frac{d\mathbf{j}}{dt} + \frac{dz}{dt}\mathbf{k} + z\frac{d\mathbf{k}}{dt}.\end{aligned}$$

Using Poisson formulas, the total derivative of the the position vector \mathbf{r} is

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} + \boldsymbol{\omega} \times \mathbf{r}.$$

The velocity of A relative to the body fixed reference frame is a derivative in the body fixed reference frame

$$\mathbf{v}_{Arel} = \frac{{}^{(RB)}d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}, \quad (4.30)$$

A general formula for the total derivative of a moving vector \mathbf{r} can be written as

$$\frac{d\mathbf{r}}{dt} = \frac{{}^{(RB)}d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}, \quad (4.31)$$

where $\frac{d\mathbf{r}}{dt} = \frac{{}^{(0)}d\mathbf{r}}{dt}$ is the derivative in the fixed reference frame (0) ($O_0x_0y_0z_0$), and $\frac{{}^{(RB)}d\mathbf{r}}{dt}$ is the derivative in the mobile reference frame (body fixed reference frame).

The velocity of the point A relative to the primary reference frame is

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{Arel} + \boldsymbol{\omega} \times \mathbf{r}, \quad (4.32)$$

Equation (4.32) expresses the velocity of a point A as the sum of three terms:

- the velocity of a point O of the rigid body,
- the velocity \mathbf{v}_{Arel} of A relative to the rigid body, and
- the velocity $\boldsymbol{\omega} \times \mathbf{r}$ of A relative to O due to the rotation of the rigid body.

The acceleration of the point A relative to the primary reference frame is obtained by taking the time derivative of Eq. (4.32)

$$\begin{aligned}\mathbf{a}_A &= \mathbf{a}_O + \mathbf{a}_{AO} \\ &= \mathbf{a}_O + \mathbf{a}_{Arel} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}),\end{aligned} \quad (4.33)$$

where

$$\mathbf{a}_{Arel} = \frac{{}^{(RB)}d^2 \mathbf{r}}{dt^2} = \frac{d^2x}{dt^2} \mathbf{i} + \frac{d^2y}{dt^2} \mathbf{j} + \frac{d^2z}{dt^2} \mathbf{k}, \quad (4.34)$$

is the acceleration of A relative to the body fixed reference frame or relative to the rigid body. The term

$$\mathbf{a}_{Cor} = 2\boldsymbol{\omega} \times \mathbf{v}_{Arel}.$$

is called the *Coriolis acceleration*.

In the case of planar motion, Eq. (4.33) becomes

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{Arel} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{r} - \boldsymbol{\omega}^2 \mathbf{r}. \quad (4.35)$$

The velocity \mathbf{v}_A and the acceleration \mathbf{a}_A of a point A are relative to the primary reference frame. The terms \mathbf{v}_{Arel} and \mathbf{a}_{Arel} are the velocity and acceleration of point A relative to the body fixed reference frame, i.e., they are the velocity and acceleration measured by an observer moving with the rigid body, Fig. 4.3.

If A is a point of the rigid body, $A \in RB$, $\mathbf{v}_{Arel} = \mathbf{0}$ and $\mathbf{a}_{Arel} = \mathbf{0}$.

Inertial reference frames

A reference frame is inertial if Newton's second law is applied in the form $\sum \mathbf{F} = m\mathbf{a}$. Figure 4.4 shows a nonaccelerating, nonrotating reference frame with the origin at O_0 , and a secondary nonrotating, earth centered reference frame with the origin at O . The nonaccelerating, nonrotating reference frame with the origin at O_0 is assumed to be an inertial reference. The acceleration of the earth, due to the gravitational attractions of the sun, moon, etc., is \mathbf{g}_O . The earth centered reference frame has the acceleration \mathbf{g}_O , too. Newton's second law for an object A of mass m , using the hypothetical nonaccelerating, nonrotating reference frame with the origin at O_0 , can be written as

$$m\mathbf{a}_A = m\mathbf{g}_A + \sum \mathbf{F}, \quad (4.36)$$

where \mathbf{a}_A is the acceleration of A relative to O_0 , \mathbf{g}_A is the resulting gravitational acceleration, and $\sum \mathbf{F}$ is the sum of all other external forces acting on A . The acceleration of A relative to O_0 is

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{Arel},$$

where \mathbf{a}_{Arel} is the acceleration of A relative to the earth centered reference frame. The acceleration of the origin O is equal to the gravitational acceleration of the earth $\mathbf{a}_O = \mathbf{g}_O$. The earth-centered reference frame does not rotate ($\boldsymbol{\omega} = \mathbf{0}$). If the object A is on or near the earth, its gravitational acceleration \mathbf{g}_A due to the attraction of the sun, etc., is nearly equal to the gravitational acceleration of the earth \mathbf{g}_O , and Eq. (4.36) becomes

$$\sum \mathbf{F} = m\mathbf{a}_{Arel}. \quad (4.37)$$

Newton's second law can be applied using a nonrotating, earth centered reference frame if the object is near the earth. In most applications, Newton's second law can be applied using an earth fixed reference frame. Figure 4.5 shows a nonrotating reference frame with its origin at the center of the earth O and a secondary earth fixed reference frame with its origin at a point B . The earth fixed reference frame with the origin at B can be assumed to be an inertial reference and $\sum \mathbf{F} = m\mathbf{a}_{Arel}$, where \mathbf{a}_{Arel} is the acceleration of A relative to the earth fixed reference frame.

The motion of an object A can be analyzed using a primary inertial reference frame with its origin at the point O , Fig. 4.6. A secondary reference frame with its origin at B undergoes an arbitrary motion with angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$. The Newton's second law for the object A of mass m is $\sum \mathbf{F} = m\mathbf{a}_A$, where \mathbf{a}_A is the acceleration of A acceleration relative to O . Newton's second law can be written in the form

$$\sum \mathbf{F} - m[\mathbf{a}_B + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel} + \boldsymbol{\alpha} \times \mathbf{r}_{BA} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{BA})] = m\mathbf{a}_{Arel}, \quad (4.38)$$

where \mathbf{a}_{Arel} is the acceleration of A relative to the secondary reference frame. The term \mathbf{a}_B is the acceleration of the origin B of the secondary reference frame relative to the primary inertial reference. The term $2\boldsymbol{\omega} \times \mathbf{v}_{Arel}$ is the Coriolis acceleration, and the term $-2m\boldsymbol{\omega} \times \mathbf{v}_{Arel}$ is called the Coriolis force. This is Newton's second law expressed in terms of a secondary reference frame undergoing an arbitrary motion relative to an inertial primary reference frame.

The classical method for obtaining the velocities and/or accelerations of links and joints is the compute the derivatives of the positions and/or velocities with respect to time.

4.2 Driver Link

For a driver link in rotational motion, Fig. 3.3(a), the following position relation can be written

$$\begin{aligned}x_B(t) &= x_A + L_{AB} \cos \phi(t), \\y_B(t) &= y_A + L_{AB} \sin \phi(t).\end{aligned}\quad (4.39)$$

Differentiating Eq. (4.39) with respect to time the following expressions are obtained

$$\begin{aligned}v_{Bx} = \dot{x}_B &= \frac{dx_B(t)}{dt} = -L_{AB}\dot{\phi} \sin \phi, \\v_{By} = \dot{y}_B &= \frac{dy_B(t)}{dt} = L_{AB}\dot{\phi} \cos \phi.\end{aligned}\quad (4.40)$$

The angular velocity of the driver link is $\omega = \dot{\phi}$.
The time derivative of Eq. (4.40) yields

$$\begin{aligned}a_{Bx} = \ddot{x}_B &= \frac{dv_{Bx}(t)}{dt} = -L_{AB}\dot{\phi}^2 \cos \phi - L_{AB}\ddot{\phi} \sin \phi, \\a_{By} = \ddot{y}_B &= \frac{dv_{By}(t)}{dt} = -L_{AB}\dot{\phi}^2 \sin \phi + L_{AB}\ddot{\phi} \cos \phi,\end{aligned}\quad (4.41)$$

where $\alpha = \ddot{\phi}$ is the angular acceleration of the driver link AB .

4.3 RRR Dyad

For the RRR dyad, Fig. 3.4, there are two quadratic equations of the form

$$\begin{aligned}[x_C(t) - x_A]^2 + [y_C(t) - y_A]^2 &= L_{AC} = L_2^2, \\[x_C(t) - x_B]^2 + [y_C(t) - y_B]^2 &= L_{BC} = L_3^2.\end{aligned}\quad (4.42)$$

Solving the above system of quadratic equations, the coordinates $x_C(t)$ and $y_C(t)$ are obtained.

The derivative of Eq. (4.42) with respect to time yields

$$\begin{aligned}(x_C - x_A)(\dot{x}_C - \dot{x}_A) + (y_C - y_A)(\dot{y}_C - \dot{y}_A) &= 0, \\(x_C - x_B)(\dot{x}_C - \dot{x}_B) + (y_C - y_B)(\dot{y}_C - \dot{y}_B) &= 0,\end{aligned}\quad (4.43)$$

From Eq. 4.43 the velocity vector of the joint C , $\mathbf{v}_C = [\dot{x}_C, \dot{y}_C]^T$, is written in matrix form

$$\mathbf{v}_C = \mathbf{M}_1 \cdot \mathbf{v}, \quad (4.44)$$

where

$$\begin{aligned} \mathbf{v} &= [\dot{x}_A, \dot{y}_A, \dot{x}_B, \dot{y}_B]^T, \\ \mathbf{M}_1 &= \mathbf{A}_1^{-1} \cdot \mathbf{A}_2, \\ \mathbf{A}_1 &= \begin{bmatrix} x_C - x_A & y_C - y_A \\ x_C - x_B & y_C - y_B \end{bmatrix}, \\ \mathbf{A}_2 &= \begin{bmatrix} x_C - x_A & y_C - y_A & 0 & 0 \\ 0 & 0 & x_C - x_B & y_C - y_B \end{bmatrix}. \end{aligned}$$

Similarly, by differentiating Eq. (4.43), the following acceleration equations are obtained

$$\begin{aligned} (\dot{x}_C - \dot{x}_A)^2 + (x_C - x_A) (\ddot{x}_C - \ddot{x}_A) + \\ (\dot{y}_C - \dot{y}_A)^2 + (y_C - y_A) (\ddot{y}_C - \ddot{y}_A) &= 0, \\ (\dot{x}_C - \dot{x}_B)^2 + (x_C - x_B) (\ddot{x}_C - \ddot{x}_B) + \\ (\dot{y}_C - \dot{y}_B)^2 + (y_C - y_B) (\ddot{y}_C - \ddot{y}_B) &= 0. \end{aligned} \quad (4.45)$$

The acceleration vector of the joint C is obtained from the above system of equations

$$\mathbf{a}_C = [\ddot{x}_C, \ddot{y}_C]^T = \mathbf{M}_1 \cdot \mathbf{a} + \mathbf{M}_2, \quad (4.46)$$

where

$$\begin{aligned} \mathbf{a} &= [\ddot{x}_A, \ddot{y}_A, \ddot{x}_B, \ddot{y}_B]^T, \\ \mathbf{M}_2 &= -\mathbf{A}_1^{-1} \cdot \mathbf{A}_3, \\ \mathbf{A}_3 &= \begin{bmatrix} (\dot{x}_C - \dot{x}_A)^2 + (\dot{y}_C - \dot{y}_A)^2 \\ (\dot{x}_C - \dot{x}_B)^2 + (\dot{y}_C - \dot{y}_B)^2 \end{bmatrix}. \end{aligned}$$

To compute the angular velocity and acceleration of the RRR dyad, the following equations are written for the angles $\phi_2(t)$ and $\phi_3(t)$

$$\begin{aligned} y_C(t) - y_A + [x_C(t) - x_A] \tan \phi_2(t) &= 0, \\ y_C(t) - y_B + [x_C(t) - x_B] \tan \phi_3(t) &= 0. \end{aligned} \quad (4.47)$$

The derivative with respect to time of Eq. (4.47) yields

$$\begin{aligned} \dot{y}_C - \dot{y}_A - (\dot{x}_C - \dot{x}_A) \tan \phi_2 - (x_C - x_A) \frac{1}{\cos^2 \phi_2} \dot{\phi}_2 &= 0, \\ \dot{y}_C - \dot{y}_B - (\dot{x}_C - \dot{x}_B) \tan \phi_3 - (x_C - x_B) \frac{1}{\cos^2 \phi_3} \dot{\phi}_3 &= 0. \end{aligned} \quad (4.48)$$

The angular velocity vector is computed as

$$\boldsymbol{\omega} = [\dot{\phi}_2, \dot{\phi}_3]^T = [\omega_2, \omega_3]^T = \boldsymbol{\Omega}_1 \cdot \mathbf{v} + \boldsymbol{\Omega}_2 \cdot \mathbf{v}_C, \quad (4.49)$$

where

$$\boldsymbol{\Omega}_1 = \begin{bmatrix} \frac{x_C - x_A}{L_2^2} & -\frac{x_C - x_A}{L_2^2} & 0 & 0 \\ 0 & 0 & \frac{x_C - x_B}{L_3^2} & -\frac{x_C - x_B}{L_3^2} \end{bmatrix},$$

$$\boldsymbol{\Omega}_2 = \begin{bmatrix} -\frac{x_C - x_A}{L_2^2} & \frac{x_C - x_A}{L_2^2} \\ -\frac{x_C - x_B}{L_3^2} & \frac{x_C - x_B}{L_3^2} \end{bmatrix}.$$

Differentiating Eq. (4.49), the angular acceleration vector, $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ is

$$\boldsymbol{\alpha} = [\ddot{\phi}_2, \ddot{\phi}_3]^T = [\alpha_2, \alpha_3]^T = \dot{\boldsymbol{\Omega}}_1 \cdot \mathbf{v} + \dot{\boldsymbol{\Omega}}_2 \cdot \mathbf{v}_C + \boldsymbol{\Omega}_1 \cdot \mathbf{a} + \boldsymbol{\Omega}_2 \cdot \mathbf{a}_C. \quad (4.50)$$

4.4 RRT Dyad

For the RRT dyad, Fig. 3.5(a), the following equations can be written for position analysis

$$\begin{aligned} [x_C(t) - x_A]^2 + [y_C(t) - y_A]^2 &= AC^2 = L_{AC}^2 = L_2^2, \\ [x_C(t) - x_B] \sin \alpha - [y_C(t) - y_B] \cos \alpha &= \pm h. \end{aligned} \quad (4.51)$$

From the above system of equations, $x_C(t)$ and $y_C(t)$ can be computed. The time derivative of Eq.(4.51) yields

$$\begin{aligned} (x_C - x_A) (\dot{x}_C - \dot{x}_A) + (y_C - y_A) (\dot{y}_C - \dot{y}_A) &= 0, \\ (x_C - x_B) \sin \alpha - (y_C - y_B) \cos \alpha &= 0. \end{aligned} \quad (4.52)$$

The solution for the velocity vector of the joint C from Eq. (4.52) is

$$\mathbf{v}_C = [\dot{x}_C, \dot{y}_C]^T = \mathbf{M}_3 \cdot \mathbf{v}, \quad (4.53)$$

where

$$\begin{aligned}\mathbf{M}_3 &= \mathbf{A}_4^{-1} \cdot \mathbf{A}_5, \\ \mathbf{A}_4 &= \begin{bmatrix} x_C - x_A & y_C - y_A \\ \sin \alpha & -\cos \alpha \end{bmatrix}, \\ \mathbf{A}_5 &= \begin{bmatrix} x_C - x_A & y_C - y_A & 0 & 0 \\ 0 & 0 & \sin \alpha & -\cos \alpha \end{bmatrix}.\end{aligned}$$

Differentiating Eq. (4.52) with respect to time

$$\begin{aligned}(\dot{x}_C - \dot{x}_A)^2 + (x_C - x_A)(\ddot{x}_C - \ddot{x}_A) + \\ (\dot{y}_C - \dot{y}_A)^2 + (y_C - y_A)(\ddot{y}_C - \ddot{y}_A) &= 0, \\ (\ddot{x}_C - \ddot{x}_B) \sin \alpha - (\ddot{y}_C - \ddot{y}_B) \cos \alpha &= 0,\end{aligned}\quad (4.54)$$

the acceleration vector \mathbf{a}_C is obtained as

$$\mathbf{a}_C = [\ddot{x}_C, \ddot{y}_C]^T = \mathbf{M}_3 \cdot \mathbf{a} + \mathbf{M}_4, \quad (4.55)$$

where

$$\begin{aligned}\mathbf{M}_4 &= -\mathbf{A}_4^{-1} \cdot \mathbf{A}_6, \\ \mathbf{A}_6 &= \begin{bmatrix} (\dot{x}_C - \dot{x}_A)^2 + (\dot{y}_C - \dot{y}_A)^2 \\ 0 \end{bmatrix}.\end{aligned}\quad (4.56)$$

The angular position of the element 2 is described by the following equation

$$y_C(t) - y_A - [x_C(t) - x_A] \tan \phi_2(t) = 0. \quad (4.57)$$

The time derivative of Eq. (4.57) yields

$$\dot{y}_C - \dot{y}_A - (\dot{x}_C - \dot{x}_A) \tan \phi_2 - (x_C - x_A) \frac{1}{\cos^2 \phi_2} \dot{\phi}_2 = 0, \quad (4.58)$$

and the angular velocity of the element 2 is

$$\omega_2 = \frac{x_C - x_A}{L_2^2} [(\dot{y}_C - \dot{y}_A) - (\dot{x}_C - \dot{x}_A) \tan \phi_2]. \quad (4.59)$$

The angular acceleration of the element 2 is $\alpha_2 = \dot{\omega}_2$.

4.5 RTR Dyad

For the RTR dyad, Fig. 3.6(a), the position relations are

$$\begin{aligned}
 [x_C(t) - x_A]^2 + [y_C(t) - y_A]^2 &= L_2^2, \\
 \tan \alpha &= \frac{\frac{y_C - y_B}{x_C - x_B} - \frac{y_C - y_A}{x_C - x_A}}{1 + \frac{y_C - y_B}{x_C - x_B} \cdot \frac{y_C - y_A}{x_C - x_A}} = \\
 &= \frac{(y_C - y_B)(x_C - x_A) - (y_C - y_A)(x_C - x_B)}{(x_C - x_B)(x_C - x_A) + (y_C - y_B)(y_C - y_A)}. \tag{4.60}
 \end{aligned}$$

The time derivative of Eq. (4.60) yields

$$\begin{aligned}
 (x_C - x_A)(\dot{x}_C - \dot{x}_A) + (y_C - y_A)(\dot{y}_C - \dot{y}_A) &= 0, \\
 \tan \alpha [(\dot{x}_C - \dot{x}_B)(x_C - x_A) + (x_C - x_B)(\dot{x}_C - \dot{x}_A)] &+ \\
 \tan \alpha [(\dot{y}_C - \dot{y}_A)(y_C - y_B) + (y_C - y_A)(\dot{y}_C - \dot{y}_B)] &+ \\
 (\dot{y}_C - \dot{y}_A)(x_C - x_B) + (y_C - y_A)(\dot{x}_C - \dot{x}_B) - & \\
 (\dot{y}_C - \dot{y}_B)(x_C - x_A) - (y_C - y_B)(\dot{x}_C - \dot{x}_A) &= 0, \tag{4.61}
 \end{aligned}$$

or in a matrix form

$$\mathbf{A}_7 \cdot \mathbf{v}_C = \mathbf{A}_8 \cdot \mathbf{v}, \tag{4.62}$$

where

$$\begin{aligned}
 \mathbf{A}_7 &= \begin{bmatrix} x_C - x_A & y_C - y_A \\ \gamma_1 & \gamma_2 \end{bmatrix}, \\
 \mathbf{A}_8 &= \begin{bmatrix} x_C - x_A & y_C - y_A & 0 & 0 \\ \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \end{bmatrix}.
 \end{aligned}$$

In addition,

$$\begin{aligned}
 \gamma_1 &= [(x_C - x_B) + (x_C - x_A)] \tan \alpha - (y_C - y_B) + (y_C - y_A), \\
 \gamma_2 &= [(y_C - y_A) + (y_C - y_B)] \tan \alpha - (x_C - x_A) + (x_C - x_B), \\
 \gamma_3 &= (x_C - x_B) \tan \alpha + (y_C - y_B), \\
 \gamma_4 &= (x_C - x_A) \tan \alpha + (y_C - y_A), \\
 \gamma_5 &= (y_C - y_B) \tan \alpha + (x_C - x_B), \\
 \gamma_6 &= (y_C - y_A) \tan \alpha - (x_C - x_A).
 \end{aligned}$$

The solution for the velocity vector, \mathbf{v}_C , of the joint C , from Eq.(4.62) is

$$\mathbf{v}_C = \mathbf{M}_5 \cdot \mathbf{v}, \quad (4.63)$$

where

$$\mathbf{M}_5 = \mathbf{A}_7^{-1} \cdot \mathbf{A}_8.$$

Differentiating Eq. (4.62), the following relation is obtained

$$\mathbf{A}_7 \cdot \mathbf{a}_C = \mathbf{A}_8 \cdot \mathbf{a} - \mathbf{A}_9, \quad (4.64)$$

where

$$\mathbf{A}_9 = \begin{bmatrix} (\dot{x}_C - \dot{x}_A)^2 + (\dot{y}_C - \dot{y}_A)^2 \\ \gamma_7 \end{bmatrix},$$

$$\gamma_7 = 2(\dot{x}_C - \dot{x}_B)(\dot{x}_C - \dot{x}_A) \tan \alpha + 2(\dot{y}_C - \dot{y}_B)(\dot{y}_C - \dot{y}_A) \tan \alpha -$$

$$2(\dot{y}_C - \dot{y}_B)(\dot{x}_C - \dot{x}_A) + 2(\dot{y}_C - \dot{y}_A)(\dot{x}_C - \dot{x}_B).$$

The acceleration vector of the joint C is

$$\mathbf{a}_C = \mathbf{M}_5 \cdot \mathbf{a} - \mathbf{M}_6, \quad (4.65)$$

where

$$\mathbf{M}_6 = \mathbf{A}_7^{-1} \cdot \mathbf{A}_9.$$

To compute the angular velocities for the RTR dyad, the following equations can be written

$$y_C(t) - y_A = [x_C(t) - x_A] \tan \phi_2$$

$$\phi_3 = \phi_2 + \alpha. \quad (4.66)$$

The time derivative of Eq. (4.66) yields

$$(\dot{y}_C - \dot{y}_A) = (\dot{x}_C - \dot{x}_A) \tan \phi_2 + (x_C - x_A) \frac{1}{\cos^2 \phi_2} \dot{\phi}_2$$

$$\dot{\phi}_3 = \dot{\phi}_2. \quad (4.67)$$

The angular velocities of the links 2 and 3 are

$$\omega_2 = \omega_3 = \frac{\cos^2 \phi_2}{x_C - x_A} [(\dot{y}_C - \dot{y}_A) - (\dot{x}_C - \dot{x}_A) \tan \phi_2]. \quad (4.68)$$

The angular accelerations are found to be

$$\alpha_2 = \alpha_3 = \dot{\omega}_2 = \dot{\omega}_3. \quad (4.69)$$

4.6 TRT Dyad

For the TRT dyad, Fig. 3.7, the two position equations are

$$\begin{aligned} [x_C(t) - x_A] \sin \alpha - [y_C(t) - y_A] \cos \alpha &= \pm d, \\ [x_C(t) - x_B] \sin \beta - [y_C(t) - y_B] \cos \beta &= \pm h. \end{aligned} \quad (4.70)$$

The derivative with respect to time of Eq. (4.70) yields

$$\begin{aligned} (\dot{x}_C - \dot{x}_A) \sin \alpha - (\dot{y}_C - \dot{y}_A) \cos \alpha + \\ (x_C - x_A) \dot{\alpha} \cos \alpha + (y_C - y_A) \dot{\alpha} \sin \alpha &= 0, \\ (\dot{x}_C - \dot{x}_B) \sin \beta - (\dot{y}_C - \dot{y}_B) \cos \beta + \\ (x_C - x_B) \dot{\beta} \cos \beta + (y_C - y_B) \dot{\beta} \sin \beta &= 0, \end{aligned} \quad (4.71)$$

or in a matrix form

$$\mathbf{A}_{10} \cdot \mathbf{v}_C = \mathbf{A}_{11} \cdot \mathbf{v}_1, \quad (4.72)$$

where

$$\begin{aligned} \mathbf{v}_1 &= [\dot{x}_A, \dot{y}_A, \dot{\alpha}, \dot{x}_B, \dot{y}_B, \dot{\beta}]^T, \\ \mathbf{A}_{10} &= \begin{bmatrix} -\sin \alpha & -\cos \alpha \\ \sin \beta & -\cos \beta \end{bmatrix}, \\ \mathbf{A}_{11} &= \begin{bmatrix} \sin \alpha & -\cos \alpha & \xi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin \beta & -\cos \beta & \xi_2 \end{bmatrix}, \\ \xi_1 &= (x_A - x_C) \cos \alpha + (y_A - y_C) \sin \alpha, \\ \xi_2 &= (x_B - x_C) \cos \beta + (y_B - y_C) \sin \beta. \end{aligned}$$

The solution of Eq. (4.72) gives the velocity of the joint C

$$\mathbf{v}_C = \mathbf{M}_7 \cdot \mathbf{v}_1, \quad (4.73)$$

where

$$\mathbf{M}_7 = \mathbf{A}_{10}^{-1} \cdot \mathbf{A}_{11}.$$

Differentiating Eq. (4.72) with respect to time gives

$$\mathbf{A}_{10} \cdot \mathbf{a}_C = \mathbf{A}_{11} \cdot \mathbf{a}_1 - \mathbf{A}_{12}, \quad (4.74)$$

where

$$\begin{aligned}\mathbf{a}_1 &= [\ddot{x}_A, \ddot{y}_A, \ddot{\alpha}, \ddot{x}_B, \ddot{y}_B, \ddot{\beta}]^T, \\ \mathbf{A}_{12} &= \begin{bmatrix} \xi_3 \\ \xi_4 \end{bmatrix}, \\ \xi_3 &= 2(\dot{x}_C - \dot{x}_A)\dot{\alpha} \cos \alpha + 2(\dot{y}_C - \dot{y}_A)\dot{\beta} \sin \alpha - \\ &\quad (x_C - x_A)\dot{\alpha}^2 \sin \alpha + (y_C - y_A)\dot{\alpha}^2 \cos \alpha, \\ \xi_4 &= 2(\dot{x}_C - \dot{x}_B)\dot{\beta} \cos \beta + 2(\dot{y}_C - \dot{y}_B)\dot{\beta} \sin \beta - \\ &\quad (x_C - x_B)\dot{\beta}^2 \sin \beta + (y_C - y_B)\dot{\beta}^2 \cos \beta.\end{aligned}$$

The solution of Eq. (4.74) gives the acceleration vector of joint C

$$\mathbf{a}_C = \mathbf{M}_7 \cdot \mathbf{a} + \mathbf{M}_8. \quad (4.75)$$

where

$$\mathbf{M}_8 = \mathbf{A}_{10}^{-1} \cdot \mathbf{A}_{12}.$$

4.7 Examples

Example 4.1. R-TRR Mechanism.

The following dimensions are given for the mechanism shown in Fig. 4.7: $AC = a = 0.100$ m and $BC = 0.300$ m. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 45^\circ$. The coordinates of the joint B are $x_B = y_B = 0.256$ m. The driver link 1 rotates with a constant speed of $n_1 = 30$ rpm. Find the velocities and the accelerations of the mechanism.

Solution

A cartesian reference frame with the origin at A is selected. The coordinates of the joint A are

$$x_A = y_A = 0.$$

the coordinates of the joint C are

$$x_C = AC = 0.100 \text{ m} \quad \text{and} \quad y_C = 0,$$

and the coordinates of the joint B are

$$x_B = 0.256 \text{ m} \quad \text{and} \quad y_B = 0.256 \text{ m}.$$

The position of the joint B is calculated from the equations

$$\tan \phi(t) = \frac{y_B(t)}{x_B(t)} \quad \text{and} \quad [x_B(t) - x_C]^2 + [y_B(t) - y_C]^2 = BC^2,$$

or

$$\begin{aligned} x_B(t) \sin \phi(t) &= y_B(t) \cos \phi(t), \\ [x_B(t) - x_C]^2 + [y_B(t) - y_C]^2 &= BC^2. \end{aligned} \quad (4.76)$$

The linear velocity of the point B on link 3 or 2 is

$$\mathbf{v}_B = \mathbf{v}_{B_3} = \mathbf{v}_{B_2} = \dot{x}_B \mathbf{i} + \dot{y}_B \mathbf{j},$$

where

$$\dot{x}_B = \frac{dx_B}{dt} \quad \text{and} \quad \dot{y}_B = \frac{dy_B}{dt}.$$

The velocity analysis is carried out differentiating Eq. (4.76)

$$\begin{aligned} \dot{x}_B \sin \phi + x_B \dot{\phi} \cos \phi &= \dot{y}_B \cos \phi - y_B \dot{\phi} \sin \phi, \\ \dot{x}_B(x_B - x_C) + \dot{y}_B(y_B - y_C) &= 0, \end{aligned}$$

or

$$\begin{aligned} \dot{x}_B \sin \phi + x_B \omega \cos \phi &= \dot{y}_B \cos \phi - y_B \omega \sin \phi, \\ \dot{x}_B(x_B - x_C) + \dot{y}_B(y_B - y_C) &= 0. \end{aligned} \quad (4.77)$$

The magnitude of the angular velocity of the driver link 1 is

$$\omega = \omega_1 = \dot{\phi} = \frac{\pi n_1}{30} = \frac{\pi (30 \text{ rpm})}{30} = 3.141 \text{ rad/s}. \quad (4.78)$$

The angular velocity of link 1 is

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 = \omega \mathbf{k} = 3.141 \mathbf{k} \text{ rad/s}.$$

The link 2 and the driver link 1 have the same angular velocity $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$. For the given numerical data Eq. (4.77) becomes

$$\begin{aligned} \dot{x}_B \sin 45^\circ + 0.256 (3.141) \cos 45^\circ &= \dot{y}_B \cos 45^\circ - 0.256 (3.141) \sin 45^\circ, \\ \dot{x}_B(0.256 - 0.1) + \dot{y}_B(0.256 - 0) &= 0. \end{aligned} \quad (4.79)$$

The solution of Eq. (4.79) gives

$$\dot{x}_B = -0.999 \text{ m/s} \quad \text{and} \quad \dot{y}_B = 0.609 \text{ m/s}.$$

The velocity of B is

$$\mathbf{v}_B = \mathbf{v}_{B_3} = \mathbf{v}_{B_2} = -0.999 \mathbf{i} + 0.609 \mathbf{j} \text{ m/s},$$

$$|\mathbf{v}_B| = |\mathbf{v}_{B_3}| = |\mathbf{v}_{B_2}| = \sqrt{(-0.999)^2 + (0.609)^2} = 1.171 \text{ m/s}.$$

The acceleration analysis is obtained using the derivative of the velocities given by Eq. (4.77)

$$\begin{aligned} \ddot{x}_B \sin \phi + \dot{x}_B \omega \cos \phi + \dot{x}_B \omega \cos \phi - x_B \omega^2 \sin \phi &= \\ \ddot{y}_B \cos \phi - \dot{y}_B \omega \sin \phi - \dot{y}_B \omega \sin \phi + y_B \omega^2 \cos \phi, \\ \ddot{x}_B(x_B - x_C) + \dot{x}_B^2 + \ddot{y}_B(y_B - y_C) + \dot{y}_B^2 &= 0. \end{aligned} \quad (4.80)$$

The magnitude of the angular acceleration of the driver link 1 is

$$\alpha = \dot{\omega} = \ddot{\phi} = 0.$$

Numerically Eq. (4.80) gives

$$\begin{aligned} \ddot{x}_B \sin 45^\circ + 2(-0.999)(3.141) \cos 45^\circ - 0.256(3.141)^2 \sin 45^\circ = \\ \ddot{y}_B \cos 45^\circ - 2(0.609)(3.141) \sin 45^\circ + 0.256(3.141)^2 \cos 45^\circ, \\ \ddot{x}_B(0.256 - 0.1) + (-0.999)^2 + \ddot{y}_B(0.256) + 0.609^2 = 0. \end{aligned} \quad (4.81)$$

The solution of Eq. (4.81) is

$$\ddot{x}_B = -1.802 \text{ m/s}^2 \quad \text{and} \quad \ddot{y}_B = -4.255 \text{ m/s}^2.$$

The acceleration of B on link 3 or 2 is

$$\mathbf{a}_B = \mathbf{a}_{B_3} = \mathbf{a}_{B_2} = \ddot{x}_B \mathbf{i} + \ddot{y}_B \mathbf{j} = -1.802 \mathbf{i} - 4.255 \mathbf{j} \quad \text{m/s}^2,$$

$$|\mathbf{a}_B| = |\mathbf{a}_{B_3}| = |\mathbf{a}_{B_2}| = \sqrt{(-1.802)^2 + (-4.255)^2} = 4.620 \quad \text{m/s}^2.$$

The slope of the link 3 (the points B and C are on the straight line BC) is

$$\tan \phi_3(t) = \frac{y_B(t) - y_C}{x_B(t) - x_C},$$

or

$$[x_B(t) - x_C] \sin \phi_3(t) = [y_B(t) - y_C] \cos \phi_3(t). \quad (4.82)$$

The angle ϕ_3 is computed as follows

$$\phi_3 = \arctan \frac{y_B - y_C}{x_B - x_C} = \arctan \frac{0.256}{0.256 - 0.1} = 1.023 \text{ rad} = 58.633^\circ$$

The derivative of Eq. (4.82) yields

$$\dot{x}_B \sin \phi_3 + (x_B - x_C) \dot{\phi}_3 \cos \phi_3 = \dot{y}_B \cos \phi_3 - (y_B - y_C) \dot{\phi}_3 \sin \phi_3,$$

or

$$\dot{x}_B \sin \phi_3 + (x_B - x_C) \omega_3 \cos \phi_3 = \dot{y}_B \cos \phi_3 - (y_B - y_C) \omega_3 \sin \phi_3, \quad (4.83)$$

where $\omega_3 = \dot{\phi}_3$. Numerically Eq. (4.83) gives

$$\begin{aligned} -0.999 \sin 58.633^\circ + (0.256 - 0.1) \omega_3 \cos 58.633^\circ = \\ 0.609 \cos 58.633^\circ - 0.256 \omega_3 \sin 58.633^\circ, \end{aligned}$$

with the solution $\omega_3 = 3.903 \text{ rad/s}$.

The angular velocity of link 3 is

$$\boldsymbol{\omega}_3 = \omega_3 \mathbf{k} = 3.903 \mathbf{k} \text{ rad/s.}$$

The angular acceleration of link 3, $\alpha_3 = \dot{\omega}_3 = \ddot{\phi}_3$, is obtained using the derivative of the Eq. (4.83)

$$\begin{aligned} & \ddot{x}_B \sin \phi_3 + \dot{x}_B \omega_3 \cos \phi_3 + \\ & \dot{x}_B \omega_3 \cos \phi_3 + (x_B - x_C) \dot{\omega}_3 \cos \phi_3 - (x_B - x_C) \omega_3^2 \sin \phi_3 = \\ & \ddot{y}_B \cos \phi_3 - \dot{y}_B \omega_3 \sin \phi_3 - \\ & \dot{y}_B \omega_3 \sin \phi_3 - (y_B - y_C) \dot{\omega}_3 \sin \phi_3 - (y_B - y_C) \omega_3^2 \cos \phi_3, \end{aligned}$$

or

$$\begin{aligned} & \ddot{x}_B \sin \phi_3 + 2 \dot{x}_B \omega_3 \cos \phi_3 + (x_B - x_C) \alpha_3 \cos \phi_3 - (x_B - x_C) \omega_3^2 \sin \phi_3 = \\ & \ddot{y}_B \cos \phi_3 - 2 \dot{y}_B \omega_3 \sin \phi_3 - (y_B - y_C) \alpha_3 \sin \phi_3 - (y_B - y_C) \omega_3^2 \cos \phi_3. \end{aligned}$$

Numerically the previous equation becomes

$$\begin{aligned} & -1.802 \sin 58.633^\circ + 2(-0.999)(3.903) \cos 58.633^\circ + \\ & (0.256 - 0.1) \alpha_3 \cos 58.633^\circ - (0.256 - 0.1)(3.903)^2 \sin 58.633^\circ = \\ & -4.255 \cos 58.633^\circ - 2(0.609)(3.903) \sin 58.633^\circ - \\ & 0.256 \alpha_3 \sin 58.633^\circ - 0.256(3.903)^2 \cos 58.633^\circ, \end{aligned}$$

with the solution $\alpha_3 = -2.252 \text{ rad/s}^2$. The angular acceleration of link 3 is

$$\boldsymbol{\alpha}_3 = \alpha_3 \mathbf{k} = -2.252 \mathbf{k} \text{ rad/s}^2.$$

The velocity of the point B_1 on link 1 is calculated with the expression of velocity field of two points (B_1 and A) on the same rigid body (link 1)

$$\begin{aligned} \mathbf{v}_{B_1} = \mathbf{v}_A + \boldsymbol{\omega}_1 \times \mathbf{r}_{AB} = \boldsymbol{\omega}_1 \times \mathbf{r}_{AB} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_1 \\ x_B & y_B & 0 \end{vmatrix} = \\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 3.141 \\ 0.256 & 0.256 & 0 \end{vmatrix} &= -0.804 \mathbf{i} + 0.804 \mathbf{j} \text{ m/s.} \end{aligned}$$

The velocity field of two points (B_1 and B_2) not situated on the same rigid body (B_1 is on link 1 and B_2 is on link 2) is calculated with

$$\mathbf{v}_{B_2} = \mathbf{v}_{B_1} + \mathbf{v}_{B_2B_1}^r = \mathbf{v}_{B_1} + \mathbf{v}_{B_{21}}^r,$$

where $\mathbf{v}_{B_{21}}^r$ is the relative velocity of the point B_2 on link 2 with respect to the point B_1 on link 1

$$\begin{aligned} \mathbf{v}_{B_{21}}^r &= \mathbf{v}_{B_2} - \mathbf{v}_{B_1} = -0.999\mathbf{i} + 0.609\mathbf{j} - (-0.804\mathbf{i} + 0.804\mathbf{j}) \\ &= -0.195\mathbf{i} - 0.195\mathbf{j} \quad \text{m/s.} \end{aligned}$$

The relation between the angular velocities of the link 2 and the link 3 is

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_3 + \boldsymbol{\omega}_{23},$$

and the relative angular velocity of the link 2 with respect to the link 3 is

$$\boldsymbol{\omega}_{23} = \boldsymbol{\omega}_2 - \boldsymbol{\omega}_3 = 3.141\mathbf{k} - 3.903\mathbf{k} = -0.762\mathbf{k} \quad \text{rad/s.}$$

The acceleration of the point B_1 on link 1 is

$$\begin{aligned} \mathbf{a}_{B_1} &= \mathbf{a}_A + \boldsymbol{\alpha}_1 \times \mathbf{r}_{AB} - \omega_1^2 \mathbf{r}_{AB} = -\omega_1^2 \mathbf{r}_{AB} = -\omega_1^2 (x_B \mathbf{i} + y_B \mathbf{j}) \\ &= -3.141^2 (0.256 \mathbf{i} + 0.256 \mathbf{j}) = -2.528\mathbf{i} - 2.528\mathbf{j} \quad \text{m/s}^2. \end{aligned}$$

The acceleration of B_2 in terms of B_1 is

$$\mathbf{a}_{B_2} = \mathbf{a}_{B_1} + \mathbf{a}_{B_{21}}^r + 2\boldsymbol{\omega}_1 \times \mathbf{v}_{B_{21}}^r,$$

where $\mathbf{a}_{B_{21}}^r$ is the relative acceleration of the point B_2 on link 2 with respect to the point B_1 on link 1 and $2\boldsymbol{\omega}_1 \times \mathbf{v}_{B_{21}}^r$ is the Coriolis acceleration

$$\begin{aligned} \mathbf{a}_{B_{21}}^c &= 2\boldsymbol{\omega}_1 \times \mathbf{v}_{B_{21}}^r = 2\boldsymbol{\omega}_2 \times \mathbf{v}_{B_{21}}^r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_1 \\ v_{B_{21}x}^r & v_{B_{21}y}^r & 0 \end{vmatrix} = \\ & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 3.141 \\ -0.195 & -0.195 & 0 \end{vmatrix} = 1.226\mathbf{i} - 1.226\mathbf{j} \quad \text{m/s}^2. \end{aligned}$$

The relative acceleration of B_2 with respect to B_1 is

$$\begin{aligned} \mathbf{a}_{B_{21}}^r &= \mathbf{a}_{B_2} - \mathbf{a}_{B_1} - \mathbf{a}_{B_{21}}^c = \\ &= -1.802\mathbf{i} - 4.255\mathbf{j} - (-2.528\mathbf{i} - 2.528\mathbf{j}) - (1.226\mathbf{i} - 1.226\mathbf{j}) = \\ &= -0.5\mathbf{i} - 0.5\mathbf{j} \quad \text{m/s}^2. \end{aligned}$$

The relative angular acceleration of the link 2 with respect to the link 3 is

$$\boldsymbol{\alpha}_{23} = \boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_3 = -\boldsymbol{\alpha}_3 = 2.252 \mathbf{k} \text{ rad/s}^2,$$

where $\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_1 = \mathbf{0}$.

Example 4.2. R-RTR-RRT Mechanism

The mechanism shown in Fig. 4.8 and has the dimensions: $AB = 0.100$ m, $AC = 0.150$ m, $CD = 0.075$ m, and $DE = 0.200$ m. The angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 45^\circ$, and the angular speed of the driver link 1 is $\omega = \omega_1 = 4.712$ rad/s. Find the velocities and the accelerations of the mechanism.

Solution

The origin of the fixed reference frame is at $C \equiv 0$. The position of the fixed joint A is

$$x_A = 0, \quad y_A = AC = 0.150 \text{ m.}$$

The position of joint B is

$$x_B(t) = x_A + AB \cos \phi(t), \quad y_B(t) = y_A + AB \sin \phi(t),$$

and for $\phi = 45^\circ$

$$x_B = 0 + 0.100 \cos 45^\circ = 0.070 \text{ m}, \quad y_B = 0.150 + 0.100 \sin 45^\circ = 0.220 \text{ m.}$$

The linear velocity vector of B is

$$\mathbf{v}_B = \dot{x}_B \mathbf{i} + \dot{y}_B \mathbf{j},$$

where

$$\dot{x}_B = \frac{dx_B}{dt} = -AB \dot{\phi} \sin \phi, \quad \dot{y}_B = \frac{dy_B}{dt} = AB \dot{\phi} \cos \phi.$$

With $\phi = 45^\circ$ and $\dot{\phi} = \omega = 4.712$ rad/s

$$\begin{aligned} \dot{x}_B &= -0.100 (4.712) \sin 45^\circ = -0.333 \text{ m/s}, \\ \dot{y}_B &= 0.100 (4.712) \cos 45^\circ = 0.333 \text{ m/s}, \\ v_B = |\mathbf{v}_B| &= \sqrt{\dot{x}_B^2 + \dot{y}_B^2} = \sqrt{(-0.333)^2 + 0.333^2} = 0.471 \text{ m/s}. \end{aligned}$$

The linear acceleration vector of B is

$$\mathbf{a}_B = \ddot{x}_B \mathbf{i} + \ddot{y}_B \mathbf{j},$$

where

$$\begin{aligned} \ddot{x}_B &= \frac{d\dot{x}_B}{dt} = -AB\dot{\phi}^2 \cos \phi - AB\ddot{\phi} \sin \phi, \\ \ddot{y}_B &= \frac{d\dot{y}_B}{dt} = -AB\dot{\phi}^2 \sin \phi + AB\ddot{\phi} \cos \phi. \end{aligned}$$

The angular acceleration of link 1 is $\ddot{\phi} = \dot{\omega} = 0$. The numerical values for the acceleration of B are

$$\begin{aligned} \ddot{x}_B &= -0.100 (4.712)^2 \cos 45^\circ = -1.569 \text{ m/s}^2, \\ \ddot{y}_B &= -0.100 (4.712)^2 \sin 45^\circ = -1.569 \text{ m/s}^2, \\ a_B = |\mathbf{a}_B| &= \sqrt{\ddot{x}_B^2 + \ddot{y}_B^2} = \sqrt{(-1.569)^2 + (-1.569)^2} = 2.220 \text{ m/s}^2. \end{aligned}$$

The velocity and acceleration of the the point B on the link 1 (or on the link 2) can also be calculated with the relations

$$\begin{aligned} \mathbf{v}_B = \mathbf{v}_{B_1} = \mathbf{v}_{B_2} = \mathbf{v}_A + \boldsymbol{\omega}_1 \times \mathbf{r}_{AB} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_1 \\ x_B - x_A & y_B - y_A & 0 \end{vmatrix} = \\ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 4.712 \\ 0.070 - 0.15 & 0.220 & 0 \end{vmatrix} &= -0.333 \mathbf{i} + 0.333 \mathbf{j} \text{ m/s}, \end{aligned}$$

$$\begin{aligned} \mathbf{a}_B = \mathbf{a}_{B_1} = \mathbf{a}_{B_2} = \mathbf{a}_A + \boldsymbol{\alpha}_1 \times \mathbf{r}_{AB} - \omega_1^2 \mathbf{r}_{AB} &= -\omega_1^2 \mathbf{r}_{AB} = \\ 4.712^2 [(0.070 - 0.15) \mathbf{i} + 0.220 \mathbf{j}] &= -1.569 \mathbf{i} - 1.569 \mathbf{j} \text{ m/s}^2, \end{aligned}$$

where $\boldsymbol{\omega}_1 = \omega_1 \mathbf{k} = \omega \mathbf{k}$ and $\boldsymbol{\alpha}_1 = \dot{\boldsymbol{\omega}}_1 = \mathbf{0}$.

The points B and C are located on the same straight line BD

$$y_B(t) - y_C - [x_B(t) - x_C] \tan \phi_3(t) = 0. \quad (4.84)$$

The angle $\phi_3 = \phi_2$ is computed as follows

$$\phi_3 = \phi_2 = \arctan \frac{y_B - y_C}{x_B - x_C},$$

and for $\phi = 45^\circ$ is obtained

$$\phi_3 = \arctan \frac{0.22}{0.07} = 72.235^\circ.$$

The derivative of Eq. (4.84) yields

$$\dot{y}_B - \dot{y}_C - (\dot{x}_B - \dot{x}_C) \tan \phi_3 - (x_B - x_C) \frac{1}{\cos^2 \phi_3} \dot{\phi}_3 = 0. \quad (4.85)$$

The angular velocity of link 3, $\omega_3 = \omega_2 = \dot{\phi}_3$, is computed as follows

$$\omega_3 = \omega_2 = \frac{\cos^2 \phi_3 [\dot{y}_B - \dot{y}_C - (\dot{x}_B - \dot{x}_C) \tan \phi_3]}{x_B - x_C},$$

and

$$\omega_3 = \frac{\cos^2 72.235^\circ (0.333 + 0.333 \tan 72.235^\circ)}{0.07} = 1.807 \text{ rad/s.}$$

The angular acceleration of link 3, $\alpha_3 = \alpha_2 = \ddot{\phi}_3$, is computed from the time derivative of Eq. (4.85)

$$\begin{aligned} \ddot{y}_B - \ddot{y}_C - (\ddot{x}_B - \ddot{x}_C) \tan \phi_3 - 2(\dot{x}_B - \dot{x}_C) \frac{1}{\cos^2 \phi_3} \dot{\phi}_3 - \\ 2(x_B - x_C) \frac{\sin \phi_3}{\cos^3 \phi_3} \dot{\phi}_3^2 - (x_B - x_C) \frac{1}{\cos^2 \phi_3} \ddot{\phi}_3 = 0. \end{aligned}$$

The solution of the previous equation is

$$\begin{aligned} \alpha_3 = \alpha_2 = [\ddot{y}_B - \ddot{y}_C - (\ddot{x}_B - \ddot{x}_C) \tan \phi_3 - 2(\dot{x}_B - \dot{x}_C) \frac{1}{\cos^2 \phi_3} \dot{\phi}_3 - \\ 2(x_B - x_C) \frac{\sin \phi_3}{\cos^3 \phi_3} \dot{\phi}_3^2] \frac{\cos^2 \phi_3}{x_B - x_C}, \end{aligned}$$

and for the given numerical data

$$\begin{aligned} \alpha_3 = \alpha_2 = [-1.569 + 1.569 \tan 72.235^\circ + 2(0.333) \frac{1}{\cos^2 72.235^\circ} (1.807) - \\ 2(0.07) \frac{\sin 72.235^\circ}{\cos^3 72.235^\circ} (1.807)^2] \frac{\cos^2 72.235^\circ}{0.07} = 1.020 \text{ rad/s}^2. \end{aligned}$$

The link 2 and the link 3 have the same angular velocity $\boldsymbol{\omega}_3 = \boldsymbol{\omega}_2 = \omega_3 \mathbf{k}$ and the same angular acceleration $\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_2 = \alpha_3 \mathbf{k}$. The relative angular velocity of link 2 relative to link 1 is

$$\boldsymbol{\omega}_{21} = \boldsymbol{\omega}_2 - \boldsymbol{\omega}_1 = (1.807 - 4.712) \mathbf{k} = -2.905 \mathbf{k} \text{ rad/s,}$$

and the relative angular acceleration of link 2 relative to link 1 is

$$\boldsymbol{\alpha}_{21} = \boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = 1.020 \mathbf{k} \text{ rad/s}^2.$$

The velocity and acceleration of the the point B on the link 3 are calculated with

$$\begin{aligned} \mathbf{v}_{B_3} &= \mathbf{v}_C + \boldsymbol{\omega}_3 \times \mathbf{r}_{CB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_3 \\ x_B & y_B & 0 \end{vmatrix} = \\ & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1.807 \\ 0.070 & 0.220 & 0 \end{vmatrix} = -0.398 \mathbf{i} + 0.127 \mathbf{j} \text{ m/s}, \end{aligned}$$

$$\begin{aligned} \mathbf{a}_{B_3} &= \mathbf{a}_C + \boldsymbol{\alpha}_3 \times \mathbf{r}_{CB} - \omega_3^2 \mathbf{r}_{CB} = \\ & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_3 \\ x_B & y_B & 0 \end{vmatrix} - \omega_3^2 (x_B \mathbf{i} + y_B \mathbf{j}) = \\ & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1.020 \\ 0.070 & 0.220 & 0 \end{vmatrix} - 1.807^2 (0.070 \mathbf{i} + 0.220 \mathbf{j}) = \\ & -0.456 \mathbf{i} - 0.649 \mathbf{j} \text{ m/s}^2. \end{aligned}$$

The velocity field of two points (B_2 and B_3) not situated on the same rigid body (B_2 is on link 2 and B_3 is on link 3) is expressed by

$$\mathbf{v}_{B_2} = \mathbf{v}_{B_3} + \mathbf{v}_{B_{23}}^r,$$

and

$$\begin{aligned} \mathbf{v}_{B_{23}}^r &= \mathbf{v}_{B_2} - \mathbf{v}_{B_3} = -0.333 \mathbf{i} + 0.333 \mathbf{j} - (-0.398 \mathbf{i} + 0.127 \mathbf{j}) \\ &= 0.065 \mathbf{i} + 0.205 \mathbf{j} \text{ m/s}. \end{aligned}$$

The expression for the Coriolis acceleration is

$$\begin{aligned} \mathbf{a}_{B_{23}}^c &= 2 \boldsymbol{\omega}_2 \times \mathbf{v}_{B_{23}}^r = 2 \boldsymbol{\omega}_3 \times \mathbf{v}_{B_{23}}^r = \\ & 2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_3 \\ v_{B_{23}x}^r & v_{B_{23}y}^r & 0 \end{vmatrix} = 2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1.807 \\ 0.065 & 0.205 & 0 \end{vmatrix} = \\ & -0.742 \mathbf{i} + 0.237 \mathbf{j} \text{ m/s}^2. \end{aligned}$$

The relative acceleration of B_2 with respect to B_3 is

$$\begin{aligned}\mathbf{a}_{B_{23}}^r &= \mathbf{a}_{B_2} - \mathbf{a}_{B_3} - \mathbf{a}_{B_{23}}^c = \\ &= -1.569 \mathbf{i} - 1.569 \mathbf{j} - (-0.456 \mathbf{i} - 0.649 \mathbf{j}) - (-0.742 \mathbf{i} + 0.237 \mathbf{j}) = \\ &= -0.5 \mathbf{i} - 0.5 \mathbf{j} \quad \text{m/s}^2.\end{aligned}$$

The position of the joint D is given by the following quadratic equations

$$\begin{aligned}[x_D(t) - x_C]^2 + [y_D(t) - y_C]^2 &= BC^2, \\ [x_D(t) - x_C] \sin \phi_3(t) - [y_D(t) - y_C] \cos \phi_3(t) &= 0,\end{aligned}$$

The previous equations are rewritten as follows

$$\begin{aligned}x_D^2(t) + y_D^2(t) &= CD^2, \\ x_D(t) \sin \phi_3(t) - y_D(t) \cos \phi_3(t) &= 0.\end{aligned}\quad (4.86)$$

For $\phi = 45^\circ$ the coordinate of joint D are

$$\begin{aligned}x_D &= \pm \frac{CD}{\sqrt{1 + \tan^2 \phi_3}} = \pm \frac{0.075}{\sqrt{1 + \tan^2 72.235^\circ}} = -0.023 \text{ m}, \\ y_D &= x_D \tan \phi_3 = -0.023 \tan 72.235^\circ = -0.071 \text{ m}.\end{aligned}$$

The negative value for x_D was selected for this position of the mechanism.

The velocity analysis is carried out differentiating Eq. (4.86)

$$\begin{aligned}x_D \dot{x}_D + y_D \dot{y}_D &= 0, \\ \dot{x}_D \sin \phi_3 + x_D \cos \phi_3 \dot{\phi}_3 - \dot{y}_D \cos \phi_3 + y_D \sin \phi_3 \dot{\phi}_3 &= 0.\end{aligned}\quad (4.87)$$

For the given data Eq. (4.87) becomes

$$\begin{aligned}-0.023 \dot{x}_D - 0.071 \dot{y}_D &= 0, \\ 0.952 \dot{x}_D - 0.023(0.305)(1.807) - 0.305 \dot{y}_D - 0.071(0.952)(1.807) &= 0.\end{aligned}$$

The solution is

$$\dot{x}_D = 0.129 \text{ m/s}, \quad \dot{y}_D = -0.041 \text{ m/s}.$$

The magnitude of the velocity of joint D is

$$v_D = |\mathbf{v}_D| = \sqrt{\dot{x}_D^2 + \dot{y}_D^2} = \sqrt{0.129^2 + (-0.041)^2} = 0.135 \text{ m/s}.$$

The acceleration analysis is obtained using the derivative of the velocity given by Eq. (4.87)

$$\begin{aligned} \dot{x}_D^2 + x_D \ddot{x}_D + \dot{y}_D^2 + y_D \ddot{y}_D &= 0, \\ \ddot{x}_D \sin \phi_3 + 2\dot{x}_D \dot{\phi}_3 \cos \phi_3 - x_D \dot{\phi}_3^2 \sin \phi_3 + x_D \ddot{\phi}_3 \cos \phi_3 - \\ \ddot{y}_D \cos \phi_3 + 2\dot{y}_D \dot{\phi}_3 \sin \phi_3 + y_D \dot{\phi}_3^2 \cos \phi_3 + y_D \ddot{\phi}_3 \sin \phi_3 &= 0, \end{aligned}$$

or

$$\begin{aligned} 0.129^2 + (-0.022)\ddot{x}_D + (-0.041)^2 + (-0.071)\ddot{y}_D &= 0, \\ \ddot{x}_D \sin 72.235^\circ + 2(0.129)(1.807) \cos 72.235^\circ - (-0.022)(1.807)^2 \sin 72.235^\circ + \\ (-0.022)(1.020) \cos 72.235^\circ - \\ \ddot{y}_D \cos 72.235^\circ + 2(-0.041)(1.807) \sin 72.235^\circ + (-0.071)(1.807)^2 \cos 72.235^\circ + \\ (-0.071)(1.020)^2 \sin 72.235^\circ &= 0. \end{aligned}$$

The solution of the previous system is

$$\ddot{x}_D = 0.147 \text{ m/s}^2, \quad \ddot{y}_D = 0.210 \text{ m/s}^2.$$

The absolute acceleration of joint D is

$$a_D = |\mathbf{a}_D| = \sqrt{\ddot{x}_D^2 + \ddot{y}_D^2} = \sqrt{(0.150)^2 + (0.212)^2} = 0.256 \text{ m/s}^2.$$

The position of joint E is determined from the following equation

$$[x_E(t) - x_D(t)]^2 + [y_E(t) - y_D(t)]^2 = DE^2,$$

and with the coordinate $y_E = 0$

$$[x_E(t) - x_D(t)]^2 + y_D^2(t) = DE^2. \quad (4.88)$$

With the given numerical values Eq. (4.88) becomes

$$(x_E + 0.023)^2 + (0.071)^2 = 0.2^2,$$

with the correct solution $x_E = 0.164 \text{ m}$.

The velocity of joint E is determined by differentiating Eq. (4.88) as follows

$$2(\dot{x}_E - \dot{x}_D)(x_E - x_D) + 2y_D \dot{y}_D = 0, \quad (4.89)$$

or

$$\dot{x}_E - \dot{x}_D = -\frac{y_D \dot{y}_D}{x_E - x_D}.$$

The solution of the above equation is

$$\dot{x}_E = 0.129 - \frac{(-0.071)(-0.041)}{0.164 + 0.023} = 0.113 \text{ m/s}.$$

The derivative of Eq. (4.89) yields

$$(\ddot{x}_E - \ddot{x}_D)(x_E - x_D) + (\dot{x}_E - \dot{x}_D)^2 + \dot{y}_D^2 + y_D \ddot{y}_D = 0,$$

with the solution

$$\ddot{x}_E = \ddot{x}_D - \frac{\dot{y}_D^2 + y_D \ddot{y}_D + (\dot{x}_E - \dot{x}_D)^2}{x_E - x_D},$$

or with numerical values

$$\ddot{x}_E = 0.150 - \frac{(-0.041)^2 + (-0.071)(0.21) + (0.112 - 0.129)^2}{0.164 + 0.023} = 0.217 \text{ m/s}^2.$$

The angle ϕ_4 is determined from the following equation

$$y_E - y_D(t) - [x_E(t) - x_D(t)] \tan \phi_4(t) = 0, \quad (4.90)$$

where $y_E = 0$. The above equation can be rewritten

$$-y_D(t) - [x_E(t) - x_D(t)] \tan \phi_4(t) = 0, \quad (4.91)$$

and the solution is

$$\phi_4 = \arctan\left(\frac{-y_D}{x_E - x_D}\right) = \arctan\left(\frac{0.071}{0.164 + 0.023}\right) = 20.923^\circ.$$

The derivative of Eq.(4.91) yields

$$-\dot{y}_D - (\dot{x}_E - \dot{x}_D) \tan \phi_4 - (x_E - x_D) \frac{1}{\cos^2 \phi_4} \dot{\phi}_4 = 0. \quad (4.92)$$

Hence

$$\begin{aligned} \omega_4 &= \dot{\phi}_4 = -\frac{\cos^2 \phi_4 [\dot{y}_D + (\dot{x}_E - \dot{x}_D) \tan \phi_4]}{x_E - x_D} \\ &= -\frac{\cos^2 20.923^\circ [-0.041 + (0.113 - 0.129) \tan 20.923^\circ]}{0.164 - (-0.022)} \\ &= 0.221 \text{ rad/s}. \end{aligned}$$

The angular acceleration of link 4 is determined by differentiating Eq. (4.92) as follows

$$-\ddot{y}_D - (\ddot{x}_E - \ddot{x}_D) \tan \phi_4 - 2(\dot{x}_E - \dot{x}_D) \frac{1}{\cos^2 \phi_4} \dot{\phi}_4 - \\ 2(x_E - x_D) \frac{\sin \phi_4}{\cos^3 \phi_4} \dot{\phi}_4^2 - (x_E - x_D) \frac{1}{\cos^2 \phi_4} \ddot{\phi}_4 = 0,$$

or

$$-0.210 - (0.217 - 0.147) \tan 20.923^\circ - 2(0.113 - 0.129) \frac{1}{\cos^2 20.923^\circ} 0.221 - \\ 2(0.164 + 0.022) \frac{\sin 20.923^\circ}{\cos^3 20.923^\circ} 0.221^2 - (0.164 + 0.022) \frac{1}{\cos^2 20.923^\circ} \ddot{\phi}_4 = 0,$$

The solution of the previous equation is

$$\alpha_4 = \ddot{\phi}_4 = -1.105 \text{ rad/s}^2.$$

4.8 Problems

- 4.1 The four-bar mechanism shown in Fig. 3.16 has the dimensions: $AB = CD = 0.04$ m and $AD = BC = 0.09$ m. The driver link AB rotates with a constant angular speed of 120 rpm. Find the velocities and the accelerations of the four-bar mechanism for the case when the angle of the driver link AB with the horizontal is $\phi = 30^\circ$.
- 4.2 The constant angular speed of the driver link 1, of the mechanism shown in Fig. 4.9, is $\omega = \omega_1 = 10$ rad/s. The distance from the link 3 to the horizontal axis Ax is $a = 55$ mm. Find the velocity and the acceleration of the point C on the link 3 for $\phi = 45^\circ$.
- 4.3 The slider crank mechanism shown in Fig. 4.10 has the dimensions: $AB = 0.1$ m and $BC = 0.2$ m. The driver link 1 rotates with a constant angular speed of $n = 60$ rpm. Find the velocity and acceleration of the slider 3 when the angle of the driver link with the horizontal is $\phi = 45^\circ$.
- 4.4 The planar mechanism considered is shown in Fig. 3.19. The following data are given: $AB=0.150$ m, $BC=0.400$ m, $CD=0.370$ m, $CE=0.230$ m, $EF=CE$, $L_a=0.300$ m, $L_b=0.450$ m, and $L_c=CD$. The constant angular speed of the driver link 1 is 60 rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 30^\circ$.
- 4.5 The R-RRR-RTT mechanism is shown in Fig. 3.20. The following data are given: $AB=0.080$ m, $BC=0.350$ m, $CE=0.200$ m, $CD=0.150$ m, $L_a=0.200$ m, $L_b=0.350$ m, $L_c=0.040$ m. The driver link 1 rotates with a constant angular speed of $n = 300$ rpm. Find the velocities and the accelerations of the mechanism when the angle of the driver link with the horizontal axis is $\phi = 155^\circ$.
- 4.6 The mechanism shown in Fig. 3.21 has the following dimensions: $AB = 60$ mm, $AD = 200$ mm, $BC = 140$ mm, $CE = 50$ mm, $EF = 170$ mm, and $a = 130$ mm. The constant angular speed of the driver link 1 is $n = 300$ rpm. Find the velocities and the accelerations of the mechanism when the angle of the driver link 1 with the horizontal is $\phi = \phi_1 = 30^\circ$.
- 4.7 The dimensions for the mechanism shown in Fig. 3.22 are: $AB = 120$ mm, $BD = 320$ mm, $BC = 110$ mm, $CD = 300$ mm, $DE = 200$ mm, $CF = 400$ mm, $AE = 320$ mm, and $b = 80$ mm. The

constant angular speed of the driver link 1 is $n = 30$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 30^\circ$.

- 4.8 The mechanism in Fig. 3.23 has the dimensions: $AB = 50$ mm, $AC = 25$ mm, $BD = 100$ mm, $DE = 140$ mm, $EF = 80$ mm, $L_a = 130$ mm, and $L_b = 30$ mm. Find the velocities and the accelerations of the mechanism if the constant angular speed of the driver link 1 is $n = 100$ rpm and for $\phi = \phi_1 = 150^\circ$.
- 4.9 The dimensions for the mechanism shown in Fig. 3.24 are: $AB = 180$ mm, $BC = 470$ mm, $AD = 430$ mm, $CD = 270$ mm, $DE = 180$ mm, $EF = 400$ mm, and $L_a = 70$ mm. The constant angular speed of the driver link 1 is $n = 220$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 45^\circ$.
- 4.10 The mechanism in Fig. 3.25 has the dimensions: $AB = 200$ mm, $AC = 600$ mm, $BD = 1000$ mm, $L_a = 150$ mm, and $L_b = 250$ mm. The driver link 1 rotates with a constant angular speed of $n = 60$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 120^\circ$.
- 4.11 Figure 3.26 shows a mechanism with the following dimensions: $AB = 250$ mm, $BD = 900$ mm, and $L_a = 300$ mm. The constant angular speed of the driver link 1 is $n = 500$ rpm. Find the velocities and the accelerations of the mechanism when the angle of the driver link 1 with the horizontal is $\phi = 240^\circ$.
- 4.12 The mechanism in Fig. 3.27 has the dimensions: $AB = 150$ mm, $AC = 350$ mm, $BD = 600$ mm, $DE = 300$ mm, $EF = 200$ mm, $L_a = 55$ mm, and $L_b = 250$ mm. The constant angular speed of the driver link 1 is $n = 30$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 120^\circ$.
- 4.13 Figure 3.28 shows a mechanism with the following dimensions: $AB = 150$ mm, $BC = 550$ mm, $CD = DE = 220$ mm, $EF = 400$ mm, $L_a = 530$ mm, and $L_b = L_c = 180$ mm. Find the velocities and the accelerations of the mechanism if the constant angular speed of the driver link 1 is $n = 1000$ rpm and for $\phi = \phi_1 = 150^\circ$.
- 4.14 Figure 3.29 shows a mechanism with the following dimensions: $AB = 250$ mm, $BC = 1200$ mm, $CE = 400$ mm, $CD = 800$ mm, $EF =$

700 mm, $L_a = 650$ mm, $L_b = 1000$ mm, and $L_c = 1200$ mm. The constant angular speed of the driver link 1 is $n = 70$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 120^\circ$.

- 4.15 Figure 3.30 shows a mechanism with the following dimensions: $AB = 100$ mm, $BC = 270$ mm, $CF = 260$ mm, $CD = 90$ mm, $DE = 300$ mm, $L_a = 350$ mm, $L_b = 200$ mm, and $L_c = 120$ mm. The constant angular speed of the driver link 1 is $n = 100$ rpm. Find the velocities and the accelerations of the mechanism when the angle of the driver link 1 with the horizontal is $\phi = 60^\circ$.
- 4.16 Figure 3.31 shows a mechanism with the following dimensions: $AB = 40$ mm, $BC = 100$ mm, $AD = 50$ mm, and $BE = 110$ mm. The constant angular speed of the driver link 1 is $n = 250$ rpm. Find the velocities and the accelerations of the mechanism if the angle of the driver link 1 with the horizontal is $\phi = 30^\circ$.
- 4.17 The dimensions of the mechanism shown in Fig. 3.32 are: $AB = 100$ mm, $BC = 200$ mm, $BE = 400$ mm, $CE = 600$ mm, $CD = 220$ mm, $EF = 800$ mm, $L_a = 250$ mm, $L_b = 150$ mm, and $L_c = 100$ mm. The constant angular speed of the driver link 1 is $n = 100$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 150^\circ$.
- 4.18 The dimensions of the mechanism shown in Fig. 3.33 are: $AB = 200$ mm, $AC = 300$ mm, $CD = 500$ mm, $DE = 250$ mm, and $L_a = 400$ mm. Find the positions of the joints and the angles of the links. The constant angular speed of the driver link 1 is $n = 40$ rpm. Find the velocities and the accelerations of the mechanism when the angle of the driver link 1 with the horizontal is $\phi = 60^\circ$.
- 4.19 The dimensions of the mechanism shown in Fig. 3.34 are: $AB = 160$ mm, $AC = 90$ mm, $CD = 150$ mm, and $DE = 400$ mm. The constant angular speed of the driver link 1 is $n = 70$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 45^\circ$.
- 4.20 The dimensions of the mechanism shown in Fig. 3.35 are: $AB = 150$ mm, $AC = 250$ mm, and $CD = 450$ mm. For the distance b select a suitable value. The constant angular speed of the driver link 1

is $n = 80$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 30^\circ$.

- 4.21 The dimensions of the mechanism shown in Fig. 3.36 are: $AB = 180$ mm, $AC = 90$ mm, and $CD = 200$ mm. The constant angular speed of the driver link 1 is $n = 180$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 60^\circ$.
- 4.22 The dimensions of the mechanism shown in Fig. 3.37 are: $AB = 180$ mm, $AC = 500$ mm, $BD = L_a = 770$ mm, and $DE = 600$ mm. The constant angular speed of the driver link 1 is $n = n_1 = 700$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 45^\circ$.
- 4.23 The dimensions of the mechanism shown in Fig. 3.38 are: $AB = 220$ mm, $AD = 600$ mm, and $BC = 250$ mm. The constant angular speed of the driver link 1 is $n = 700$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = \phi_1 = 120^\circ$. Select a suitable value for the distance a .
- 4.24 Referring to Example 3.1. The mechanism in Fig. 3.11(a) has the dimensions: $AB = 0.20$ m, $AD = 0.40$ m, $CD = 0.70$ m, $CE = 0.30$ m, and $y_E = 0.35$ m. The constant angular speed of the driver link 1 is $n = 1000$ rpm. Find the velocities and the accelerations of the mechanism for the given input angle $\phi = \phi_1 = 60^\circ$.
- 4.25 Referring to Example 3.2. The mechanism in Fig. 3.12 has the dimensions: $AB = 0.02$ m, $BC = 0.03$ m, $CD = 0.06$ m, $AE = 0.05$ m, and $L_a = 0.02$ m. The constant angular speed of the driver link 1 is $n = 600$ rpm. Find the velocities and the accelerations of the mechanism for the given input angle $\phi = \phi_1 = \pi/3$.
- 4.26 Referring to Example 3.3. The mechanism in Fig. 3.15 has the dimensions: $AC = 0.100$ m, $BC = 0.300$ m, $BD = 0.900$ m, and $L_a = 0.100$ m. The constant angular speed of the driver link 1 is $n = 100$ rpm. Find the velocities and the accelerations of the mechanism for $\phi = 30^\circ$.

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Figure captions

Figure 4.1 Mobile reference frame ($\mathbf{i}, \mathbf{j}, \mathbf{k}$) that moves with the rigid body (RB)

Figure 4.2 Motion of a point A that moves relative to a rigid body (RB)

Figure 4.3 Velocity (\mathbf{v}_{Arel}) and acceleration (\mathbf{a}_{Arel}) of A relative to the rigid body

Figure 4.4 Nonaccelerating, nonrotating reference frame with the origin at O_0 , and a secondary nonrotating, earth centered reference frame with the origin at O

Figure 4.5. Nonrotating reference frame with the origin at the center of the earth O and a secondary earth fixed reference frame with the origin at B .

Figure 4.6. Primary inertial reference frame with the origin at O and a secondary reference frame with the origin at B

Figure 4.7 R-TRR mechanism for Example 4.1

Figure 4.8 R-RTR-RRT mechanism for Example 4.2

Figure 4.9 Mechanism for Problem 4.2

Figure 4.10 Mechanism for Problem 4.3