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# 1 Introduction

## 1.1 Vector Algebra

### Vector Terminology

*Scalars* are mathematics quantities that can be fully defined by specifying their magnitude in suitable units of measure. The mass is a scalar and can be expressed in kilograms, the time is a scalar and can be expressed seconds, and the temperature can be expressed in degrees.

*Vectors* are quantities that require the specification of: magnitude, orientation, and sense. The characteristics of a vector are the magnitude, the orientation, and the sense.

The *magnitude* of a vector is specified by a positive number and a unit having appropriate dimensions. No unit is stated if the dimensions are those of a pure number.

The *orientation* of a vector is specified by the relationship between the vector and given reference lines and/or planes.

The *sense* of a vector is specified by the order of two points on a line parallel to the vector.

Orientation and sense together determine the *direction* of a vector.

The *line of action* of a vector is a hypothetical infinite straight line collinear with the vector.

Displacement, velocity, and force are examples of vectors.

To distinguish vectors from scalars it is customary to denote vectors by boldface letters. Thus the vector shown in Fig. 1.1(a) is denoted by  $\mathbf{r}$  or  $\mathbf{r}_{AB}$ . The symbol  $|\mathbf{r}| = r$  represents the magnitude (or module, or absolute value) of the vector  $\mathbf{r}$ . In handwritten work a distinguishing mark is used for vectors, such as an arrow over the symbol,  $\vec{r}$  or  $\vec{AB}$ , a line over the symbol,  $\bar{r}$ , or an underline,  $\underline{r}$ .

The vectors are depicted by either straight or curved arrows. A vector represented by a straight arrow has the direction indicated by the arrow. The direction of a vector represented by a curved arrow is the same as the direction in which a right-handed screw moves when the axis of the screw is normal to the plane in which the arrow is drawn and the screw is rotated as indicated by the arrow.

Figure 1.1(b) shows representations of vectors. Sometimes vectors are represented by means of a straight or curved arrow together with a measure number. In this case the vector is regarded as having the direction indicated

by the arrow if the measure number is positive, and the opposite direction if it is negative.

A *bound* (or *fixed*) vector is a vector associated with a particular point  $P$  in space (Fig. 1.2). The point  $P$  is the *point of application* of the vector, and the line passing through  $P$  and parallel to the vector is the line of action of the vector. The point of application may be represented as the tail, Fig. 1.2(a), or the head of the vector arrow, Fig. 1.2(b).

A *free* vector is not associated with a particular point or line in space.

A *transmissible* (or *sliding*) vector is a vector that can be moved along his line of action without change of meaning.

To move the body in Fig. 1.3 the force vector  $\mathbf{F}$  can be applied anywhere along the line  $\Delta$  or may be applied at specific points  $A, B, C$ . The force vector  $\mathbf{F}$  is a transmissible vector because the resulting motion is the same in all cases.

The force  $\mathbf{F}$  applied at  $B$  will cause a different deformation of the body than the same force  $\mathbf{F}$  applied at a different point  $C$ . The points  $B$  and  $C$  are on the body. If one is interested in the deformation of the body, the force  $\mathbf{F}$  positioned at  $C$  is a bound vector.

The operations of vector analysis deal only with the characteristics of vectors and apply, therefore, to both bound and free vectors. Vector analysis is a branch of mathematics that deals with quantities that have both magnitude and direction.

### Vector Equality

Two vectors  $\mathbf{a}$  and  $\mathbf{b}$  are said to be equal to each other when they have the same characteristics

$$\mathbf{a} = \mathbf{b}.$$

Equality does not imply physical equivalence. For instance, two forces represented by equal vectors do not necessarily cause identical motions of a body on which they act.

### Product of a Vector and a Scalar

**Definition.** The product of a vector  $\mathbf{v}$  and a scalar  $s$ ,  $s\mathbf{v}$  or  $\mathbf{v}s$ , is a vector having the following characteristics:

1. Magnitude.

$$|s\mathbf{v}| \equiv |\mathbf{v}s| = |s||\mathbf{v}|,$$

where  $|s|$  denotes the absolute value (or magnitude, or module) of the scalar  $s$ .

2. Orientation.  $s\mathbf{v}$  is parallel to  $\mathbf{v}$ . If  $s = 0$ , no definite orientation is attributed to  $s\mathbf{v}$ .

3. Sense. If  $s > 0$ , the sense of  $s\mathbf{v}$  is the same as that of  $\mathbf{v}$ . If  $s < 0$ , the sense of  $s\mathbf{v}$  is opposite to that of  $\mathbf{v}$ . If  $s = 0$ , no definite sense is attributed to  $s\mathbf{v}$ .

### Zero Vectors

**Definition.** A *zero vector* is a vector that does not have a definite direction and whose magnitude is equal to zero. The symbol used to denote a zero vector is  $\mathbf{0}$ .

### Unit Vectors

**Definition.** A *unit vector* (versor) is a vector with the magnitude equal to 1.

Given a vector  $\mathbf{v}$ , a unit vector  $\mathbf{u}$  having the same direction as  $\mathbf{v}$  is obtained by forming the quotient of  $\mathbf{v}$  and  $|\mathbf{v}|$ :

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

### Vector Addition

The sum of a vector  $\mathbf{v}_1$  and a vector  $\mathbf{v}_2$ :  $\mathbf{v}_1 + \mathbf{v}_2$  or  $\mathbf{v}_2 + \mathbf{v}_1$  is a vector whose characteristics are found by either graphical or analytical processes. The vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  add according to the parallelogram law:  $\mathbf{v}_1 + \mathbf{v}_2$  is equal to the diagonal of a parallelogram formed by the graphical representation of the vectors (Fig. 1.4(a)). The vector  $\mathbf{v}_1 + \mathbf{v}_2$  is called the *resultant* of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . The vectors can be added by moving them successively to parallel positions so that the head of one vector connects to the tail of the next vector. The resultant is the vector whose tail connects to the tail of the first vector, and whose head connects to the head of the last vector (Fig. 1.4(b)).

The sum  $\mathbf{v}_1 + (-\mathbf{v}_2)$  is called the *difference* of  $\mathbf{v}_1$  and  $\mathbf{v}_2$  and is denoted by  $\mathbf{v}_1 - \mathbf{v}_2$  (Figs. 1.4(c) and 1.4 (d)).

The sum of  $n$  vectors  $\mathbf{v}_i$ ,  $i = 1, \dots, n$ ,

$$\sum_{i=1}^n \mathbf{v}_i \text{ or } \mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_n$$

is called the *resultant* of the vectors  $\mathbf{v}_i$ ,  $i = 1, \dots, n$ .

The vector addition is:

1. commutative, that is, the characteristics of the resultant are independent of the order in which the vectors are added (commutativity)

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1.$$

2. associative, that is, the characteristics of the resultant are not affected by the manner in which the vectors are grouped (associativity)

$$\mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3.$$

3. distributive, that is, the vector addition obeys the following laws of distributivity

$$\begin{aligned} \mathbf{v} \sum_{i=1}^p s_i &= \sum_{i=1}^p (\mathbf{v}s_i), \text{ for } s_i \neq 0, s_i \in \mathcal{R}, \\ s \sum_{i=1}^n \mathbf{v}_i &= \sum_{i=1}^n (s\mathbf{v}_i), \text{ for } s \neq 0, s \in \mathcal{R}, \end{aligned}$$

where  $\mathcal{R}$  is the set of real numbers.

Every vector can be regarded as the sum of  $n$  vectors ( $n = 2, 3, \dots$ ) of which all but one can be selected arbitrarily.

### Resolution of Vectors and Components

Let  $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$  be any three unit vectors not parallel to the same plane (noncollinear vectors)

$$|\mathbf{i}_1| = |\mathbf{i}_2| = |\mathbf{i}_3| = 1.$$

For a given vector  $\mathbf{v}$  (Fig. 1.5), there exist three unique scalars  $v_1, v_2, v_3$ , such that  $\mathbf{v}$  can be expressed as

$$\mathbf{v} = v_1\mathbf{i}_1 + v_2\mathbf{i}_2 + v_3\mathbf{i}_3.$$

The opposite action of addition of vectors is the *resolution* of vectors. Thus, for the given vector  $\mathbf{v}$  the vectors  $v_1\mathbf{i}_1, v_2\mathbf{i}_2$ , and  $v_3\mathbf{i}_3$  sum to the original vector. The vector  $v_k\mathbf{i}_k$  is called the  $\mathbf{i}_k$  *component* of  $\mathbf{v}$  and  $v_k$  is called the  $\mathbf{i}_k$  *scalar component* of  $\mathbf{v}$ , where  $k = 1, 2, 3$ . A vector is often replaced by its components since the components are equivalent to the original vector.

Every vector equation  $\mathbf{v} = \mathbf{0}$ , where  $\mathbf{v} = v_1\mathbf{i}_1 + v_2\mathbf{i}_2 + v_3\mathbf{i}_3$ , is equivalent to three scalar equations  $v_1 = 0, v_2 = 0, v_3 = 0$ .

If the unit vectors  $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$  are mutually perpendicular they form a *cartesian reference frame*. For a cartesian reference frame the following notation is used (Fig. 1.6):

$$\mathbf{i}_1 \equiv \mathbf{i}, \mathbf{i}_2 \equiv \mathbf{j}, \mathbf{i}_3 \equiv \mathbf{k},$$

and

$$\mathbf{i} \perp \mathbf{j}, \mathbf{i} \perp \mathbf{k}, \mathbf{j} \perp \mathbf{k}.$$

The symbol  $\perp$  denotes perpendicular.

When a vector  $\mathbf{v}$  is expressed in the form  $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$  where  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are mutually perpendicular unit vectors (cartesian reference frame or orthogonal reference frame), the magnitude of  $\mathbf{v}$  is given by

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

The vectors  $\mathbf{v}_x = v_x\mathbf{i}$ ,  $\mathbf{v}_y = v_y\mathbf{j}$ , and  $\mathbf{v}_z = v_z\mathbf{k}$  are the *orthogonal* or *rectangular component vectors* of the vector  $\mathbf{v}$ . The measures  $v_x, v_y, v_z$  are the *orthogonal* or *rectangular scalar components* of the vector  $\mathbf{v}$ .

If  $\mathbf{v}_1 = v_{1x}\mathbf{i} + v_{1y}\mathbf{j} + v_{1z}\mathbf{k}$  and  $\mathbf{v}_2 = v_{2x}\mathbf{i} + v_{2y}\mathbf{j} + v_{2z}\mathbf{k}$ , then the sum of the vectors is

$$\mathbf{v}_1 + \mathbf{v}_2 = (v_{1x} + v_{2x})\mathbf{i} + (v_{1y} + v_{2y})\mathbf{j} + (v_{1z} + v_{2z})\mathbf{k}.$$

### Angle Between Two Vectors

Two vectors  $\mathbf{a}$  and  $\mathbf{b}$  are considered. One can move either vector parallel to itself (leaving its sense unaltered) until their initial points (tails) coincide. The *angle* between  $\mathbf{a}$  and  $\mathbf{b}$  is the angle  $\theta$  in Figs. 1.7(a) and 1.7(b). The angle between  $\mathbf{a}$  and  $\mathbf{b}$  is denoted by the symbols  $(\mathbf{a}, \mathbf{b})$  or  $(\mathbf{b}, \mathbf{a})$ . Figure 1.7(c) represents the case  $(\mathbf{a}, \mathbf{b}) = 0$ , and Fig. 1.7(d) represents the case  $(\mathbf{a}, \mathbf{b}) = 180^\circ$ .

The direction of a vector  $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$  relative to a cartesian reference,  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ , is given by the cosines of the angles formed by the vector and the respective unit vectors. These are called *direction cosines* and are denoted as (Fig. 1.8)

$$\cos(\mathbf{v}, \mathbf{i}) = \cos \alpha = l; \cos(\mathbf{v}, \mathbf{j}) = \cos \beta = m; \cos(\mathbf{v}, \mathbf{k}) = \cos \gamma = n.$$

The following relations exist

$$v_x = |\mathbf{v}| \cos \alpha; v_y = |\mathbf{v}| \cos \beta; v_z = |\mathbf{v}| \cos \gamma,$$

$$l^2 + m^2 + n^2 = 1, \quad (v_x^2 + v_y^2 + v_z^2 = v^2).$$

### Scalar (Dot) Product of Vectors

**Definition.** The scalar (dot) product of a vector  $\mathbf{a}$  and a vector  $\mathbf{b}$  is

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = |\mathbf{a}| |\mathbf{b}| \cos(\mathbf{a}, \mathbf{b}).$$

For any two vectors  $\mathbf{a}$  and  $\mathbf{b}$  and any scalar  $s$

$$(\mathbf{sa}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\mathbf{sb}) = \mathbf{sa} \cdot \mathbf{b}.$$

If

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k},$$

and

$$\mathbf{b} = b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k},$$

where  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are mutually perpendicular unit vectors, then

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z.$$

The following relationships exist

$$\begin{aligned} \mathbf{i} \cdot \mathbf{i} &= \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1, \\ \mathbf{i} \cdot \mathbf{j} &= \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0. \end{aligned}$$

Every vector  $\mathbf{v}$  can be expressed in the form

$$\mathbf{v} = \mathbf{i} \cdot \mathbf{v} \mathbf{i} + \mathbf{j} \cdot \mathbf{v} \mathbf{j} + \mathbf{k} \cdot \mathbf{v} \mathbf{k}.$$

The vector  $\mathbf{v}$  can always be expressed as

$$\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}.$$

Dot multiply both sides by  $\mathbf{i}$

$$\mathbf{i} \cdot \mathbf{v} = v_x \mathbf{i} \cdot \mathbf{i} + v_y \mathbf{i} \cdot \mathbf{j} + v_z \mathbf{i} \cdot \mathbf{k}.$$

But,

$$\mathbf{i} \cdot \mathbf{i} = 1, \quad \text{and} \quad \mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = 0.$$

Hence,

$$\mathbf{i} \cdot \mathbf{v} = v_x.$$

Similarly,

$$\mathbf{j} \cdot \mathbf{v} = v_y \quad \text{and} \quad \mathbf{k} \cdot \mathbf{v} = v_z.$$

The associative, commutative, and distributive laws of elementary algebra are valid for the dot multiplication (product) of vectors.

### Vector (Cross) Product of Vectors

**Definition.** The vector (cross) product of a vector  $\mathbf{a}$  and a vector  $\mathbf{b}$  is the vector (Fig. 1.9)

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin(\mathbf{a}, \mathbf{b}) \mathbf{n}$$

where  $\mathbf{n}$  is a unit vector whose direction is the same as the direction of advance of a right-handed screw rotated from  $\mathbf{a}$  toward  $\mathbf{b}$ , through the angle  $(\mathbf{a}, \mathbf{b})$ , when the axis of the screw is perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ .

The magnitude of  $\mathbf{a} \times \mathbf{b}$  is given by

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin(\mathbf{a}, \mathbf{b}).$$

If  $\mathbf{a}$  is parallel to  $\mathbf{b}$ ,  $\mathbf{a} \parallel \mathbf{b}$ , then  $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ . The symbol  $\parallel$  denotes parallel. The relation  $\mathbf{a} \times \mathbf{b} = \mathbf{0}$  implies only that the product  $|\mathbf{a}| |\mathbf{b}| \sin(\mathbf{a}, \mathbf{b})$  is equal to zero, and this is the case whenever  $|\mathbf{a}| = 0$ , or  $|\mathbf{b}| = 0$ , or  $\sin(\mathbf{a}, \mathbf{b}) = 0$ . For any two vectors  $\mathbf{a}$  and  $\mathbf{b}$  and any real scalar  $s$ ,

$$(s\mathbf{a}) \times \mathbf{b} = s(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (s\mathbf{b}) = s\mathbf{a} \times \mathbf{b}.$$

The sense of the unit vector  $\mathbf{n}$  which appears in the definition of  $\mathbf{a} \times \mathbf{b}$  depends on the order of the factors  $\mathbf{a}$  and  $\mathbf{b}$  in such a way that

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}.$$

Vector multiplication obeys the following law of distributivity (Varignon theorem)

$$\mathbf{a} \times \sum_{i=1}^n \mathbf{v}_i = \sum_{i=1}^n (\mathbf{a} \times \mathbf{v}_i).$$

The cross product is not commutative, but the associative law and the distributive law are valid for cross products.

A set of mutually perpendicular unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  is called *right-handed* if  $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ . A set of mutually perpendicular unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  is called *left-handed* if  $\mathbf{i} \times \mathbf{j} = -\mathbf{k}$ .

If

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k},$$

and

$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k},$$

where  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  are mutually perpendicular unit vectors, then  $\mathbf{a} \times \mathbf{b}$  can be expressed in the following determinant form:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

The determinant can be expanded by minors of the elements of the first row:

$$\begin{aligned} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} &= \mathbf{i} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \\ &= \mathbf{i}(a_y b_z - a_z b_y) - \mathbf{j}(a_x b_z - a_z b_x) + \mathbf{k}(a_x b_y - a_y b_x) \\ &= (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}. \end{aligned}$$

### Scalar Triple Product of Three Vectors

**Definition.** The scalar triple product of three vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  is

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}.$$

It does not matter whether the dot is placed between  $\mathbf{a}$  and  $\mathbf{b}$ , and the cross between  $\mathbf{b}$  and  $\mathbf{c}$ , or vice versa, that is,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}.$$

A change in the order of the factors appearing in a scalar triple product at most changes the sign of the product, that is,

$$[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{a}, \mathbf{b}, \mathbf{c}],$$

and

$$[\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}].$$

If  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  are parallel to the same plane, or if any two of the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  are parallel to each other, then  $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0$ .

The scalar triple product  $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$  can be expressed in the following determinant form

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

### Vector Triple Product of Three Vector

**Definition.** The vector triple product of three vectors  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  is the vector  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ .

The parentheses are essential because  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$  is not, in general, equal to  $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ .

For any three vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ ,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{c} \mathbf{b} - \mathbf{a} \cdot \mathbf{b} \mathbf{c}.$$

### Derivative of a Vector

The derivative of a vector is defined in exactly the same way as is the derivative of a scalar function. The derivative of a vector has some of the properties of the derivative of a scalar function.

The derivative of the sum of two vector functions  $\mathbf{a}$  and  $\mathbf{b}$  is

$$\frac{d}{dt}(\mathbf{a} + \mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}.$$

The time derivative of the product of a scalar function  $f$  and a vector function  $\mathbf{a}$  is

$$\frac{d(f\mathbf{a})}{dt} = \frac{df}{dt}\mathbf{a} + f\frac{d\mathbf{a}}{dt}.$$

## 1.2 Centroids

### Position Vector

The position vector of a point  $P$  relative to a point  $M$  is a vector  $\mathbf{r}_{MP}$  having the following characteristics, Fig. 1.10

- magnitude ( $|\mathbf{r}_{MP}| = r_{MP}$ ) the length of line  $MP$  ;
- orientation parallel to line  $MP$ ;
- sense  $MP$  (from point  $M$  to point  $P$ ).

The vector  $\mathbf{r}_{MP}$  is shown as an arrow connecting  $M$  to  $P$ . The position of a point  $P$  relative to  $O$  is a zero vector.

Let  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  be mutually perpendicular unit vectors (cartesian reference frame) with the origin at  $O$ , Fig. 1.10. The axes of the cartesian reference frame are  $x$ ,  $y$ ,  $z$ . The unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  are parallel to  $x$ ,  $y$ ,  $z$ , and they have the senses of the positive  $x$ ,  $y$ ,  $z$  axes. The coordinates of the origin  $O$  are  $x = y = z = 0$ , i.e.,  $O(0, 0, 0)$ . The coordinates of a point  $P$  are  $x = x_P$ ,  $y = y_P$ ,  $z = z_P$ , i.e.,  $P(x_P, y_P, z_P)$ . The position vector of  $P$  relative to the origin  $O$  is

$$\mathbf{r}_{OP} = \mathbf{r}_P = x_P \mathbf{i} + y_P \mathbf{j} + z_P \mathbf{k}.$$

The position vector of the point  $P$  relative to a point  $M$ ,  $M \neq O$  of coordinates  $(x_M, y_M, z_M)$  is

$$\mathbf{r}_{MP} = (x_P - x_M) \mathbf{i} + (y_P - y_M) \mathbf{j} + (z_P - z_M) \mathbf{k}.$$

The distance  $d$  between  $P$  and  $M$  is given by

$$d = |\mathbf{r}_P - \mathbf{r}_M| = |\mathbf{r}_{MP}| = \sqrt{(x_P - x_M)^2 + (y_P - y_M)^2 + (z_P - z_M)^2}.$$

### First Moment

The position vector of a point  $P$  relative to a point  $O$  is  $\mathbf{r}_P$  and a scalar associated with  $P$  is  $s$ , e.g., the mass  $m$  of a particle situated at  $P$ . The *first moment* of a point  $P$  with respect to a point  $O$  is the vector  $\mathbf{M} = s \mathbf{r}_P$ . The scalar  $s$  is called the *strength* of  $P$ .

### Centroid of a Set of Points

The set of  $n$  points  $P_i$ ,  $i = 1, 2, \dots, n$ , is  $\{S\}$ , Fig. 1.11

$$\{S\} = \{P_1, P_2, \dots, P_n\} = \{P_i\}_{i=1,2,\dots,n}.$$

The strengths of the points  $P_i$  are  $s_i$ ,  $i = 1, 2, \dots, n$ , i.e.,  $n$  scalars, all having the same dimensions, and each associated with one of the points of  $\{S\}$ .

The *centroid* of the set  $\{S\}$  is the point  $C$  with respect to which the sum of the first moments of the points of  $\{S\}$  is equal to zero.

The position vector of  $C$  relative to an arbitrarily selected reference point  $O$  is  $\mathbf{r}_C$ , Fig. 1.11. The position vector of  $P_i$  relative to  $O$  is  $\mathbf{r}_i$ . The position vector of  $P_i$  relative to  $C$  is  $\mathbf{r}_i - \mathbf{r}_C$ . The sum of the first moments of the points  $P_i$  with respect to  $C$  is  $\sum_{i=1}^n s_i(\mathbf{r}_i - \mathbf{r}_C)$ . If  $C$  is to be centroid of  $\{S\}$ , this sum is equal to zero

$$\sum_{i=1}^n s_i(\mathbf{r}_i - \mathbf{r}_C) = \sum_{i=1}^n s_i\mathbf{r}_i - \mathbf{r}_C \sum_{i=1}^n s_i = 0.$$

The position vector  $\mathbf{r}_C$  of the centroid  $C$ , relative to an arbitrarily selected reference point  $O$ , is given by

$$\mathbf{r}_C = \frac{\sum_{i=1}^n s_i\mathbf{r}_i}{\sum_{i=1}^n s_i}.$$

If  $\sum_{i=1}^n s_i = 0$  the centroid is not defined.

The centroid  $C$  of a set of points of given strength is a unique point, its location being independent of the choice of reference point  $O$ .

The cartesian coordinates of the centroid  $C(x_C, y_C, z_C)$  of a set of points  $P_i, i = 1, \dots, n$ , of strengths  $s_i, i = 1, \dots, n$ , are given by the expressions

$$x_C = \frac{\sum_{i=1}^n s_i x_i}{\sum_{i=1}^n s_i}, \quad y_C = \frac{\sum_{i=1}^n s_i y_i}{\sum_{i=1}^n s_i}, \quad z_C = \frac{\sum_{i=1}^n s_i z_i}{\sum_{i=1}^n s_i}.$$

The *plane of symmetry* of a set is the plane where the centroid of the set lies, the points of the set being arranged in such a way that corresponding to every point on one side of the plane of symmetry there exists a point of equal strength on the other side, the two points being equidistant from the plane.

A set  $\{S'\}$  of points is called a *subset* of a set  $\{S\}$  if every point of  $\{S'\}$  is a point of  $\{S\}$ . The centroid of a set  $\{S\}$  may be located using the *method of decomposition*:

- divide the system  $\{S\}$  into subsets;

- find the centroid of each subset;
- assign to each centroid of a subset a strength proportional to the sum of the strengths of the points of the corresponding subset;
- determine the centroid of this set of centroids.

### Centroid of a Curve, Surface, or Solid

The position vector of the centroid  $C$  of a curve, surface, or solid relative to a point  $O$  is

$$\mathbf{r}_C = \frac{\int_D \mathbf{r} d\tau}{\int_D d\tau},$$

where,  $D$  is a curve, surface, or solid,  $\mathbf{r}$  denotes the position vector of a typical point of  $D$ , relative to  $O$ , and  $d\tau$  is the length, area, or volume of a differential element of  $D$ . Each of the two limits in this expression is called an “integral over the domain  $D$  (curve, surface, or solid).”

The integral  $\int_D d\tau$  gives the total length, area, or volume of  $D$ , that is

$$\int_D d\tau = \tau.$$

The position vector of the centroid is

$$\mathbf{r}_C = \frac{1}{\tau} \int_D \mathbf{r} d\tau.$$

Let  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  be mutually perpendicular unit vectors (cartesian reference frame) with the origin at  $O$ . The coordinates of  $C$  are  $x_C$ ,  $y_C$ ,  $z_C$  and

$$\mathbf{r}_C = x_C \mathbf{i} + y_C \mathbf{j} + z_C \mathbf{k}.$$

It results that

$$x_C = \frac{1}{\tau} \int_D x d\tau, \quad y_C = \frac{1}{\tau} \int_D y d\tau, \quad z_C = \frac{1}{\tau} \int_D z d\tau.$$

### Mass Center of a Set of Particles

The *mass center* of a set of particles  $\{S\} = \{P_1, P_2, \dots, P_n\} = \{P_i\}_{i=1,2,\dots,n}$  is the centroid of the set of points at which the particles are situated with the strength of each point being taken equal to the mass of the corresponding

particle,  $s_i = m_i$ ,  $i = 1, 2, \dots, n$ . For the system of  $n$  particles the following relation can be written

$$\left( \sum_{i=1}^n m_i \right) \mathbf{r}_C = \sum_{i=1}^n m_i \mathbf{r}_i,$$

and the position vector of the mass center  $C$  is

$$\mathbf{r}_C = \frac{\sum_{i=1}^n m_i \mathbf{r}_i}{m}, \quad (1.1)$$

where  $m$  is the total mass of the system.

### Mass Center of a Curve, Surface, or Solid

The position vector of the mass center  $C$  of a continuous body  $D$ , curve, surface, or solid, relative to a point  $O$  is

$$\mathbf{r}_C = \frac{1}{m} \int_D \mathbf{r} \rho \, d\tau,$$

or using the orthogonal cartesian coordinates

$$x_C = \frac{1}{m} \int_D x \rho \, d\tau, \quad y_C = \frac{1}{m} \int_D y \rho \, d\tau, \quad z_C = \frac{1}{m} \int_D z \rho \, d\tau,$$

where,  $\rho$  is the mass density of the body: mass per unit of length if  $D$  is a curve, mass per unit area if  $D$  is a surface, and mass per unit of volume if  $D$  is a solid,  $\mathbf{r}$  is the position vector of a typical point of  $D$ , relative to  $O$ ,  $d\tau$  is the length, area, or volume of a differential element of  $D$ ,  $m = \int_D \rho \, d\tau$  is the total mass of the body, and  $x_C, y_C, z_C$  are the coordinates of  $C$ .

If the mass density  $\rho$  of a body is the same at all points of the body,  $\rho = \text{constant}$ , the density, as well as the body, are said to be *uniform*. The mass center of a uniform body coincides with the centroid of the figure occupied by the body.

The *method of decomposition* may be used to locate the mass center of a continuous body  $B$ :

- divide the body  $B$  into a number of bodies, which may be particles, curves, surfaces, or solids;
- locate the mass center of each body;

- assign to each mass center a strength proportional to the mass of the corresponding body (e.g., the weight of the body);
- locate the centroid of this set of mass centers.

### First Moment of an Area

A planar surface of area  $A$  and a reference frame  $xOy$  in the plane of the surface are shown in Fig. 1.12. The first moment of area  $A$  about the  $x$  axis is

$$M_x = \int_A y \, dA, \quad (1.2)$$

and the first moment about the  $y$  axis is

$$M_y = \int_A x \, dA. \quad (1.3)$$

The first moment of area gives information of the shape, size, and orientation of the area.

The entire area  $A$  can be concentrated at a position  $C(x_C, y_C)$ , the centroid. The coordinates  $x_C$  and  $y_C$  are the centroidal coordinates. To compute the centroidal coordinates the moments of the distributed area are equated with that of the concentrated area about both axes

$$A y_C = \int_A y \, dA, \quad \implies \quad y_C = \frac{\int_A y \, dA}{A} = \frac{M_x}{A}, \quad (1.4)$$

$$A x_C = \int_A x \, dA, \quad \implies \quad x_C = \frac{\int_A x \, dA}{A} = \frac{M_y}{A}. \quad (1.5)$$

The location of the centroid of an area is independent of the reference axes employed, i.e., the centroid is a property only of the area itself.

If the axes  $xy$  have their origin at the centroid,  $O \equiv C$ , then these axes are called *centroidal axes*. The first moments about centroidal axes are zero. All axes going through the centroid of an area are called centroidal axes for that area, and the first moments of an area about any of its centroidal axes are zero. The perpendicular distance from the centroid to the centroidal axis must be zero.

In Fig. 1.13 is shown a plane area with the axis of symmetry collinear with the axis  $y$ . The area  $A$  can be considered as composed of area elements in symmetric pairs such as shown in Fig. 1.13. The first moment of such a

pair about the axis of symmetry  $y$  is zero. The entire area can be considered as composed of such symmetric pairs and the coordinate  $x_C$  is zero

$$x_C = \frac{1}{A} \int_A x \, dA = 0.$$

Thus, *the centroid of an area with one axis of symmetry must lie along the axis of symmetry.* The axis of symmetry then is a centroidal axis, which is another indication that the first moment of area must be zero about the axis of symmetry. With two orthogonal axes of symmetry, the centroid must lie at the intersection of these axes. For such areas as circles and rectangles, the centroid is easily determined by inspection.

In many problems, the area of interest can be considered formed by the addition or subtraction of simple areas. For simple areas the centroids are known by inspection. The areas made up of such simple areas are *composite* areas. For composite areas

$$x_C = \frac{\sum_i A_i x_{Ci}}{A},$$

$$y_C = \frac{\sum_i A_i y_{Ci}}{A},$$

where  $x_{Ci}$  and  $y_{Ci}$  (with proper signs) are the centroidal coordinates to simple area  $A_i$ , and where  $A$  is the total area.

### 1.3 Moments and Couples

#### Moment of a Bound Vector About a Point

**Definition.** The moment of a bound vector  $\mathbf{v}$  about a point  $A$  is the vector

$$\mathbf{M}_A^{\mathbf{v}} = \mathbf{r}_{AB} \times \mathbf{v}, \quad (1.6)$$

where  $\mathbf{r}_{AB}$  is the position vector of  $B$  relative to  $A$ , and  $B$  is any point of line of action,  $\Delta$ , of the vector  $\mathbf{v}$ , Fig. 1.14.

The vector  $\mathbf{M}_A^{\mathbf{v}} = \mathbf{0}$  if and only if the line of action of  $\mathbf{v}$  passes through  $A$  or  $\mathbf{v} = \mathbf{0}$ .

The magnitude of  $\mathbf{M}_A^{\mathbf{v}}$  is

$$|\mathbf{M}_A^{\mathbf{v}}| = M_A^{\mathbf{v}} = |\mathbf{r}_{AB}| |\mathbf{v}| \sin \theta,$$

where  $\theta$  is the angle between  $\mathbf{r}_{AB}$  and  $\mathbf{v}$  when they are placed tail to tail. The perpendicular distance from  $A$  to the line of action of  $\mathbf{v}$  is

$$d = |\mathbf{r}_{AB}| \sin \theta,$$

and the magnitude of  $\mathbf{M}_A^{\mathbf{v}}$  is

$$|\mathbf{M}_A^{\mathbf{v}}| = M_A^{\mathbf{v}} = d |\mathbf{v}|.$$

The vector  $\mathbf{M}_A^{\mathbf{v}}$  is perpendicular to both  $\mathbf{r}_{AB}$  and  $\mathbf{v}$ :  $\mathbf{M}_A^{\mathbf{v}} \perp \mathbf{r}_{AB}$  and  $\mathbf{M}_A^{\mathbf{v}} \perp \mathbf{v}$ . The vector  $\mathbf{M}_A^{\mathbf{v}}$  being perpendicular to  $\mathbf{r}_{AB}$  and  $\mathbf{v}$  is perpendicular to the plane containing  $\mathbf{r}_{AB}$  and  $\mathbf{v}$ .

The moment given by Eq. (1.6) does not depend on the point  $B$  of the line of action of  $\mathbf{v}$ ,  $\Delta$ , where  $\mathbf{r}_{AB}$  intersects  $\Delta$ . Instead of using the point  $B$  the point  $B'$ , Fig. 1.14 can be used. The vector  $\mathbf{r}_{AB} = \mathbf{r}_{AB'} + \mathbf{r}_{B'B}$  where the vector  $\mathbf{r}_{B'B}$  is parallel to  $\mathbf{v}$ ,  $\mathbf{r}_{B'B} \parallel \mathbf{v}$ . Therefore,

$$\mathbf{M}_A^{\mathbf{v}} = \mathbf{r}_{AB} \times \mathbf{v} = (\mathbf{r}_{AB'} + \mathbf{r}_{B'B}) \times \mathbf{v} = \mathbf{r}_{AB'} \times \mathbf{v} + \mathbf{r}_{B'B} \times \mathbf{v} = \mathbf{r}_{AB'} \times \mathbf{v},$$

because  $\mathbf{r}_{B'B} \times \mathbf{v} = \mathbf{0}$ .

### Moment of a Bound Vector About a Line

**Definition.** The moment  $\mathbf{M}_\Omega^{\mathbf{v}}$  of a bound vector  $\mathbf{v}$  about a line  $\Omega$  is the  $\Omega$  resolute ( $\Omega$  component) of the moment  $\mathbf{v}$  about any point on  $\Omega$ , Fig. 1.15.

The  $\mathbf{M}_\Omega^{\mathbf{v}}$  is the  $\Omega$  resolute of  $\mathbf{M}_A^{\mathbf{v}}$

$$\begin{aligned} \mathbf{M}_\Omega^{\mathbf{v}} &= \mathbf{n} \cdot \mathbf{M}_A^{\mathbf{v}} \mathbf{n} \\ &= \mathbf{n} \cdot (\mathbf{r} \times \mathbf{v}) \mathbf{n} \\ &= [\mathbf{n}, \mathbf{r}, \mathbf{v}] \mathbf{n}, \end{aligned}$$

where  $\mathbf{n}$  is a unit vector parallel to  $\Omega$ , and  $\mathbf{r}$  is the position vector of a point on the line of action of  $\mathbf{v}$  relative to a point on  $\Omega$ .

The magnitude of  $\mathbf{M}_\Omega^{\mathbf{v}}$  is given by

$$|\mathbf{M}_\Omega^{\mathbf{v}}| = |[\mathbf{n}, \mathbf{r}, \mathbf{v}]|.$$

The moment of a vector about a line is a free vector.

If a line  $\Omega$  is parallel to the line of action  $\Delta$  of a vector  $\mathbf{v}$ , then  $[\mathbf{n}, \mathbf{r}, \mathbf{v}] \mathbf{n} = \mathbf{0}$  and  $\mathbf{M}_\Omega^{\mathbf{v}} = \mathbf{0}$ .

If a line  $\Omega$  intersects the line of action  $\Delta$  of  $\mathbf{v}$ , then  $\mathbf{r}$  can be chosen in such a way that  $\mathbf{r} = \mathbf{0}$  and  $\mathbf{M}_\Omega^{\mathbf{v}} = \mathbf{0}$ .

If a line  $\Omega$  is perpendicular to the line of action  $\Delta$  of a vector  $\mathbf{v}$ , and  $d$  is the shortest distance between these two lines, then

$$|\mathbf{M}_\Omega^{\mathbf{v}}| = d|\mathbf{v}|.$$

### Moments of a System of Bound Vectors

**Definition.** The moment of a system  $\{S\}$  of bound vectors  $\mathbf{v}_i$ ,  $\{S\} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = \{\mathbf{v}_i\}_{i=1,2,\dots,n}$  about a point  $A$  is

$$\mathbf{M}_A^{\{S\}} = \sum_{i=1}^n \mathbf{M}_A^{\mathbf{v}_i}.$$

**Definition.** The moment of a system  $\{S\}$  of bound vectors  $\mathbf{v}_i$ ,  $\{S\} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = \{\mathbf{v}_i\}_{i=1,2,\dots,n}$  about a line  $\Omega$  is

$$\mathbf{M}_\Omega^{\{S\}} = \sum_{i=1}^n \mathbf{M}_\Omega^{\mathbf{v}_i}.$$

The moments  $\mathbf{M}_A^{\{S\}}$  and  $\mathbf{M}_P^{\{S\}}$  of a system  $\{S\}$ ,  $\{S\} = \{\mathbf{v}_i\}_{i=1,2,\dots,n}$ , of bound vectors,  $\mathbf{v}_i$ , about two points  $A$  and  $P$ , are related to each other as follows,

$$\mathbf{M}_A^{\{S\}} = \mathbf{M}_P^{\{S\}} + \mathbf{r}_{AP} \times \mathbf{R}, \quad (1.7)$$

where  $\mathbf{r}_{AP}$  is the position vector of  $P$  relative to  $A$ , and  $\mathbf{R}$  is the resultant of  $\{S\}$ .

**Proof.** Let  $B_i$  a point on the line of action of the vector  $\mathbf{v}_i$ ,  $\mathbf{r}_{AB_i}$  and  $\mathbf{r}_{PB_i}$  the position vectors of  $B_i$  relative to  $A$  and  $P$ , Fig. 1.16. Thus,

$$\begin{aligned} \mathbf{M}_A^{\{S\}} &= \sum_{i=1}^n \mathbf{M}_A^{\mathbf{v}_i} = \sum_{i=1}^n \mathbf{r}_{AB_i} \times \mathbf{v}_i \\ &= \sum_{i=1}^n (\mathbf{r}_{AP} + \mathbf{r}_{PB_i}) \times \mathbf{v}_i = \sum_{i=1}^n (\mathbf{r}_{AP} \times \mathbf{v}_i + \mathbf{r}_{PB_i} \times \mathbf{v}_i) \\ &= \sum_{i=1}^n \mathbf{r}_{AP} \times \mathbf{v}_i + \sum_{i=1}^n \mathbf{r}_{PB_i} \times \mathbf{v}_i \end{aligned}$$

$$\begin{aligned}
&= \mathbf{r}_{AP} \times \sum_{i=1}^n \mathbf{v}_i + \sum_{i=1}^n \mathbf{r}_{PB_i} \times \mathbf{v}_i \\
&= \mathbf{r}_{AP} \times \mathbf{R} + \sum_{i=1}^n \mathbf{M}_P^{\mathbf{v}_i} \\
&= \mathbf{r}_{AP} \times \mathbf{R} + \mathbf{M}_P^{\{S\}}.
\end{aligned}$$

If the resultant  $\mathbf{R}$  of a system  $\{S\}$  of bound vectors is not equal to zero,  $\mathbf{R} \neq \mathbf{0}$ , the points about which  $\{S\}$  has a minimum moment  $\mathbf{M}_{min}$  lie on a line called *central axis*, ( $CA$ ), of  $\{S\}$ , which is parallel to  $\mathbf{R}$  and passes through a point  $P$  whose position vector  $\mathbf{r}$  relative to an arbitrarily selected reference point  $O$  is given by

$$\mathbf{r} = \frac{\mathbf{R} \times \mathbf{M}_O^{\{S\}}}{\mathbf{R}^2}.$$

The minimum moment  $\mathbf{M}_{min}$  is given by

$$\mathbf{M}_{min} = \frac{\mathbf{R} \cdot \mathbf{M}_O^{\{S\}}}{\mathbf{R}^2} \mathbf{R}.$$

## Couples

**Definition.** A *couple* is a system of bound vectors whose resultant is equal to zero and whose moment about some point is not equal to zero.

A system of vectors is not a vector, therefore couples are not vectors.

A couple consisting of only two vectors is called a *simple couple*. The vectors of a simple couple have equal magnitudes, parallel lines of action, and opposite senses.

Writers use the word “couple” to denote the simple couple.

The moment of a couple about a point is called the *torque* of the couple,  $\mathbf{M}$  or  $\mathbf{T}$ . The moment of a couple about one point is equal to the moment of the couple about any other point i.e., it is unnecessary to refer to a specific point. The moment of a couple is a free vector.

The torques are vectors and the magnitude of a torque of a simple couple is given by

$$|\mathbf{M}| = d|\mathbf{v}|,$$

where  $d$  is the distance between the lines of action of the two vectors comprising the couple, and  $\mathbf{v}$  is one of these vectors.

**Proof.** In Fig. 1.17, the torque  $\mathbf{M}$  is the sum of the moments of  $\mathbf{v}$  and  $-\mathbf{v}$  about any point. The moments about point  $A$

$$\mathbf{M} = \mathbf{M}_A^{\mathbf{v}} + \mathbf{M}_A^{-\mathbf{v}} = \mathbf{r} \times \mathbf{v} + \mathbf{0}.$$

Hence,

$$|\mathbf{M}| = |\mathbf{r} \times \mathbf{v}| = |\mathbf{r}||\mathbf{v}| \sin(\mathbf{r}, \mathbf{v}) = d|\mathbf{v}|.$$

The direction of the torque of a simple couple can be determined by inspection:  $\mathbf{M}$  is perpendicular to the plane determined by the lines of action of the two vectors comprising the couple, and the sense of  $\mathbf{M}$  is the same as that of  $\mathbf{r} \times \mathbf{v}$ .

The moment of a couple about a line  $\Omega$  is equal to the  $\Omega$  resolute of the torque of the couple.

The moments of a couple about two parallel lines are equal to each other.

### Equivalence of Systems

**Definition.** Two systems  $\{S\}$  and  $\{S'\}$  of bound vectors are said to be *equivalent* when:

1) the resultant of  $\{S\}$ ,  $\mathbf{R}$ , is equal to the resultant of  $\{S'\}$ ,  $\mathbf{R}'$

$$\mathbf{R} = \mathbf{R}'$$

2) there exists at least one point about which  $\{S\}$  and  $\{S'\}$  have equal moments

$$\text{exists } P : \mathbf{M}_P^{\{S\}} = \mathbf{M}_P^{\{S'\}}.$$

Figures 1.18(a) and 1.18(b) each show a rod subjected to the action of a pair of forces. The two pairs of forces are equivalent, but their effects on the rod are different from each other. The word “equivalence” is not to be regarded as implying physical equivalence.

For given a line  $\Omega$  and two equivalent systems  $\{S\}$  and  $\{S'\}$  of bound vectors, the sum of the  $\Omega$  resolutes of the vectors in  $\{S\}$  is equal to the sum of the  $\Omega$  resolutes of the vectors in  $\{S'\}$ .

The moments of two equivalent systems of bound vectors, about point, are equal to each other.

The moments of two equivalent systems  $\{S\}$  and  $\{S'\}$  of bound vectors, about *any* line  $\Omega$ , are equal to each other.

*Transitivity of the equivalence relation.* If  $\{S\}$  is equivalent to  $\{S'\}$ , and  $\{S'\}$  is equivalent to  $\{S''\}$ , then  $\{S\}$  is equivalent to  $\{S''\}$ .

Every system  $\{S\}$  of bound vectors with the resultant  $\mathbf{R}$  can be replaced with a system consisting of a couple  $C$  and a single bound vector  $\mathbf{v}$  whose line of action passes through an arbitrarily selected *base point*  $O$ . The torque  $\mathbf{M}$  of  $C$  depends on the choice of base point  $\mathbf{M} = \mathbf{M}_O^{\{S\}}$ . The vector  $\mathbf{v}$  is independent of the choice of base point,  $\mathbf{v} = \mathbf{R}$ .

A couple  $C$  can be replaced with any system of couples the sum of whose torque is equal to the torque of  $C$ .

When a system of bound vectors consists of a couple of torque  $\mathbf{M}$  and a single vector parallel to  $\mathbf{M}$ , it is called a *wrench*.

### **Force Vector and Moment of a Force**

Force is a vector quantity, having both magnitude and direction. Force is commonly explained in terms of Newton's three laws of motion set forth in his *Principia Mathematica* (1687). Newton's first principle: a body that is at rest or moving at a uniform rate in a straight line will remain in that state until some force is applied to it. Newton's second law of motion states that a particle acted on by forces whose resultant is not zero will move in such a way that the time rate of change of its momentum will at any instant be proportional to the resultant force. Newton's third law states that when one body exerts a force on another body, the second body exerts an equal force on the first body. This is the principle of action and reaction.

Because force is a vector quantity can be represented graphically as a directed line segment. The representation of forces by vectors implies that they are concentrated either at a single point or along a single line. The force of gravity is invariably distributed throughout the volume of a body. Nonetheless, when the equilibrium of a body is the primary consideration, it is generally valid as well as convenient to assume that the forces are concentrated at a single point. In the case of gravitational force, the total weight of a body may be assumed to be concentrated at its centre of gravity.

Force is measured in newtons (N); a force of 1 N will accelerate a mass of one kilogram at a rate of one meter per second per second. The newton is a unit of the International System (SI), for measuring force.

Using the English system, the force is measured in pounds. One pound of force imparts to a one-pound object an acceleration of 32.17 feet per second squared.

The force vector  $\mathbf{F}$  can be expressed in terms of a cartesian reference

frame, with the unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$ , Fig. 1.19(a)

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}. \quad (1.8)$$

The components of the force in the  $x$ ,  $y$ , and  $z$  directions are  $F_x$ ,  $F_y$ , and  $F_z$ . The resultant of two forces  $\mathbf{F}_1 = F_{1x}\mathbf{i} + F_{1y}\mathbf{j} + F_{1z}\mathbf{k}$  and  $\mathbf{F}_2 = F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k}$  is the vector sum of those forces

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = (F_{1x} + F_{2x})\mathbf{i} + (F_{1y} + F_{2y})\mathbf{j} + (F_{1z} + F_{2z})\mathbf{k}. \quad (1.9)$$

A moment is defined as the moment of a force about (with respect to) a point. The moment of the force  $\mathbf{F}$  about the point  $O$  is the cross product vector

$$\begin{aligned} \mathbf{M}_O^{\mathbf{F}} &= \mathbf{r} \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \\ &= (r_y F_z - r_z F_y)\mathbf{i} + (r_z F_x - r_x F_z)\mathbf{j} + (r_x F_y - r_y F_x)\mathbf{k}. \end{aligned} \quad (1.10)$$

where  $\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$  is a position vector directed from the point about which the moment is taken ( $O$  in this case) to any point  $A$  on the line of action of the force, Fig. 1.19(a). If the coordinates of  $O$  are  $x_O, y_O, z_O$  and the coordinates of  $A$  are  $x_A, y_A, z_A$  then  $\mathbf{r} = \mathbf{r}_{OA} = (x_A - x_O)\mathbf{i} + (y_A - y_O)\mathbf{j} + (z_A - z_O)\mathbf{k}$  and the the moment of the force  $\mathbf{F}$  about the point  $O$  is

$$\mathbf{M}_O^{\mathbf{F}} = \mathbf{r}_{OA} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_A - x_O & y_A - y_O & z_A - z_O \\ F_x & F_y & F_z \end{vmatrix}.$$

The magnitude of  $\mathbf{M}_O^{\mathbf{F}}$  is

$$|\mathbf{M}_O^{\mathbf{F}}| = M_O^{\mathbf{F}} = r F |\sin \theta|,$$

where  $\theta = \angle(\mathbf{r}, \mathbf{F})$  is the angle between vectors  $\mathbf{r}$  and  $\mathbf{F}$ , and  $r = |\mathbf{r}|$  and  $F = |\mathbf{F}|$  are the magnitudes of the vectors.

The line of action of  $\mathbf{M}_O^{\mathbf{F}}$  is perpendicular to the plane containing  $\mathbf{r}$  and  $\mathbf{F}$  ( $\mathbf{M}_O^{\mathbf{F}} \perp \mathbf{r}$  &  $\mathbf{M}_O^{\mathbf{F}} \perp \mathbf{F}$ ) and the sense is given by the right-hand rule.

The moment of the force  $\mathbf{F}$  about another point  $P$  is

$$\mathbf{M}_P^{\mathbf{F}} = \mathbf{r}_{PA} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_A - x_P & y_A - y_P & z_A - z_P \\ F_x & F_y & F_z \end{vmatrix},$$

where  $x_P, y_P, z_P$  are the coordinates of the point  $P$ .

The system of two forces,  $\mathbf{F}_1$  and  $\mathbf{F}_2$ , which have equal magnitudes  $|\mathbf{F}_1| = |\mathbf{F}_2|$ , opposite senses  $\mathbf{F}_1 = -\mathbf{F}_2$ , and parallel directions ( $\mathbf{F}_1 \parallel \mathbf{F}_2$ ) is a couple. The resultant force of a couple is zero  $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0}$ . The resultant moment  $\mathbf{M} \neq \mathbf{0}$  about an arbitrary point is

$$\mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2,$$

or

$$\mathbf{M} = \mathbf{r}_1 \times (-\mathbf{F}_2) + \mathbf{r}_2 \times \mathbf{F}_2 = (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}_2 = \mathbf{r} \times \mathbf{F}_2, \quad (1.11)$$

where  $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$  is a vector from any point on the line of action of  $\mathbf{F}_1$  to any point of the line of action of  $\mathbf{F}_2$ . The direction of the torque of the couple is perpendicular to the plane of the couple and the magnitude is given by, Fig. 1.19(b)

$$|\mathbf{M}| = M = r F_2 |\sin \theta| = h F_2, \quad (1.12)$$

where  $h = r |\sin \theta|$  is the perpendicular distance between the lines of action. The resultant moment of a couple is independent of the point with respect to which moments are taken.

### Representing Systems by Equivalent Systems

To simplify the analysis of the forces and moments acting on a given system one can represent the system by an equivalent one, a less complicated one. The actual forces and moments can be replaced with a total force and a total moment.

In Fig. 1.20 is shown an arbitrary system of forces and moments, {system 1}, and a point  $P$ . This system can be represented by a system, {system 2}, consisting of a single force  $\mathbf{F}$  acting at  $P$  and a single couple of torque  $\mathbf{M}$ . The conditions for equivalence are

$$\sum \mathbf{F}^{\{\text{system 2}\}} = \sum \mathbf{F}^{\{\text{system 1}\}} \implies \mathbf{F} = \sum \mathbf{F}^{\{\text{system 1}\}},$$

and

$$\sum \mathbf{M}_P^{\{\text{system 2}\}} = \sum \mathbf{M}_P^{\{\text{system 1}\}} \implies \mathbf{M} = \sum \mathbf{M}_P^{\{\text{system 1}\}}.$$

These conditions are satisfied if  $\mathbf{F}$  equals the sum of the forces in {system 1}, and  $\mathbf{M}$  equals the sum of the moments about  $P$  in {system 1}. Thus no

matter how complicated a system of forces and moments may be, it can be represented by a single force acting at a given point and a single couple. Three particular cases occur frequently in practice.

*1. Force represented by a force and a couple.*

A force  $\mathbf{F}_P$  acting at a point  $P$  {system 1} in Fig. 1.20 can be represented by a force  $\mathbf{F}$  acting at a different point  $Q$  and a couple of torque  $\mathbf{M}$  {system 2}. The moment of {system 1} about point  $Q$  is  $\mathbf{r}_{QP} \times \mathbf{F}_P$ , where  $\mathbf{r}_{QP}$  is the vector from  $Q$  to  $P$ . The conditions for equivalence are

$$\sum \mathbf{M}_P^{\{\text{system 2}\}} = \sum \mathbf{M}_P^{\{\text{system 1}\}} \implies \mathbf{F} = \mathbf{F}_P,$$

and

$$\sum \mathbf{M}_Q^{\{\text{system 2}\}} = \sum \mathbf{M}_Q^{\{\text{system 1}\}} \implies \mathbf{M} = \mathbf{M}_Q^{\mathbf{F}_P} = \mathbf{r}_{QP} \times \mathbf{F}_P.$$

The systems are equivalent if the force  $\mathbf{F}$  equals the force  $\mathbf{F}_P$  and the couple of torque  $\mathbf{M}_Q^{\mathbf{F}_P}$  equals the moment of  $\mathbf{F}_P$  about  $Q$ .

*2. Concurrent forces represented by a force.*

A system of concurrent forces whose lines of action intersect at a point  $P$  {system 1}, Fig. 1.21(a), can be represented by a single force whose line of action intersects  $P$ , {system 2}. The sums of the forces in the two systems are equal if

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \dots + \mathbf{F}_n.$$

The sum of the moments about  $P$  equals zero for each system, so the systems are equivalent if the force  $\mathbf{F}$  equals the sum of the forces in {system 1}.

*3. Parallel forces represented by a force.*

A system of parallel forces whose sum is not zero can be represented by a single force  $\mathbf{F}$ , Fig. 1.21(b).

*4. System represented by a wrench.*

In general any system of forces and moments can be represented by a single force acting at a given point and a single couple.

Figure 1.22 shows an arbitrary force  $\mathbf{F}$  acting at a point  $P$  and an arbitrary couple of torque  $\mathbf{M}$ , {system 1}. This system can be represented by a

simpler one i.e., one may represent the force  $\mathbf{F}$  acting at a different point  $Q$  and the component of  $\mathbf{M}$  that is parallel to  $\mathbf{F}$ . A coordinate system is chosen so that  $\mathbf{F}$  is along the  $y$  axis

$$\mathbf{F} = F\mathbf{j},$$

and  $\mathbf{M}$  is contained in the  $xy$  plane

$$\mathbf{M} = M_x\mathbf{i} + M_y\mathbf{j}.$$

The equivalent system, {system 2}, consists of the force  $\mathbf{F}$  acting at a point  $Q$  on the  $z$  axis

$$\mathbf{F} = F\mathbf{j},$$

and the component of  $\mathbf{M}$  parallel to  $\mathbf{F}$

$$\mathbf{M}_p = M_y\mathbf{j}.$$

The distance  $PQ$  is chosen so that  $|\mathbf{r}_{PQ}| = PQ = M_x/F$ . The {system 1} is equivalent to {system 2}.

The sum of the forces in each system is the same  $\mathbf{F}$ .

The sum of the moments about  $P$  in {system 1} is  $\mathbf{M}$ , and the sum of the moments about  $P$  in {system 2} is

$$\sum \mathbf{M}_P^{\{\text{system 2}\}} = \mathbf{r}_{PQ} \times \mathbf{F} + M_y\mathbf{j} = (-PQ\mathbf{k}) \times (F\mathbf{j}) + M_y\mathbf{j} = M_x\mathbf{i} + M_y\mathbf{j} = \mathbf{M}.$$

The system of the force  $\mathbf{F} = F\mathbf{j}$  and the couple  $\mathbf{M}_p = M_y\mathbf{j}$  that is parallel to  $\mathbf{F}$  is a wrench. A wrench is the simplest system that can be equivalent to an arbitrary system of forces and moments.

The representation of a given system of forces and moments by a wrench requires the following steps

1. Choose a convenient point  $P$  and represent the system by a force  $\mathbf{F}$  acting at  $P$  and a couple  $\mathbf{M}$  (Fig. 1.23(a)).
2. Determine the components of  $\mathbf{M}$  parallel and normal to  $\mathbf{F}$  (Fig. 1.23(b))

$$\mathbf{M} = \mathbf{M}_p + \mathbf{M}_n, \quad \text{where } \mathbf{M}_p \parallel \mathbf{F}.$$

3. The wrench consists of the force  $\mathbf{F}$  acting at a point  $Q$  and the parallel component  $\mathbf{M}_p$  (Fig. 1.23(c)). For equivalence, the following condition must be satisfied

$$\mathbf{r}_{PQ} \times \mathbf{F} = \mathbf{M}_n,$$

where  $\mathbf{M}_n$  is the normal component of  $\mathbf{M}$ .

In general, the {system 1} cannot be represented by a force  $\mathbf{F}$  alone.

## 1.4 Equilibrium

### Equilibrium Equations

A body is in *equilibrium* when it is stationary or in steady translation relative to an inertial reference frame. The following conditions are satisfied when a body, acted upon by a system of forces and moments, is in equilibrium:

1. the sum of the forces is zero

$$\sum \mathbf{F} = \mathbf{0}. \quad (1.13)$$

2. the sum of the moments about any point is zero

$$\sum \mathbf{M}_P = \mathbf{0}, \quad \forall P. \quad (1.14)$$

If the sum of the forces acting on a body is zero and the sum of the moments about one point is zero, then the sum of the moments about every point is zero.

**Proof.** The body shown in Figure 1.24, is subjected to forces  $\mathbf{F}_{Ai}$ ,  $i = 1, \dots, n$ , and moments  $\mathbf{M}_j$ ,  $j = 1, \dots, m$ . The sum of the forces is zero

$$\sum \mathbf{F} = \sum_{i=1}^n \mathbf{F}_{Ai} = \mathbf{0},$$

and the sum of the moments about a point  $P$  is zero

$$\sum \mathbf{M}_P = \sum_{i=1}^n \mathbf{r}_{PAi} \times \mathbf{F}_{Ai} + \sum_{j=1}^m \mathbf{M}_j = \mathbf{0},$$

where  $\mathbf{r}_{PAi} = \vec{PA}_i$ ,  $i = 1, \dots, n$ . The sum of the moments about any other point  $Q$  is

$$\begin{aligned} \sum \mathbf{M}_Q &= \sum_{i=1}^n \mathbf{r}_{QAi} \times \mathbf{F}_{Ai} + \sum_{j=1}^m \mathbf{M}_j = \\ &= \sum_{i=1}^n (\mathbf{r}_{QP} + \mathbf{r}_{PAi}) \times \mathbf{F}_{Ai} + \sum_{j=1}^m \mathbf{M}_j = \\ &= \mathbf{r}_{QP} \times \sum_{i=1}^n \mathbf{F}_{Ai} + \sum_{i=1}^n \mathbf{r}_{PAi} \times \mathbf{F}_{Ai} + \sum_{j=1}^m \mathbf{M}_j = \end{aligned}$$

$$\begin{aligned} \mathbf{r}_{QP} \times \mathbf{0} + \sum_{i=1}^n \mathbf{r}_{PAi} \times \mathbf{F}_{Ai} + \sum_{j=1}^m \mathbf{M}_j = \\ \sum_{i=1}^n \mathbf{r}_{PAi} \times \mathbf{F}_{Ai} + \sum_{j=1}^m \mathbf{M}_j = \sum \mathbf{M}_P = \mathbf{0}. \end{aligned}$$

A body subjected to concurrent forces  $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$  and no couples. If the sum of the concurrent forces is zero,

$$\mathbf{F}_1 + \mathbf{F}_2 + \dots + \mathbf{F}_n = \mathbf{0},$$

the sum of the moments of the forces about the concurrent point is zero, so the sum of the moments about every point is zero. The only condition imposed by equilibrium on a set of concurrent forces is that their sum is zero.

### Free-Body Diagrams

Free-body diagrams are used to determine forces and moments acting on simple bodies in equilibrium.

The beam in Figure 1.25(a) has a pin support at the left end  $A$  and a roller support at the right end  $B$ . The beam is loaded by a force  $F$  and a moment  $M$  at  $C$ . To obtain the free-body diagram first the beam is isolated from its supports. Next, the reactions exerted on the beam by the supports are shown on the the free-body diagram, Figure 1.25(b). Once the free-body diagram is obtained one can apply the equilibrium equations.

The steps required to determine the reactions on bodies are

1. draw the free-body diagram, isolating the body from its supports and showing the forces and the reactions;
2. apply the equilibrium equations to determine the reactions.

For two-dimensional systems, the forces and moments are related by three scalar equilibrium equations

$$\sum F_x = 0, \tag{1.15}$$

$$\sum F_y = 0, \tag{1.16}$$

$$\sum M_P = 0, \quad \forall P. \tag{1.17}$$

One can obtain more than one equation from Eq. (1.17) by evaluating the sum of the moments about more than one point. The additional equations

will not be independent of Eqs. (1.15)-(1.17). One cannot obtain more than three independent equilibrium equations from a two-dimensional free-body diagram, which means one can solve for at most three unknown forces or moments.

For three-dimensional systems, the forces and moments are related by six scalar equilibrium equations

$$\sum F_x = 0, \quad (1.18)$$

$$\sum F_y = 0, \quad (1.19)$$

$$\sum F_z = 0, \quad (1.20)$$

$$\sum M_x = 0, \quad (1.21)$$

$$\sum M_y = 0, \quad (1.22)$$

$$\sum M_z = 0. \quad (1.23)$$

The sums of the moments about any point can be evaluated. Although one can obtain other equations by summing the moments about additional points, they will not be independent of these equations. For a three-dimensional free-body diagram six independent equilibrium equations are obtained and one can solve for at most six unknown forces or moments.

A body has *redundant supports* when the body has more supports than the minimum number necessary to maintain it in equilibrium. Redundant supports are used whenever possible for strength and safety. Each support added to a body results in additional reactions. The difference between the number of reactions and the number of independent equilibrium equations is called the *degree of redundancy*.

A body has *improper supports* if it will not remain in equilibrium under the action of the loads exerted on it. The body with improper supports will move when the loads are applied.

#### *Two-force and three-force members*

A body is a *two-force member* if the system of forces and moments acting on the body is equivalent to two forces acting at different points.

For example a body is subjected to two forces  $\mathbf{F}_A$  and  $\mathbf{F}_B$  at  $A$  and  $B$ . If the body is in equilibrium the sum of the forces equals zero only if  $\mathbf{F}_A = -\mathbf{F}_B$ . Furthermore, the forces  $\mathbf{F}_A$  and  $-\mathbf{F}_B$  form a couple, so the sum of the moments is not zero unless the lines of action of the forces lie along the line through the points  $A$  and  $B$ . Thus for equilibrium the two forces

are equal in magnitude, are opposite in direction, and have the same line of action.

A body is a *three-force member* if the system of forces and moments acting on the body is equivalent to three forces acting at different points.

If a three-force member is in equilibrium, the three forces are coplanar and are either parallel or concurrent.

**Proof.** Let the forces  $\mathbf{F}_1$ ,  $\mathbf{F}_2$ , and  $\mathbf{F}_3$  act on the body at  $A_1$ ,  $A_2$ , and  $A_3$ . Let  $\pi$  be the plane containing the three points of application  $A_1$ ,  $A_2$ , and  $A_3$ . Let  $\Delta = A_1A_2$  be the line through the points of application of  $\mathbf{F}_1$  and  $\mathbf{F}_2$ . Since the moments due to  $\mathbf{F}_1$  and  $\mathbf{F}_2$  about  $\Delta$  are zero, the moment due to  $\mathbf{F}_3$  about  $\Delta$  must equal zero

$$[\mathbf{n} \cdot (\mathbf{r} \times \mathbf{F}_3)] \mathbf{n} = [\mathbf{F}_3 \cdot (\mathbf{n} \times \mathbf{r})] \mathbf{n} = \mathbf{0},$$

where  $\mathbf{n}$  is the unit vector of  $\Delta$ . This equation requires that  $\mathbf{F}_3$  be perpendicular to  $\mathbf{n} \times \mathbf{r}$ , which means that  $\mathbf{F}_3$  is contained in  $\pi$ . The same procedure can be used to show that  $\mathbf{F}_1$  and  $\mathbf{F}_2$  are contained in  $\pi$ , so the forces  $\mathbf{F}_1$ ,  $\mathbf{F}_2$ , and  $\mathbf{F}_3$  are coplanar.

If the three coplanar forces are not parallel, there will be points where their lines of action intersect. Suppose that the lines of action of two forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$  intersect at a point  $P$ . Then the moments of  $\mathbf{F}_1$  and  $\mathbf{F}_2$  about  $P$  are zero. The sum of the moments about  $P$  is zero only if the line of action of the third force,  $\mathbf{F}_3$ , also passes through  $P$ . Therefore either the forces are concurrent or they are parallel.

The analysis of a body in equilibrium can often be simplified by recognizing the two-force or three-force member.

## 1.5 Dry Friction

If a body rests on an incline plane, the friction force exerted on it by the surface prevents it from sliding down the incline. The question is: what is the steepest incline on which the body can rest?

A body is placed on a horizontal surface. The body is pushed with a small horizontal force  $F$ . If the force  $F$  is sufficiently small, the body does not move. Figure 1.26 shows the free-body diagram of the body, where the force  $W$  is the weight of the body, and  $N$  is the normal force exerted by the surface. The force  $F$  is the horizontal force, and  $F_f$  is the friction force exerted by the surface. Friction force arises in part from the interactions

of the roughness, or asperities, of the contacting surfaces. The body is in equilibrium and  $F_f = F$ .

The force  $F$  is slowly increased. As long as the body remains in equilibrium, the friction force  $F_f$  must increase correspondingly, since it equals the force  $F$ . The body slips on the surface. The friction force, after reaching the maximum value, cannot maintain the body in equilibrium. The force applied to keep the body moving on the surface is smaller than the force required to cause it to slip. More force is required to start the body sliding on a surface than to keep it sliding is explained in part by the necessity to break the asperities of the contacting surfaces before sliding can begin.

The theory of dry friction, or *Coulomb friction*, predicts:

- the maximum friction forces that can be exerted by dry, contacting surfaces that are stationary relative to each other;
- the friction forces exerted by the surfaces when they are in relative motion, or sliding.

### Static Coefficient of Friction

The magnitude of the maximum friction force,  $F_f$ , that can be exerted between two plane dry surfaces in contact is

$$F_f = \mu_s N, \quad (1.24)$$

where  $\mu_s$  is a constant, the *static coefficient of friction*, and  $N$  is the normal component of the contact force between the surfaces. The value of the static coefficient of friction,  $\mu_s$ , depends on:

- the materials of the contacting surfaces;
- the conditions of the contacting surfaces: smoothness and degree of contamination.

Typical values of  $\mu_s$  for various materials are shown in Table 1.1.

Table 1.1. Typical values of the static coefficient of friction.

Materials	$\mu_s$
metal on metal	0.15 - 0.20
metal on wood	0.20 - 0.60
metal on masonry	0.30 - 0.70
wood on wood	0.25 - 0.50
masonry on masonry	0.60 - 0.70
rubber on concrete	0.50 - 0.90

Equation (1.24) gives the maximum friction force that the two surfaces can exert without causing it to slip. If the static coefficient of friction  $\mu_s$  between the body and the surface is known the largest value of  $F$  one can apply to the body without causing it to slip is  $F = F_f = \mu_s N$ . Equation (1.24) determines the magnitude of the maximum friction force but not its direction. The friction force resists the impending motion.

### Kinetic coefficient of friction

The magnitude of the friction force between two plane dry contacting surfaces that are in motion relative to each other is

$$F_f = \mu_k N, \quad (1.25)$$

where  $\mu_k$  is the *kinetic coefficient of friction* and  $N$  is the normal force between the surfaces. The value of the kinetic coefficient of friction is generally smaller than the value of the static coefficient of friction,  $\mu_s$ .

To keep the body in Fig. 1.26 in uniform motion (sliding on the surface) the force exerted must be  $F = F_f = \mu_k N$ . The friction force resists the relative motion, when two surfaces are sliding relative to each other.

The body  $RB$  shown in Fig. 1.27(a) is moving on the the fixed surface  $0$ . The direction of motion of  $RB$  is the positive axis  $x$ . The friction force on the body  $RB$  acts in the direction opposite to its motion, and the friction force on the fixed surface is in the opposite direction, Fig. 1.27(b).

### Angles of friction

The *angle of friction*,  $\theta$ , is the angle between the friction force,  $F_f = |\mathbf{F}_f|$ , and the normal force,  $N = |\mathbf{N}|$ , to the surface, Fig. 1.28. The magnitudes of the normal force and friction force, and  $\theta$  are related by

$$\begin{aligned} F_f &= R \sin \theta, \\ N &= R \cos \theta, \end{aligned}$$

where  $R = |\mathbf{R}| = |\mathbf{N} + \mathbf{F}_f|$ .

The value of the angle of friction when slip is impending is called the *static angle of friction*,  $\theta_s$

$$\tan \theta_s = \mu_s.$$

The value of the angle of friction when the surfaces are sliding relative to each other is called the the *kinetic angle of friction*,  $\theta_k$

$$\tan \theta_k = \mu_k.$$

## 1.6 Problems

- 1.1 a) Find the angle made by the vector  $\mathbf{v} = -10\mathbf{i} + 5\mathbf{j}$  with the positive  $x$ -axis and determine the unit vector in the direction of  $\mathbf{v}$ . b) Determine the magnitude of the resultant  $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$  and the angle which  $\mathbf{v}$  makes with the positive  $x$ -axis, where the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are shown in Fig. 1.29. The magnitudes of the vectors are  $|\mathbf{v}_1| = v_1 = 5$ ,  $|\mathbf{v}_2| = v_2 = 10$ , and the angles of the vectors with the positive  $x$ -axis are  $\theta_1 = 30^\circ$ ,  $\theta_2 = 60^\circ$ .
- 1.2 The following planar vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are given in  $xOy$  plane as shown in Fig. 1.30. The magnitude of the vectors are  $a = P$ ,  $b = 2P$ , and  $c = P\sqrt{2}$ . The angles in the Fig 1.30 are  $\alpha = 45^\circ$ ,  $\beta = 120^\circ$ , and  $\gamma = 30^\circ$ . Determine the magnitude of the resultant  $\mathbf{v} = \mathbf{a} + \mathbf{b} + \mathbf{c}$  and the angle which  $\mathbf{v}$  makes with the positive  $x$ -axis.
- 1.3 The cube in Fig. 1.31 has the sides equal to  $l$ . Find the direction cosines of the resultant  $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$ .
- 1.4 The following spatial vectors are given  $\mathbf{v}_1 = -3\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ ,  $\mathbf{v}_2 = 3\mathbf{i} + 3\mathbf{k}$ , and  $\mathbf{v}_3 = 1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ . Find the expressions  $\mathbf{E}_1 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ ,  $\mathbf{E}_2 = \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$ ,  $\mathbf{E}_3 = (\mathbf{v}_1 \times \mathbf{v}_2) \times \mathbf{v}_3$ , and  $E_4 = (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3$ .
- 1.5 Find the angle between the vectors  $\mathbf{v}_1 = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$  and  $\mathbf{v}_2 = 4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ . Find the expressions  $\mathbf{v}_1 \times \mathbf{v}_2$  and  $\mathbf{v}_1 \cdot \mathbf{v}_2$ .
- 1.6 The following vectors are given  $\mathbf{v}_1 = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$ ,  $\mathbf{v}_2 = 1\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ , and  $\mathbf{v}_3 = -2\mathbf{i} + 2\mathbf{k}$ . Find the vector triple product of  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$ , and explain the result.
- 1.7 Solve the vectorial equation  $\mathbf{x} \times \mathbf{a} = \mathbf{x} \times \mathbf{b}$ , where  $\mathbf{a}$  and  $\mathbf{b}$  are two known given vectors.
- 1.8 Solve the vectorial equation  $\mathbf{v} = \mathbf{a} \times \mathbf{x}$ , where  $\mathbf{v}$  and  $\mathbf{a}$  are two known given vectors.
- 1.9 Solve the vectorial equation  $\mathbf{a} \cdot \mathbf{x} = m$ , where  $\mathbf{a}$  is a known given vector and  $m$  is a known given scalar.
- 1.10 The forces  $\mathbf{F}_1$ ,  $\mathbf{F}_2$ ,  $\mathbf{F}_3$ , and  $\mathbf{F}_4$ , shown in Fig. 1.32, act on the sides of a cube (the side of the cube is  $l$ ). The magnitude of the forces are

$\mathbf{F}_1 = \mathbf{F}_2 = F$ , and  $\mathbf{F}_3 = \mathbf{F}_4 = F\sqrt{2}$ . Represent the given system of forces by an equivalent system at  $O$ .

1.11 Figure 1.33 represents the vectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ ,  $\mathbf{v}_3$ , and  $\mathbf{v}_4$  acting on a cube with the side  $l$ . The magnitude of the forces are  $\mathbf{v}_1 = V$  and  $\mathbf{v}_2 = \mathbf{v}_3 = \mathbf{v}_4 = 2V$ . Find the equivalent system at  $O$ .

1.12 Repeat the previous problem for Fig. 1.34.

1.13 The parallelepiped shown in Fig. 1.35 has the sides  $l=1\text{m}$ ,  $w=2\text{ m}$ , and  $h=3\text{ m}$ . The magnitude of the forces are  $\mathbf{F}_1 = \mathbf{F}_2 = 10\text{ N}$ , and  $\mathbf{F}_3 = \mathbf{F}_4 = 20\text{ N}$ . Find the equivalent wrench of the system.

1.14 A uniform rectangular plate of length  $l$  and width  $w$  is held open by a cable, Fig. 1.36. The plate is hinged about an axis parallel to the plate edge of length  $l$ . Points  $A$  and  $B$  are at the extreme ends of this hinged edge. Points  $D$  and  $C$  are at the ends of the other edge of length  $l$  and are respectively adjacent to points  $A$  and  $B$ . Points  $D$  and  $C$  move as the plate opens. In the closed position, the plate is in a horizontal plane. When held open by a cable, the plate has rotated through an angle  $\theta$  relative to the closed position. The supporting cable runs from point  $D$  to point  $E$  where point  $E$  is located a height  $h$  directly above the point  $B$  on the hinged edge of the plate. The cable tension required to hold the plate open is  $T$ . Find the projection of the tension force onto the diagonal axis  $AC$  of the plate.

Numerical application:  $l = 1.0\text{ m}$ ,  $w = 0.5\text{ m}$ ,  $\theta = 45^\circ$ ,  $h = 1.0\text{ m}$ , and  $T = 100\text{ N}$ .

1.15 A smooth sphere of mass  $m$  is resting against a vertical surface and an inclined surface that makes an angle  $\theta$  with the horizontal, as shown in Fig. 1.37. Find the forces exerted on the sphere by the two contacting surfaces.

Numerical application: a)  $m = 10\text{ kg}$ ,  $\theta = 30^\circ$ , and  $g = 9.8\text{ m/s}^2$ ; b)  $m = 2\text{ slugs}$ ,  $\theta = 60^\circ$ , and  $g = 32.2\text{ ft/sec}^2$ .

1.16 The links 1 and 2 shown in Fig. 1.38 are each connected to the ground at  $A$  and  $C$ , and to each other at  $B$  using frictionless pins. The length of link 1 is  $AB = l$ . The angle between the links is  $\angle ABC = \theta$ . A force of magnitude  $P$  is applied at the point  $D$  ( $AD = 2l/3$ ) of the link 1.

The force makes an angle  $\theta$  with the horizontal. Find the force exerted by the lower link 2 on the upper link 1.

Numerical application: a)  $l=1$  m,  $\theta=30^\circ$ , and  $P =1000$  N; b)  $l=2$  ft,  $\theta=45^\circ$ , and  $P = 500$  lb.

- 1.17 The block of mass  $m$  rests on a rough horizontal surface and is acted upon by a force,  $F$ , that makes an angle  $\theta$  with the horizontal, as shown in Fig. 1.39. The coefficient of static friction between the surface and the block is  $\mu_s$ . Find the magnitude of the force  $F$  required to cause the block to begin to slide.

Numerical application: a)  $m =2$  kg,  $\theta=60^\circ$ ,  $\mu_s = 0.4$ , and  $g = 9.8$  m/s<sup>2</sup>; b)  $m =10$  slugs,  $\theta=30^\circ$ ,  $\mu_s = 0.3$ , and  $g =32.2$  ft/sec<sup>2</sup>.

- 1.18 Find the  $x$ -coordinate of the centroid of the plane region bounded by the curves  $y = x^2$  and  $y = \sqrt{x}$ , ( $x > 0$ ).

- 1.19 The shaft shown in Fig. 1.40 turns in the bearings  $A$  and  $B$ . The dimensions of the shaft are  $a = 6$  in. and  $b = 3$  in. The forces on the gear attached to the shaft are  $F_t = 900$  lb and  $F_r = 500$  lb. The gear forces act at a radius  $R = 4$  in. from the axis of the shaft. Find the loads applied to the bearings.

- 1.20 The shaft shown in Fig. 1.41 turns in the bearings  $A$  and  $B$ . The dimensions of the shaft are  $a = 120$  mm and  $b = 30$  mm. The forces on the gear attached to the shaft are  $F_t = 4500$  N,  $F_r = 2500$  N, and  $F_a = 1000$  N. The gear forces act at a radius  $R = 100$  mm from the shaft axis. Determine the bearings loads.

- 1.21 The dimensions of the shaft, shown Fig. 1.42, are  $a = 2$  in. and  $l = 5$  in. The force on the disk with the radius  $r_1 = 5$  in. is  $F_1 = 600$  lb and the force on the disk with the radius  $r_2 = 2.5$  in. is  $F_2 = 1200$  lb. Determine the forces on the bearings at  $A$  and  $B$ .

- 1.22 The dimensions of the shaft, shown Fig. 1.43, are  $a = 50$  mm and  $l = 120$  mm. The force on the disk with the radius  $r_1 = 50$  mm is  $F_1 = 2000$  N and the force on the disk with the radius  $r_2 = 100$  mm is  $F_2 = 4000$  N. Determine the bearing loads at  $A$  and  $B$ .

1.23 The force on the gear in Fig. 1.44 is  $F = 1.5$  kN and the radius of the gear is  $R = 60$  mm. The dimensions of the shaft are  $l = 300$  mm and  $a = 60$  mm. Determine the bearing loads at  $A$  and  $B$ .

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## Figure captions

Figure 1.1. Vector representations: (a) straight arrow and (b) straight and curved arrows

Figure 1.2. Bound or fixed vector: (a) point of application represented as the tail of the vector arrow and (b) point of application represented as the head of the vector arrow

Figure 1.3. Transmissible vector: the force vector  $\mathbf{F}$  can be applied anywhere along the line  $\Delta$

Figure 1.4. Vector addition: (a) parallelogram law, (b) moving the vectors successively to parallel positions, Vector difference: (c) parallelogram law, (d) moving the vectors successively to parallel positions

Figure 1.5. Resolution of a vector  $\mathbf{v}$  and components

Figure 1.6. Cartesian reference frame and the orthogonal scalar components  $v_x, v_y, v_z$

Figure 1.7. The angle  $\theta$  between the vectors  $\mathbf{a}$  and  $\mathbf{b}$ : (a)  $0 < \theta < 90^\circ$ , (b)  $90^\circ < \theta < 180^\circ$ , and (c)  $\theta = 0^\circ$ , and (d)  $\theta = 180^\circ$

Figure 1.8. Direction cosines

Figure 1.9. Vector (cross) product of the vector  $\mathbf{a}$  and the vector  $\mathbf{b}$

Figure 1.10. Position vector

Figure 1.11. Centroid of a set of points

Figure 1.12. Centroid of a planar surface of area

Figure 1.13. Plane area with axis of symmetry

Figure 1.14. Moment of a bound vector about a point

Figure 1.15. Moment of a bound vector about a line

Figure 1.16. Moments of a system of bound vectors

Figure 1.17. Couple

Figure 1.18. Equivalent systems (not physical equivalence): (a) tension and (b) compression

Figure 1.19. Moment of a force: (a) moment of a force about a point and (b) torque of the couple

Figure 1.20. Equivalent systems

Figure 1.21. System of forces: (a) concurrent forces, and (b) parallel forces

Figure 1.22. System represented by a wrench

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Figure 1.41. Shaft with gear for Problem 1.20

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Figure 1.43. Shaft with two disks for Problem 1.22

Figure 1.44. Shaft with gear for Problem 1.23