

Chapter 4

Moments of Inertia

4.1 Introduction

A system of n particle P_i , $i = 1, 2, \dots, n$ is considered. The mass of the particle P_i is m_i as shown in Fig. 4.1

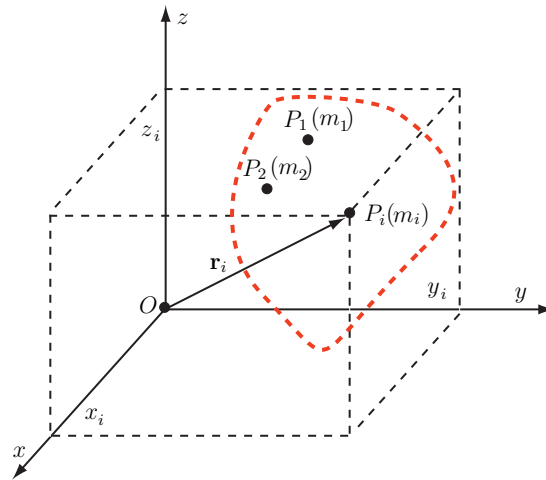


Fig. 4.1 Particle P_i with the mass m_i

The position vector of the particle P_i is

$$\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}.$$

The *moments of inertia* of the system about the planes xOy , yOz , and zOx are

$$I_{xOy} = \sum_i m_i z_i^2,$$

$$\begin{aligned}
 I_{yOz} &= \sum_i m_i x_i^2, \\
 I_{zOx} &= \sum_i m_i y_i^2.
 \end{aligned}
 \tag{4.1}$$

The *moments of inertia* of the system about x , y , and z axes are

$$\begin{aligned}
 I_{xx} &= A = \sum_i m_i (y_i^2 + z_i^2), \\
 I_{yy} &= B = \sum_i m_i (z_i^2 + x_i^2), \\
 I_{zz} &= C = \sum_i m_i (x_i^2 + y_i^2).
 \end{aligned}
 \tag{4.2}$$

The *moment of inertia* of the system about the origin O is

$$I_O = \sum_i m_i (x_i^2 + y_i^2 + z_i^2).
 \tag{4.3}$$

The *products of inertia* of the system about the axes xy , yz and zx are

$$\begin{aligned}
 I_{yz} &= D = \sum_i m_i y_i z_i, \\
 I_{zx} &= E = \sum_i m_i z_i x_i, \\
 I_{xy} &= F = \sum_i m_i x_i y_i.
 \end{aligned}
 \tag{4.4}$$

Between the different moments of inertia one can write the relations

$$\begin{aligned}
 I_O &= I_{xOy} + I_{yOz} + I_{zOx} \\
 &= \frac{1}{2} (I_{xx} + I_{yy} + I_{zz}),
 \end{aligned}$$

and

$$I_{xx} = I_{yOz} + I_{zOx}.$$

For a continuous domain D , the previous relations become

$$\begin{aligned}
 I_{xOy} &= \int_D z^2 dm, \quad I_{yOz} = \int_D x^2 dm, \quad I_{zOx} = \int_D y^2 dm, \\
 I_{xx} &= \int_D (y^2 + z^2) dm, \quad I_{yy} = \int_D (x^2 + z^2) dm, \quad I_{zz} = \int_D (x^2 + y^2) dm, \\
 I_O &= \int_D (x^2 + y^2 + z^2) dm, \\
 I_{xy} &= \int_D xy dm, \quad I_{xz} = \int_D xz dm, \quad I_{yz} = \int_D yz dm.
 \end{aligned}
 \tag{4.5}$$

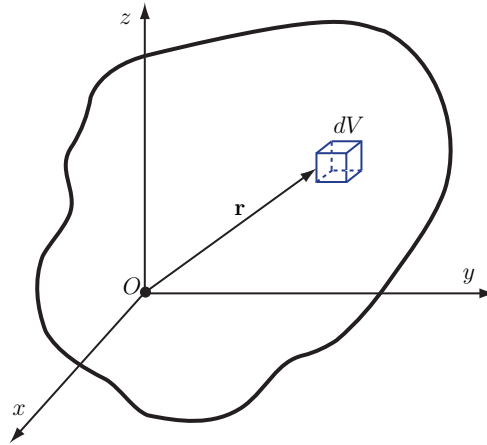


Fig. 4.2 Rigid body in space with mass m and differential volume dV

The infinitesimal mass element dm can have the values

$$\begin{aligned} dm &= \rho_v dV, \\ dm &= \rho_A dA, \\ dm &= \rho_l dl, \end{aligned}$$

where ρ_v , ρ_A and ρ_l are the volume density, area density and length density.

4.2 Moment of Inertia of a Rigid Body

For a rigid body with mass m , density ρ , and volume V , as shown in Fig. 4.2, the moments of inertia are defined as follows

$$\begin{aligned} I_{xx} &= \int_V \rho (y^2 + z^2) dV, \\ I_{yy} &= \int_V \rho (z^2 + x^2) dV, \\ I_{zz} &= \int_V \rho (x^2 + y^2) dV, \end{aligned} \tag{4.6}$$

and the products of inertia

$$I_{xy} = I_{yx} = \int_V \rho xy dV,$$

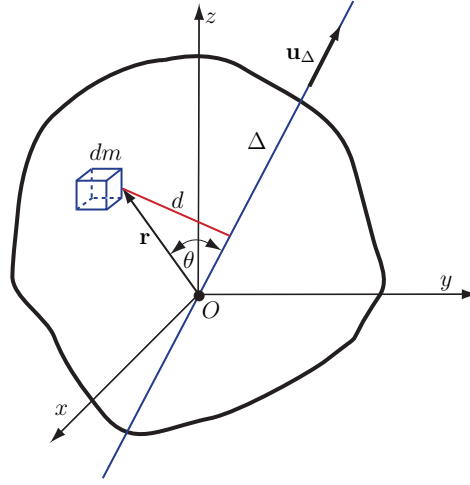


Fig. 4.3 Rigid body and an arbitrary axis Δ of unit vector \mathbf{u}_Δ

$$\begin{aligned} I_{xz} = I_{zx} &= \int_V \rho xz dV, \\ I_{yz} = I_{zy} &= \int_V \rho yz dV. \end{aligned} \quad (4.7)$$

The moment of inertia given in Eq. (4.6) is just the *second moment* of the mass distribution with respect to a cartesian axis. For example, I_{xx} is the integral of summation of the infinitesimal mass elements ρdV , each multiplied by the square of its distance from the x axis.

The effective value of this distance for a certain body is known as its *radius of gyration* with respect to the given axis.

The *radius of gyration* corresponding to I_{jj} is defined as

$$k_j = \sqrt{\frac{I_{jj}}{m}},$$

where m is the total mass of the rigid body, and where the symbol j can be replaced by x , y or z . The *inertia matrix* of a rigid body is represented by the matrix

$$[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}.$$

Moment of inertia about an arbitrary axis

Consider the rigid body shown in Fig. 4.3. The reference frame x , y , z has the origin at O . The direction of an arbitrary axis Δ through O is defined by the unit vector \mathbf{u}_Δ

$$\mathbf{u}_\Delta = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}$$

where $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the direction cosines.

The moment of inertia about the Δ axis, for a differential mass element dm of the body is by definition

$$I_{\Delta} = \int_D d^2 dm,$$

where d is the perpendicular distance from dm to Δ . The position of the mass element dm is located using the position vector \mathbf{r} and then $d = r \sin \theta$, which represents the magnitude of the cross product $\mathbf{u}_{\Delta} \times \mathbf{r}$.

The moment of inertia can be expressed as

$$I_{\Delta} = \int_D |\mathbf{u}_{\Delta} \times \mathbf{r}|^2 dm = \int_D (\mathbf{u}_{\Delta} \times \mathbf{r}) \cdot (\mathbf{u}_{\Delta} \times \mathbf{r}) dm.$$

If the position vector is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then

$$\mathbf{u}_{\Delta} \times \mathbf{r} = (z \cos \beta - y \cos \gamma)\mathbf{i} + (x \cos \gamma - z \cos \alpha)\mathbf{j} + (y \cos \alpha - x \cos \beta)\mathbf{k}$$

After substituting and performing the dot-product operation, one can write the moment of inertia as

$$\begin{aligned} I_{\Delta} &= \int_D \left[(z \cos \beta - y \cos \gamma)^2 + (x \cos \gamma - z \cos \alpha)^2 + (y \cos \alpha - x \cos \beta)^2 \right] dm \\ &= \cos^2 \alpha \int_D (y^2 + z^2) dm + \cos^2 \beta \int_D (z^2 + x^2) dm + \cos^2 \gamma \int_D (x^2 + y^2) dm \\ &\quad - 2 \cos \alpha \cos \beta \int_D xy dm - 2 \cos \beta \cos \gamma \int_D yz dm - 2 \cos \gamma \cos \alpha \int_D zx dm. \end{aligned}$$

The moment of inertia with respect to the Δ axis as

$$\begin{aligned} I_{\Delta} &= I_{xx} \cos^2 \alpha + I_{yy} \cos^2 \beta + I_{zz} \cos^2 \gamma \\ &\quad - 2I_{xy} \cos \alpha \cos \beta - 2I_{yz} \cos \beta \cos \gamma - 2I_{zx} \cos \gamma \cos \alpha. \end{aligned} \quad (4.8)$$

4.3 Translation of Coordinate Axes

The defining equations for the moments and products of inertia, as given by Eqs. (4.6) and (4.7), do not require that the origin of the cartesian coordinate system be taken at the mass center. Next one can calculate the moments and products of inertia for a given body with respect to a set of parallel axes that do not pass through the mass center. Consider the body shown in Fig. 4.4. The mass center is located at the origin $O' \equiv C$ of the primed system $x'y'z'$. The coordinate of O' with respect to the unprimed system xyz is (x_c, y_c, z_c) . An infinitesimal volume element dV is located at (x, y, z) in the unprimed system and at (x', y', z') in the primed system. These coordinates are related by the equations

$$x = x' + x_c,$$

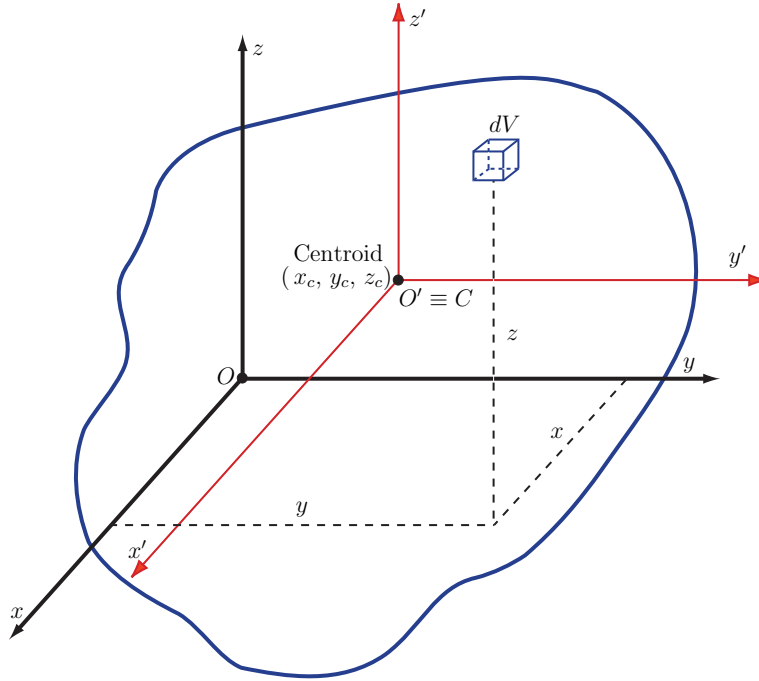


Fig. 4.4 Rigid body and centroidal axes $x'y'z'$: $x = x' + x_c$, $y = y' + y_c$, $z = z' + z_c$

$$\begin{aligned} y &= y' + y_c, \\ z &= z' + z_c. \end{aligned} \quad (4.9)$$

The moment of inertia about the x axis can be written in terms of primed coordinates by using Eqs. (4.6) and (4.9)

$$\begin{aligned} I_{xx} &= \int_V \rho \left[(y' + y_c)^2 + (z' + z_c)^2 \right] dV \\ &= I_{Cx'x'} + 2y_c \int_V \rho y' dV + 2z_c \int_V \rho z' dV + m(y_c^2 + z_c^2), \end{aligned} \quad (4.10)$$

where m is the total mass of the rigid body, and the origin of the primed coordinate system was chosen at the mass center. One can write

$$\int_V \rho x' dV = \int_V \rho y' dV = \int_V \rho z' dV = 0, \quad (4.11)$$

and therefore the two integrals on the right hand side of Eq. (4.10) are zero. In a similar way one can obtain I_{yy} and I_{zz} . The results are summarized as follows

$$\begin{aligned} I_{xx} &= I_{C x' x'} + m (y_c^2 + z_c^2), \\ I_{yy} &= I_{C y' y'} + m (x_c^2 + z_c^2), \\ I_{zz} &= I_{C z' z'} + m (x_c^2 + y_c^2), \end{aligned} \quad (4.12)$$

or, in general,

$$I_{kk} = I_{C k' k'} + m d^2, \quad (4.13)$$

where d is the distance between a given unprimed axis and a parallel primed axis passing through the mass center C . Equation (4.13) represents the *parallel axis theorem*.

The products of inertia are obtained in a similar manner, using Eqs. (4.7) and (4.9)

$$\begin{aligned} I_{xy} &= \int_V \rho (x' + x_c) (y' + y_c) dV \\ &= I_{C x' y'} + x_c \int_V \rho y' dV + y_c \int_V \rho x' dV + m x_c y_c. \end{aligned}$$

The two integrals on the previous equation are zero. The other products of inertia can be calculated in a similar manner and the results written as follows

$$\begin{aligned} I_{xy} &= I_{C x' y'} + m x_c y_c, \\ I_{xz} &= I_{C x' z'} + m x_c z_c, \\ I_{yz} &= I_{C y' z'} + m y_c z_c. \end{aligned} \quad (4.14)$$

From Eqs. (4.12) and (4.14), result that a translation of axes away from the mass center results in an increase in the moments of inertia. The products of inertia may increase or decrease, depending upon the particular case.

4.4 Principal Axes

Next the changes in the moments and product of inertia of a rigid body due to a rotation of coordinate axes are considered, as shown in Fig. 4.5. The origin of the coordinate axes is located at the fixed point O . In general the origin O is not the mass center C of the rigid body. From the definitions of the moments of inertia given in Eq. (4.6) it results that the moments of inertia cannot be negative. Furthermore,

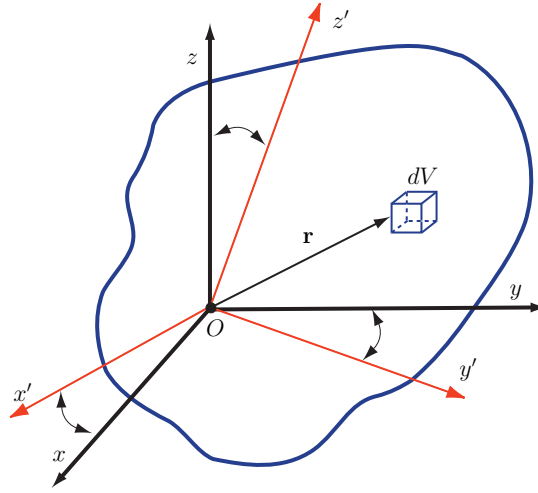


Fig. 4.5 Rotation of coordinate axes

$$I_{xx} + I_{yy} + I_{zz} = 2 \int_V \rho r^2 dV, \quad (4.15)$$

where r is the square of the distance from the origin O ,

$$r^2 = x^2 + y^2 + z^2.$$

The distance r corresponding to any mass element ρdV of the rigid body does not change with a rotation of axes from xyz to $x'y'z'$ (Fig. 4.5). Therefore the sum of the moments of inertia is invariant with respect to a coordinate system rotation. In terms of matrix notation, the sum of the moments of inertia is just the sum of the elements on the principal diagonal of the inertia matrix and is known as the trace of that matrix. So the trace of the inertia matrix is unchanged by a coordinate rotation, because the trace of any square matrix is invariant under an orthogonal transformation.

Next the products of inertia are considered. A coordinate rotation of axes can result in a change in the signs of the products of inertia. A 180° rotation about the x axis, for example, reverses the signs of I_{xy} and I_{xz} , while the sign of I_{yz} is unchanged. This occurs because the directions of the positive y and z axes are reversed. On the other hand, a 90° rotation about the x axis reverses the sign of I_{yz} . It can be seen that the moments and products of inertia vary smoothly with changes in the orientation of the coordinate system because the direction cosines vary smoothly. Therefore an orientation can always be found for which a given product of inertia is zero. It is always possible to find an orientation of the coordinate system relative to a given rigid body such that all products of inertia are zero simultaneously, that is, the inertia matrix is *diagonal*. The three mutually orthogonal coordinate axes are known as *principal axes* in this case, and the corresponding moments of inertia are

the *principal moments of inertia*. The three planes formed by the principal axes are called *principal planes*.

If I is a principal moment of inertia, then I satisfies the cubic characteristic equation

$$\begin{vmatrix} I_{xx} - I & -I_{xy} & -I_{xz} \\ -I_{yz} & I_{yy} - I & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} - I \end{vmatrix} = 0. \quad (4.16)$$

Equation (4.16) is used to determine the associated principal moments of inertia.

Suppose that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are mutually perpendicular unit vectors each parallel to a principal axis of the rigid body relative to O . The principal moments of inertia associated to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ for the rigid body relative to O are I_1, I_2 , and I_3 . The inertia matrix, in this case, is

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}.$$

When the point O under consideration is the mass center of the rigid body one speaks of *central principal moments of inertia*.

4.5 Ellipsoid of Inertia

The ellipsoid of inertia for a given body and reference point is a plot of the moment of inertia of the body for all possible axis orientations through the reference point. This graph in space has the form of an ellipsoid surface. Consider a rigid body in rotational motion about an axis Δ . The *ellipsoid of inertia* with respect to an arbitrary point O is the geometrical locus of the points Q , where Q is the extremity of the vector \overrightarrow{OQ} with the module $|\overrightarrow{OQ}| = \frac{1}{\sqrt{I_\Delta}}$, and where I_Δ is the moment of inertia about the instantaneous axis of rotation Δ , as shown in Fig. 4.6. The segment OQ is calculated with

$$|\overrightarrow{OQ}| = \frac{1}{\sqrt{I_\Delta}} = \frac{1}{k_0 m},$$

where k_0 is the radius of gyration of the body about the given axis and m is the total mass. For a cartesian system of axes the equation of the ellipsoid surface centered at O is

$$I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 + 2I_{xy}xy + 2I_{xz}xz + 2I_{yz}yz = 1. \quad (4.17)$$

The radius of gyration of a given rigid body depends upon the location of the axis relative to the body, and is not depended upon the position of the body in space. The ellipsoid of inertia is fixed in the body and rotates with it. The x_1, y_1 , and z_1 axes are assumed to be the principal axes of the ellipsoid, as shown in Fig. 4.6. For the principal axes, the equation of the inertia ellipsoid, Eq. (4.17), takes the following simple form

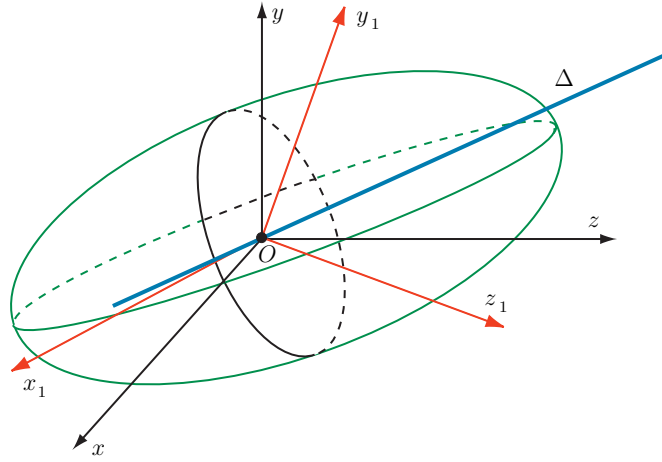


Fig. 4.6 Ellipsoid of inertia

$$I_1 x_1^2 + I_2 y_1^2 + I_3 z_1^2 = 1. \quad (4.18)$$

The previous equation is of the same form as Eq. (4.17) for the case where the x_1 , y_1 , and z_1 axes are the principal axes of the rigid body and all products of inertia vanish. Therefore I_1 , I_2 , and I_3 are the principal moments of inertia of the rigid body and, furthermore, the principal axes of the body coincide with those of the ellipsoid of inertia. From Eq. (4.18) the lengths of the principal semiaxes of the ellipsoid of inertia are

$$\begin{aligned} l_1 &= \frac{1}{\sqrt{I_1}}, \\ l_2 &= \frac{1}{\sqrt{I_2}}, \\ l_3 &= \frac{1}{\sqrt{I_3}}. \end{aligned}$$

From the parallel-axis theorem, Eq. (4.13), one can remark that the minimum moment of inertia about the mass center is also the smallest possible moment of inertia for the given body with respect to any reference point.

If the point O is the same as the mass center ($O \equiv C$), the ellipsoid is named *principal ellipsoid of inertia*.

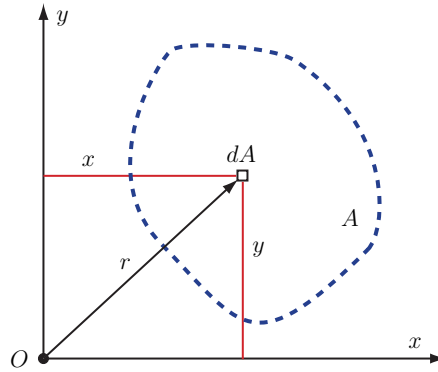


Fig. 4.7 Moments of inertia for area A about x and y axes

4.6 Moments of Inertia for Areas

The moment of inertia (*second moment*) of the area A about x and y axes, see Fig. 4.7, denoted as I_{xx} and I_{yy} , respectively, are

$$I_{xx} = \int_A y^2 dA, \quad (4.19)$$

$$I_{yy} = \int_A x^2 dA. \quad (4.20)$$

The second moment of area cannot be negative.

The entire area may be concentrated at a single point (k_x, k_y) to give the same second moment of area for a given reference. The distances k_x and k_y are called the radii of gyration. Thus,

$$A k_x^2 = I_{xx} = \int_A y^2 dA \implies k_x^2 = \frac{\int_A y^2 dA}{A} = \frac{I_{xx}}{A},$$

$$A k_y^2 = I_{yy} = \int_A x^2 dA \implies k_y^2 = \frac{\int_A x^2 dA}{A} = \frac{I_{yy}}{A}. \quad (4.21)$$

This point (k_x, k_y) depends on the shape of the area and on the position of the reference. The centroid location is independent of the reference position.

The *product of inertia* for an area A is defined as

$$I_{xy} = \int_A xy dA. \quad (4.22)$$

This quantity may be positive or negative and relates an area directly to a set of axes.

If the area under consideration has an axis of symmetry, the product of area for this axis is zero. Consider the area in Fig. 4.8, which is symmetrical about the ver-

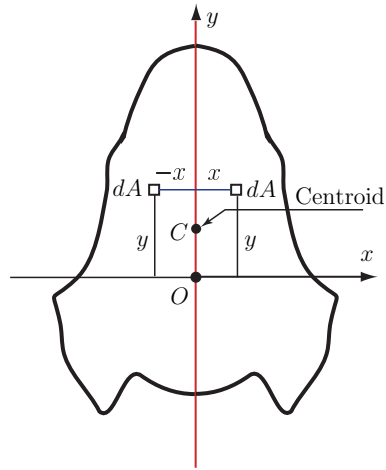


Fig. 4.8 Area, A , with an axis of symmetry Oy

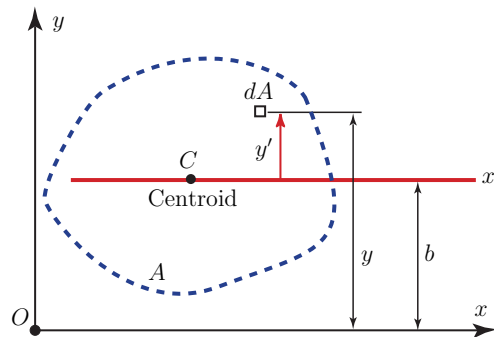


Fig. 4.9 Area and centroidal axis $Cx'x' \parallel x$

tical axis y . The planar cartesian frame is xOy . The centroid is located somewhere along the symmetrical axis y . Two differential elements of area that are positioned as mirror images about the y axis are shown in Fig. 4.8. The contribution to the product of area of each elemental area is $xy \, dA$, but with opposite signs, and so the result is zero. The entire area is composed of such elemental area pairs, and the product of area is zero. The product of inertia for an area I_{xy} is zero ($I_{xy} = 0$) if either the x or y axis is an axis of symmetry for the area.

Transfer theorem or parallel-axis theorem

The x axis in Fig. 4.9 is parallel to an axis x' and it is at a distance b from the axis x' . The axis x' is going through the centroid C of the A area, and it is a centroidal axis. The second moment of area about the x axis is

$$I_{xx} = \int_A y^2 \, dA = \int_A (y' + b)^2 \, dA,$$

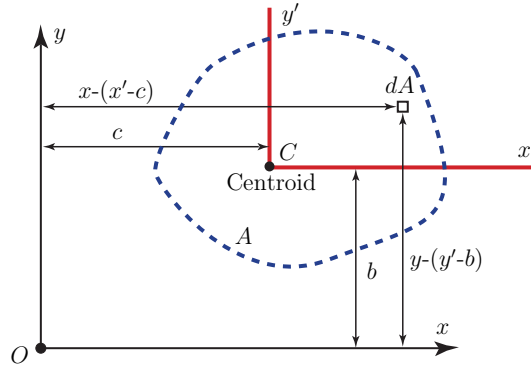


Fig. 4.10 Centroidal axes $x'y'$ parallel to reference axes xy : $Cx'x' ||_{xx}$ and $Cy'y' ||_{yy}$

where the distance $y = y' + b$. Carrying out the operations

$$I_{xx} = \int_A y'^2 dA + 2b \int_A y' dA + Ab^2.$$

The first term of the right-hand side is by definition $I_{x'y'}$,

$$I_{Cx'x'} = \int_A y'^2 dA.$$

The second term involves the first moment of area about the x' axis, and it is zero because the x' axis is a centroidal axis

$$\int_A y' dA = 0.$$

The second moment of the area A about any axis I_{xx} is equal to the second moment of the area A about a parallel axis at centroid $I_{Cx'x'}$ plus Ab^2 , where b is the perpendicular distance between the axis for which the second moment is being computed and the parallel centroidal axis

$$I_{xx} = I_{Cx'x'} + Ab^2.$$

With the transfer theorem, the second moments or products of area about any axis can be computed in terms of the second moments or products of area about a parallel set of axes going through the centroid of the area in question.

In handbooks the areas and second moments about various centroidal axes are listed for many of the practical configurations, and using the parallel-axis theorem second moments can be calculated for axes not at the centroid.

In Fig. 4.10 are shown two references, one $x'y'$ at the centroid C and the other xy arbitrary but positioned parallel relative to $x'y'$. The coordinates of the centroid $C(x_C, y_C)$ of area A measured from the reference x, y are c and b , $x_C = c$, $y_C = b$. The centroid coordinates must have the proper signs. The product of area about the

noncentroidal axes xy is

$$I_{xy} = \int_A xy \, dA = \int_A (x' + c)(y' + b) \, dA,$$

or

$$I_{xy} = \int_A x' y' \, dA + c \int_A y' \, dA + b \int_A x' \, dA + A b c.$$

The first term of the right-hand side is by definition $I_{x'y'}$

$$I_{x'y'} = \int_A x' y' \, dA.$$

The next two terms of the right-hand side are zero since x' and y' are centroidal axes

$$\int_A y' \, dA = 0 \text{ and } \int_A x' \, dA = 0.$$

Thus, the parallel-axis theorem for products of area is as follows.

The product of area for any set of axes I_{xy} is equal to the product of area for a parallel set of axes at centroid $I_{C x' y'}$ plus $A c b$, where c and b are the coordinates of the centroid of area A ,

$$I_{xy} = I_{C x' y'} + A c b.$$

With the transfer theorem, the second moments or products of area can be found about any axis in terms of second moments or products of area about a parallel set of axes going through the centroid of the area.

Polar Moment of Area

In Fig. 4.7, there is a reference xy associated with the origin O . Summing I_{xx} and I_{yy} ,

$$\begin{aligned} I_{xx} + I_{yy} &= \int_A y^2 \, dA + \int_A x^2 \, dA \\ &= \int_A (x^2 + y^2) \, dA = \int_A r^2 \, dA, \end{aligned}$$

where $r^2 = x^2 + y^2$. The distance r^2 is independent of the orientation of the reference, and the sum $I_{xx} + I_{yy}$ is independent of the orientation of the coordinate system. Therefore, the sum of second moments of area about orthogonal axes is a function only of the position of the origin O for the axes.

The polar moment of area about the origin O is

$$I_O = I_{xx} + I_{yy}. \quad (4.23)$$

The polar moment of area is an *invariant* of the system. The group of terms $I_{xx} I_{yy} - I_{xy}^2$ is also invariant under a rotation of axes. The polar radius of gyration is

$$k_O = \sqrt{\frac{I_O}{A}}. \quad (4.24)$$

Principal Axes

In Fig. 4.11, an area A is shown with a reference xy having its origin at O . Another reference $x'y'$ with the same origin O is rotated with an angle α from xy (counterclockwise as positive). The relations between the coordinates of the area elements dA for the two references are

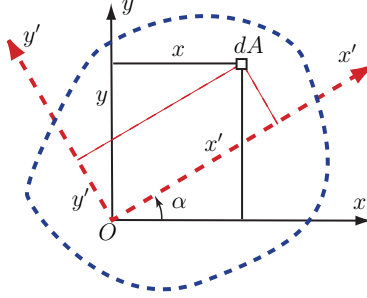


Fig. 4.11 Reference xy and reference $x'y'$ rotated with an angle α

$$\begin{aligned} x' &= x \cos \alpha + y \sin \alpha, \\ y' &= -x \sin \alpha + y \cos \alpha. \end{aligned}$$

The second moment $I_{x'x'}$ can be expressed as

$$\begin{aligned} I_{x'x'} &= \int_A (y')^2 dA = \int_A (-x \sin \alpha + y \cos \alpha)^2 dA \\ &= \sin^2 \alpha \int_A x^2 dA - 2 \sin \alpha \cos \alpha \int_A xy dA + \cos^2 \alpha \int_A y^2 dA \\ &= I_{yy} \sin^2 \alpha + I_{xx} \cos^2 \alpha - 2I_{xy} \sin \alpha \cos \alpha. \end{aligned} \quad (4.25)$$

Using the trigonometric identities

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}, \quad \sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}, \quad \sin 2\alpha = \frac{\sin \alpha \cos \alpha}{2},$$

Eq. (4.25) becomes

$$I_{x'x'} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2} \cos 2\alpha - I_{xy} \sin 2\alpha. \quad (4.26)$$

Replacing α with $\alpha + \pi/2$ in Eq. (4.26) and using the trigonometric relations

$$\cos(2\alpha + \pi) = -\cos 2\alpha, \quad \sin(2\alpha + \pi) = -\sin 2\alpha,$$

the second moment $I_{y'y'}$ is

$$I_{y'y'} = \frac{I_{xx} + I_{yy}}{2} - \frac{I_{xx} - I_{yy}}{2} \cos 2\alpha + I_{xy} \sin 2\alpha. \quad (4.27)$$

The product of area $I_{x'y'}$ is computed in a similar manner

$$I_{x'y'} = \int_A x'y' dA = \frac{I_{xx} - I_{yy}}{2} \sin 2\alpha + I_{xy} \cos 2\alpha. \quad (4.28)$$

If I_{xx} , I_{yy} , and I_{xy} are known for a reference xy with an origin O then the second moments and products of area for every set of axes at O can be computed. Next,

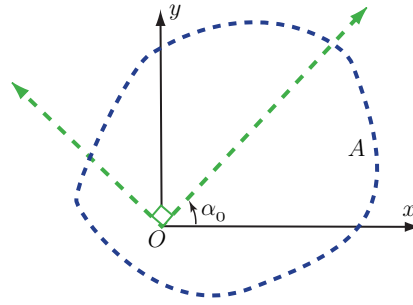


Fig. 4.12 Principal axis of area

it is assumed that I_{xx} , I_{yy} , and I_{xy} are known for a reference xy . The sum of the second moments of area is constant for any reference with origin at O . The *minimum* second moment of area corresponds to an axis at *right angles* to the axis having the *maximum* second moment, as shown in Fig. 4.12. This particular set of axes is called *principal axis* of area and the corresponding moments of inertia with respect to these axes are called *principal moments of inertia*.

The second moments of area can be expressed as functions of the angle variable α . The maximum second moment may be determined by setting the partial derivative of $I_{x'x'}$ with respect to α equal to zero. Thus

$$\frac{\partial I_{x'x'}}{\partial \alpha} = (I_{xx} - I_{yy})(-\sin 2\alpha) - 2I_{xy} \cos 2\alpha = 0, \quad (4.29)$$

or

$$(I_{yy} - I_{xx}) \sin 2\alpha_0 - 2I_{xy} \cos 2\alpha_0 = 0,$$

where α_0 is the value of α which defines the orientation of principal axes. Hence,

$$\tan 2\alpha_0 = \frac{2I_{xy}}{I_{yy} - I_{xx}}. \quad (4.30)$$

The angle α_0 corresponds to an extreme value of $I_{x'y'}$ (i.e., to a maximum or minimum value). There are two roots for $2\alpha_0$, which are π radians apart, that will satisfy the previous equation. Thus,

$$2\alpha_{0_1} = \tan^{-1} \frac{2I_{xy}}{I_{yy} - I_{xx}} \implies \alpha_{0_1} = \frac{1}{2} \tan^{-1} \frac{2I_{xy}}{I_{yy} - I_{xx}},$$

and

$$2\alpha_{0_2} = \tan^{-1} \frac{2I_{xy}}{I_{yy} - I_{xx}} + \pi \implies \alpha_{0_2} = \frac{1}{2} \tan^{-1} \frac{2I_{xy}}{I_{yy} - I_{xx}} + \frac{\pi}{2}.$$

This means that there are two axes orthogonal to each other having extreme values for the second moment of area at O . One of the axes is the maximum second moment of area, and the minimum second moment of area is on the other axis. These axes are the principal axes.

With $\alpha = \alpha_0$, the product of area $I_{x'y'}$ becomes

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2} \sin 2\alpha_0 + I_{xy} \cos 2\alpha_0. \quad (4.31)$$

For $\alpha_0 = \alpha_{0_1}$ the sine and cosine expressions are

$$\sin 2\alpha_{0_1} = \frac{2I_{xy}}{\sqrt{(I_{yy} - I_{xx})^2 + 4I_{xy}^2}}, \quad \cos 2\alpha_{0_1} = \frac{-(I_{xx} - I_{yy})}{\sqrt{(I_{yy} - I_{xx})^2 + 4I_{xy}^2}}.$$

For $\alpha_0 = \alpha_{0_2}$ the sine and cosine expressions are

$$\sin 2\alpha_{0_2} = \frac{-2I_{xy}}{\sqrt{(I_{yy} - I_{xx})^2 + 4I_{xy}^2}}, \quad \cos 2\alpha_{0_2} = \frac{I_{xx} - I_{yy}}{\sqrt{(I_{yy} - I_{xx})^2 + 4I_{xy}^2}}.$$

Equation (4.31) and $\alpha_0 = \alpha_{0_1}$ give

$$I_{x'y'} = -(I_{yy} - I_{xx}) \frac{I_{xy}}{[(I_{yy} - I_{xx})^2 + 4I_{xy}^2]^{1/2}} + I_{xy} \frac{I_{yy} - I_{xx}}{[(I_{yy} - I_{xx})^2 + 4I_{xy}^2]^{1/2}} = 0.$$

In a similar way Eq. (4.31) and $\alpha_0 = \alpha_{0_2}$ give $I_{x'y'} = 0$. The product of area corresponding to the principal axes is zero.

The maximum or minimum moment of inertia for the area are

$$I_{1,2} = I_{max,min} = \frac{I_{xx} + I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}. \quad (4.32)$$

If I is a principal moment of inertia, then I satisfies the quadratic characteristic equation

$$\begin{vmatrix} I_{xx} - I & I_{xy} \\ I_{yz} & I_{yy} - I \end{vmatrix} = 0. \quad (4.33)$$

4.7 Examples

Example 4.1

A rectangular planar plate with the sides $b=1$ m and $h=2$ m is shown in Fig. 4.13.

a) Find the product of inertia and the moments of inertia with respect to the axes of the reference frame xy with the origin at O .

b) Determine the product of inertia and the moments of inertia with respect to the centroidal axes that are located at the mass center C of the rectangle and are parallel to its sides.

c) Another reference uv with the same origin O is rotated with an angle $\alpha=45^\circ$ from xy (counterclockwise as positive). Find the inertia matrix of the plate with respect to uv axes.

d) Find the principal moments and the principal directions with the reference frame xy with the origin at O .

Solution

a) The differential element of area is $dA = dxdy$. The product of inertia of the rectangle about the xy axes is

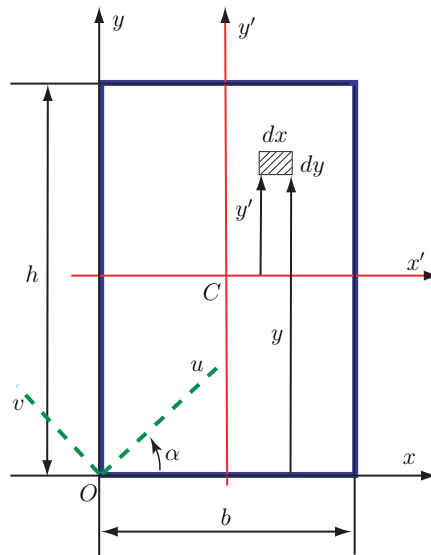


Fig. 4.13 Example 4.1

$$I_{xy} = \int_A xy dA = \int_0^h \int_0^b xy dx dy = \int_0^b x dx \int_0^h y dy = \frac{b^2}{2} \frac{h^2}{2} = \frac{b^2 h^2}{4} = \frac{1^2 (2^2)}{4} = 1 \text{ m}^4.$$

The moment of inertia of the rectangle about x axis is

$$I_{xx} = \int_A y^2 dA = \int_0^h \int_0^b y^2 dx dy = \int_0^b dx \int_0^h y^2 dy = b \frac{h^3}{3} = \frac{bh^3}{3} = \frac{(1)2^3}{3} = 2.666 \text{ m}^4.$$

The moment of inertia of the rectangle about y axis is

$$I_{yy} = \int_A x^2 dA = \int_0^h \int_0^b x^2 dx dy = \int_0^b x^2 dx \int_0^h dy = \frac{b^3}{3} h = \frac{hb^3}{3} = \frac{(3)1^3}{3} = 0.666 \text{ m}^4.$$

The moment of inertia of the rectangle about z axis (the polar moment about O) is

$$I_O = I_{zz} = I_{xx} + I_{yy} = \frac{A}{3}(b^2 + h^2) = \frac{1(2)}{3}(1^2 + 2^2) = 3.33 \text{ m}^4.$$

The inertia matrix of the plane figure with respect to xy axes is represented by

$$[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} \frac{bh^3}{3} & \frac{b^2h^2}{4} & 0 \\ \frac{b^2h^2}{4} & \frac{hb^3}{3} & 0 \\ 0 & 0 & \frac{A}{3}(b^2 + h^2) \end{bmatrix} = \begin{bmatrix} 2.666 & 1 & 0 \\ 1 & 0.666 & 0 \\ 0 & 0 & 3.33 \end{bmatrix}.$$

b) The product of inertia of the rectangle about the $x'y'$ axes is

$$I_{x'y'} = \int_A xy dA = \int_{-h/2}^{h/2} \int_{-b/2}^{b/2} xy dx dy = \int_{-h/2}^{h/2} x dx \int_{-b/2}^{b/2} y dy = 0.$$

The same results is obtained using the parallel axis-theorem

$$I_{xy} = I_{x'y'} + \frac{b}{2} \frac{h}{2} A,$$

or

$$I_{x'y'} = I_{xy} - \frac{bh}{4}(bh) = \frac{b^2h^2}{4} - \frac{b^2h^2}{4} = 0.$$

The moment of inertia of the rectangle about x' axis is

$$\begin{aligned} I_{x'x'} &= \int_A y^2 dA = \int_{-h/2}^{h/2} \int_{-b/2}^{b/2} y^2 dx dy = \int_{-h/2}^{h/2} dx \int_{-b/2}^{b/2} y^2 dy \\ &= \{x\}_{-h/2}^{h/2} \left\{ \frac{y^3}{3} \right\}_{-b/2}^{b/2} = \frac{bh^3}{12} = \frac{(1)2^3}{12} = 0.666 \text{ m}^4. \end{aligned}$$

Using the parallel axis-theorem the moment of inertia of the rectangle about y' axis is

$$I_{y'y'} = I_{yy} - \left(\frac{b}{2}\right)^2 A = \frac{hb^3}{3} - \frac{hb^3}{4} = \frac{hb^3}{12} = \frac{(2)1^3}{12} = 0.166 \text{ m}^4.$$

The moment of inertia of the rectangle about z' axis (the centroid polar moment) is

$$I_C = I_{z'z'} = I_{x'x'} + I_{y'y'} = \frac{A}{12}(b^2 + h^2) = \frac{1(2)}{12}(1^2 + 2^2) = 0.833 \text{ m}^4.$$

The inertia matrix of the plane figure with respect to centroidal axes $x'y'$ is represented by

$$[I_C] = \begin{bmatrix} I_{x'x'} & -I_{x'y'} & -I_{x'z'} \\ -I_{y'x'} & I_{y'y'} & -I_{y'z'} \\ -I_{z'x'} & -I_{z'y'} & I_{z'z'} \end{bmatrix} = \begin{bmatrix} \frac{bh^3}{12} & 0 & 0 \\ 0 & \frac{hb^3}{12} & 0 \\ 0 & 0 & \frac{A}{12}(b^2 + h^2) \end{bmatrix} = \begin{bmatrix} 0.666 & 0 & 0 \\ 0 & 1.666 & 0 \\ 0 & 0 & 0.833 \end{bmatrix}.$$

c) The moment of inertia of the rectangle about u axis is

$$I_{uu} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2} \cos 2\alpha - I_{xy} \sin 2\alpha = \frac{2.666 + 0.666}{2} + \frac{2.666 - 0.666}{2} \cos 2(45^\circ) - (1) \sin 2(45^\circ) = 0.666 \text{ m}^4.$$

The moment of inertia of the rectangle about v axis is

$$I_{vv} = \frac{I_{xx} + I_{yy}}{2} - \frac{I_{xx} - I_{yy}}{2} \cos 2\alpha + I_{xy} \sin 2\alpha = \frac{2.666 + 0.666}{2} - \frac{2.666 - 0.666}{2} \cos 2(45^\circ) + (1) \sin 2(45^\circ) = 2.666 \text{ m}^4.$$

The product of inertia of the rectangle about uv axis is

$$I_{uv} = \frac{I_{xx} - I_{yy}}{2} \sin 2\alpha + I_{xy} \cos 2\alpha = \frac{2.666 - 0.666}{2} \sin 2(45^\circ) + (1) \cos 2(45^\circ) = 1 \text{ m}^4.$$

The polar moment of inertia of the rectangle about O is

$$I_O = I_{zz} = I_{uu} + I_{vv} = I_{xx} + I_{yy} = 0.666 + 2.666 = 3.33 \text{ m}^4.$$

The inertia matrix of the plane figure with respect to uv axes is

$$[I_\alpha] = \begin{bmatrix} I_{uu} & -I_{uv} & 0 \\ -I_{vu} & I_{vv} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} = \begin{bmatrix} 0.666 & 1 & 0 \\ 1 & 2.666 & 0 \\ 0 & 0 & 3.33 \end{bmatrix}.$$

d) The maximum or minimum moment of inertia for the area are

$$I_{1,2} = I_{max,min} = \frac{I_{xx} + I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2},$$

$$I_1 = I_{max} = \frac{I_{xx} + I_{yy}}{2} + \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2} =$$

$$\frac{2.666 + 0.666}{2} + \sqrt{\left(\frac{2.666 - 0.666}{2}\right)^2 + 1^2} = 3.080 \text{ m}^4,$$

$$I_2 = I_{min} = \frac{I_{xx} + I_{yy}}{2} - \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2} =$$

$$\frac{2.666 + 0.666}{2} - \sqrt{\left(\frac{2.666 - 0.666}{2}\right)^2 + 1^2} = 0.252 \text{ m}^4.$$

The polar moment of inertia of the rectangle about O is

$$I_O = I_{zz} = I_1 + I_2 = I_{uu} + I_{vv} = I_{xx} + I_{yy} = 3.0806 + 0.252 = 3.33 \text{ m}^4.$$

The principal directions are obtained from

$$\tan 2\alpha_0 = \frac{2I_{xy}}{I_{yy} - I_{xx}},$$

or

$$\alpha_0 = \frac{1}{2} \tan^{-1} \frac{2I_{xy}}{I_{yy} - I_{xx}} = \frac{1}{2} \tan^{-1} \frac{2(1)}{0.666 - 2.666} = -22.5^\circ.$$

The principal directions are

$$\alpha_1 = -22.5^\circ \text{ and } \alpha_2 = \alpha_1 + \pi/2 = 67.5^\circ.$$

Example 4.2

Determine the moment of inertia for the slender rod, shown in Fig. 4.14, with respect to axes of reference with the origin at the end O and with respect to centroidal axes. The length of the rod is l , the density is ρ , and the cross-sectional area is A . Express the results in terms of the total mass, m , of the rod.

Solution

The mass of the rod is $m = \rho l A$ and the density will be $\rho = m/(lA)$. The differential element of mass is $dm = \rho A dx$. The moment of inertia of the slender rod about the y or z axes is

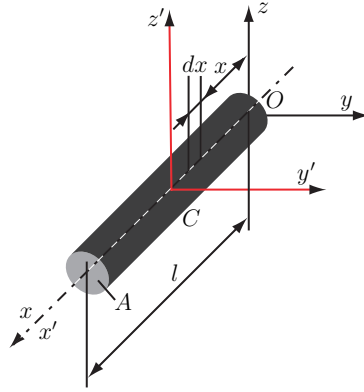


Fig. 4.14 Example 4.2

$$I_{yy} = I_{zz} = \int_0^l x^2 \rho A dx = \int_0^l \frac{m}{lA} A x^2 dx = \frac{m}{l} \int_0^l x^2 dx = \frac{ml^2}{3}.$$

The x axis is a symmetry axis and that is why $I_{xx} = 0$. The moment of inertia of the slender rod about the centroidal axes y' or z' is calculated with the parallel-axis theorem

$$I_{y'y'} = I_{z'z'} = I_{yy} - \left(\frac{l}{2}\right)^2 m = \frac{ml^2}{3} - \frac{ml^2}{4} = \frac{ml^2}{12}.$$

Example 4.3

Find the polar moment of inertia of the planar flywheel shown in Fig. 4.15(a). The radii of the wheel are R_1 and R_2 ($R_1 < R_2$). Calculate the moments of inertia of the area of a circle with radius R about a diametral axis and about the polar axis through the center as shown in Fig. 4.15(b).

Solution

The polar moment of inertia is given by the equation

$$I_O = \int_A r^2 dA,$$

where r is the distance from the pole O to an arbitrary point on the wheel and the differential element of area is

$$dA = r d\alpha dr.$$

The polar moment of inertia is

$$I_O = \int_{R_1}^{R_2} \int_0^{2\pi} r^3 dr d\alpha = \int_{R_1}^{R_2} r^3 dr \int_0^{2\pi} d\alpha = \frac{R_2^4 - R_1^4}{4} (2\pi) = \frac{R_2^4 - R_1^4}{2} \pi.$$

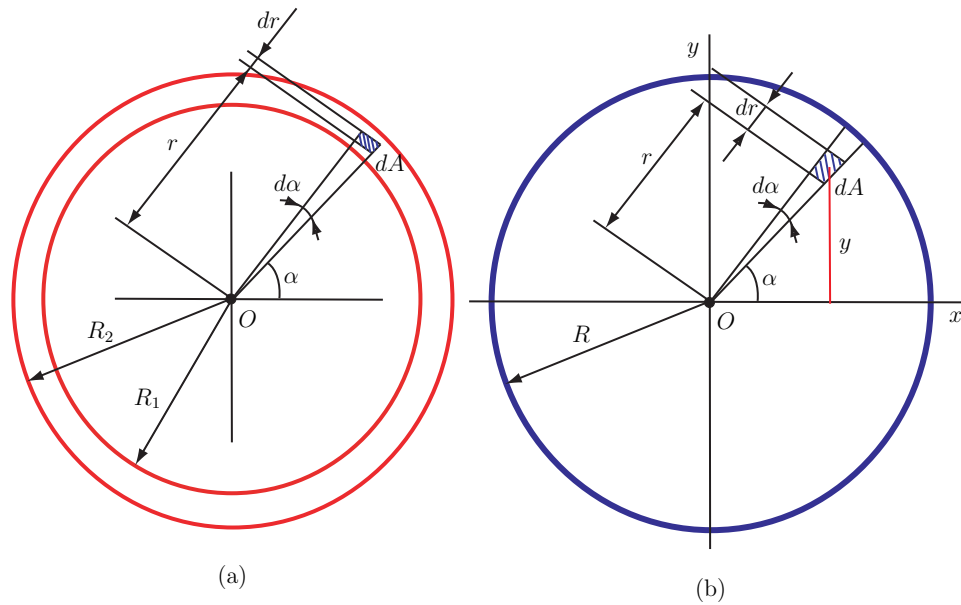


Fig. 4.15 Example 4.3

The area of the wheel is $A = \pi(R_2^2 - R_1^2)$ and

$$I_O = A \frac{R_2^2 + R_1^2}{2}. \quad (4.34)$$

If $R_1 = 0$ and $R_2 = R$ the polar moment of inertia of the circular area of radius R is, Fig. 4.15(b)

$$I_O = \frac{AR^2}{2} = \frac{\pi R^4}{2}.$$

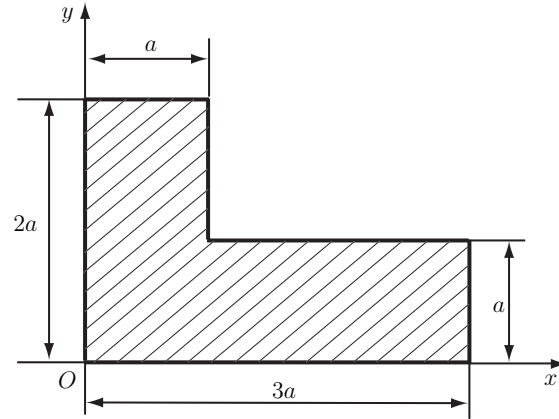
By symmetry for the circular area, shown in Fig. 4.15(b), the moment of inertia about a diametral axis is $I_{xx} = I_{yy}$ and $I_O = I_{xx} + I_{yy} \implies I_{xx} = I_{yy} = I_O/2 = \pi R^4/4$. The results can be obtained using the integration

$$\begin{aligned} I_{xx} &= \int_A y^2 dA = \int_0^R \int_0^{2\pi} (r \sin \alpha)^2 r dr d\alpha = \int_0^{2\pi} \frac{R^4}{4} (\sin \alpha)^2 d\alpha \\ &= \frac{R^4}{4} \frac{1}{2} \left\{ \alpha - \frac{\sin 2\alpha}{2} \right\}_0^{2\pi} = \frac{\pi R^4}{4} \end{aligned}$$

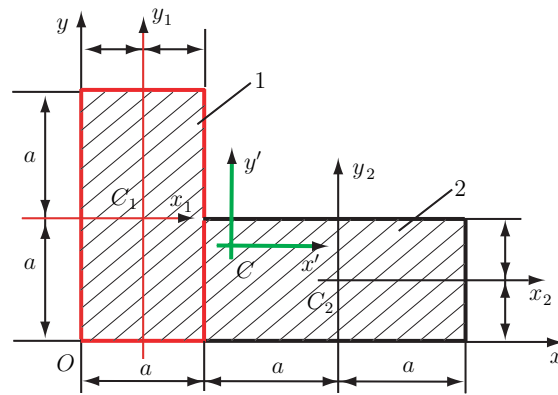
Example 4.4

Find the moments of inertia and products of inertia for the area shown in Fig. 4.16(a), with respect to the xy axes and with respect to the centroidal $x'y'$ axes

that pass through the mass center C . Find the principal moments of inertia for the area and the principal directions.



(a)



(b)

Fig. 4.16 Example 4.4

Solution

The plate is composed of two element area: the rectangular area 1 and the rectangular area 2, Fig. 4.16(b). The x and y coordinates of the mass center C are

$$x_C = \frac{x_{C_1} A_1 + x_{C_2} A_2}{A_1 + A_2} = \frac{(a/2)(2a^2) + (2a)(2a^2)}{2a^2 + 2a^2} = \frac{5a}{4}$$

$$y_C = \frac{y_{C_1} A_1 + y_{C_2} A_2}{A_1 + A_2} = \frac{(a)(2a^2) + (a/2)(2a^2)}{2a^2 + 2a^2} = \frac{3a}{4}$$

The product of inertia for the area shown in Fig. 4.16(a) about xy axes is given by

$$\begin{aligned} I_{xy} &= \int_0^a \int_0^{2a} xy \, dx \, dy + \int_a^{3a} \int_0^a xy \, dx \, dy = \int_0^a x \, dx \int_0^{2a} y \, dy + \int_a^{3a} x \, dx \int_0^a y \, dy \\ &= \frac{a^2 (4a^2)}{4} + \frac{(9a^2 - a^2) (a^2)}{4} = 3a^4. \end{aligned}$$

The same result is obtained if parallel-axis theorem is used

$$\begin{aligned} I_{xy} &= I_{C_1x_1y_1} + (-x_{C_1})(-y_{C_1})A_1 + I_{C_2x_2y_2} + (-x_{C_2})(-y_{C_2})A_2 \\ &= 0 + \left(-\frac{a}{2}\right)(-a)(2a^2) + 0 + (-2a)\left(-\frac{a}{2}\right)(2a^2) = 3a^4. \end{aligned}$$

The product of inertia for the area about xz and yz axes are $I_{xz} = I_{yz} = 0$.
The moment of inertia of the figure with respect to x axis is

$$\begin{aligned} I_{xx} &= I_{C_1x_1x_1} + (y_{C_1})^2 A_1 + I_{C_2x_2x_2} + (y_{C_2})^2 A_2 \\ &= \frac{a(2a)^3}{12} + a^2(2a^2) + \frac{2a(a)^3}{12} + \left(\frac{a}{2}\right)^2 (2a^2) = \frac{10a^4}{3}. \end{aligned}$$

The moment of inertia of the figure with respect to y axis is

$$\begin{aligned} I_{yy} &= I_{C_1y_1y_1} + (x_{C_1})^2 A_1 + I_{C_2y_2y_2} + (x_{C_2})^2 A_2 \\ &= \frac{(2a)a^3}{12} + \left(\frac{a}{2}\right)^2 (2a^2) + \frac{a(2a)^3}{12} + (2a)^2 (2a^2) = \frac{28a^4}{3}. \end{aligned}$$

The moment of inertia of the area with respect to z axis is

$$I_{zz} = I_{xx} + I_{yy} = \frac{10a^4}{3} + \frac{28a^4}{3} = \frac{38a^4}{3}.$$

The inertia matrix of the plane figure is represented by the matrix

$$[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} \frac{10a^4}{3} & -3a^4 & 0 \\ -3a^4 & \frac{28a^4}{3} & 0 \\ 0 & 0 & \frac{38a^4}{3} \end{bmatrix}.$$

Using the parallel-axes theorem the moments of inertia of the area with respect to the centroidal axes $x'y'z'$ are

$$\begin{aligned} I_{x'y'} &= I_{xx} - (x_C)^2 A = \frac{10a^4}{3} - \left(\frac{3a}{4}\right)^2 (4a^2) = \frac{13a^4}{12}, \\ I_{y'z'} &= I_{yy} - (y_C)^2 A = \frac{28a^4}{3} - \left(\frac{5a}{4}\right)^2 (4a^2) = \frac{37a^4}{12}, \end{aligned}$$

$$I_{z'z'} = I_{x'x'} + I_{y'y'} = \frac{13a^4}{12} + \frac{37a^4}{12} = \frac{25a^4}{6},$$

$$I_{x'y'} = I_{xy} - (-x_C)(-y_C)A = 3a^4 - \left(-\frac{3a}{4}\right)\left(-\frac{5a}{4}\right)(4a^2) = -\frac{3a^4}{4},$$

$$I_{x'z'} = I_{y'z'} = 0.$$

The centroidal inertia matrix of the plane figure is

$$[I'] = \begin{bmatrix} I_{x'x'} & -I_{x'y'} & -I_{x'z'} \\ -I_{y'x'} & I_{y'y'} & -I_{y'z'} \\ -I_{z'x'} & -I_{z'y'} & I_{z'z'} \end{bmatrix} = \begin{bmatrix} \frac{13a^4}{12} & \frac{3a^4}{4} & 0 \\ \frac{3a^4}{4} & \frac{37a^4}{12} & 0 \\ 0 & 0 & \frac{25a^4}{6} \end{bmatrix}.$$

The principal moments of inertia for the area are

$$I_{1,2} = I_{max,min} = \frac{I_{x'x'} + I_{y'y'}}{2} \pm \sqrt{\left(\frac{I_{x'x'} - I_{y'y'}}{2}\right)^2 + I_{x'y'}^2} =$$

$$= \frac{25a^4}{12} \pm \frac{a^4}{2} \sqrt{4 + 9/4} = \frac{25a^4}{12} \pm \frac{5a^4}{4}.$$

$$I_1 = \frac{10a^4}{3} \quad \text{and} \quad I_2 = \frac{5a^4}{6}.$$

The invariant of the system is: $I_{x'x'} + I_{y'y'} = I_1 + I_2$. The principal directions are obtained from

$$\tan 2\alpha = \frac{2I_{y'y'}}{I_{y'y'} - I_{x'x'}} = \frac{-2\frac{3a^4}{4}}{\frac{24a^4}{12} - \frac{13a^4}{12}} = -\frac{3}{4} \implies$$

$$\alpha_1 = \frac{1}{2} \tan^{-1}(-3/4) = -36.869^\circ \quad \text{and} \quad \alpha_2 = \frac{\pi}{2} + \frac{1}{2} \tan^{-1}(-3/4) = 53.131^\circ.$$

Example 4.5

Find the inertia matrix of the area delimited by the curve $y^2 = 2px$, from $x = 0$ to $x = a$ as shown in Fig 4.17(a), about the axes of the cartesian frame with the origin at O . Calculate the centroidal inertia matrix.

Solution

From Fig 4.17(a) when $x = a$ the value of y coordinate is $y = b$ and $b^2 = 2pa \implies 2p = b^2/a$. The expression of the function is

$$y^2 = 2px = \frac{b^2}{a}x$$

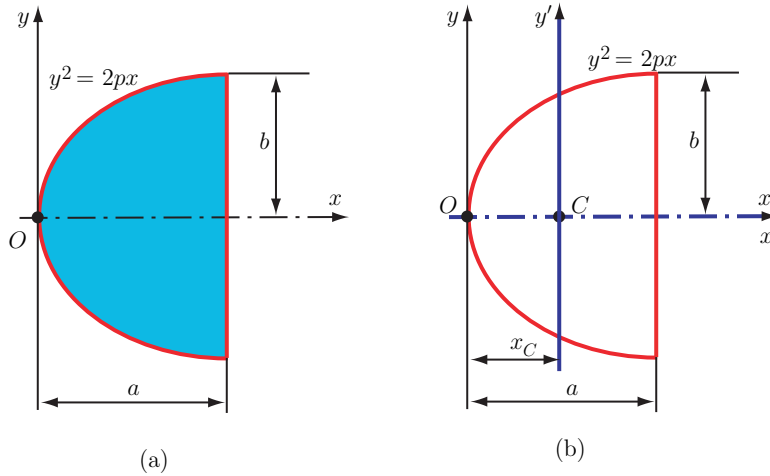


Fig. 4.17 Example 4.5

The differential element of area is $dA = dx dy$ and the area of the figure is

$$A = \int_A dx dy = \int_0^a \int_{-\sqrt{2px}}^{\sqrt{2px}} dx dy = \int_0^a dx \int_{-\sqrt{2px}}^{\sqrt{2px}} dy =$$

$$\int_0^a dx \{y\}_{-\sqrt{2px}}^{\sqrt{2px}} = \int_0^a 2\sqrt{2px} dx = 2\sqrt{2p} \int_0^a x^{1/2} dx = \frac{2b}{\sqrt{a}} \left\{ \frac{x^{3/2}}{3/2} \right\}_0^a = \frac{4ab}{3}.$$

The moment of inertia of the area with respect to x axis is

$$I_{xx} = \int_A y^2 dx dy = \int_0^a \int_{-\sqrt{2px}}^{\sqrt{2px}} y^2 dx dy = \int_0^a dx \int_{-\sqrt{2px}}^{\sqrt{2px}} y^2 dy =$$

$$\int_0^a dx \left\{ \frac{y^3}{3} \right\}_{-\sqrt{2px}}^{\sqrt{2px}} = \frac{2}{3} \int_0^a (2px)^{3/2} dx =$$

$$\frac{2}{3} (2p)^{3/2} \int_0^a x^{3/2} dx = \frac{2}{3} (2p)^{3/2} \left\{ \frac{x^{5/2}}{5/2} \right\}_0^a = \frac{4ab^3}{15} = \frac{4ab}{3} \frac{b^2}{5},$$

or

$$I_{xx} = \frac{b^2 A}{5}.$$

The moment of inertia of the area with respect to y axis is

$$I_{yy} = \int_A x^2 dx dy = \int_0^a \int_{-\sqrt{2px}}^{\sqrt{2px}} x^2 dx dy = \int_0^a x^2 dx \int_{-\sqrt{2px}}^{\sqrt{2px}} dy =$$

$$\int_0^a x^2 dx \{y\}_{-\sqrt{2px}}^{\sqrt{2px}} = 2 \int_0^a x^2 (2px)^{1/2} dx =$$

$$\frac{2b}{\sqrt{a}} \int_0^a x^{5/2} dx = \frac{2b}{\sqrt{a}} \left\{ \frac{x^{7/2}}{7/2} \right\}_0^a = \frac{4a^3 b}{7} = \frac{4ab}{3} \frac{3a^2}{7},$$

or

$$I_{yy} = \frac{3a^2 A}{7}.$$

The moment of inertia of the area with respect to z axis is

$$I_{zz} = I_{xx} + I_{yy} = \frac{b^2 A}{5} + \frac{3a^2 A}{7} = A \left(\frac{b^2}{5} + \frac{3a^2}{7} \right).$$

The product of inertia of the area with respect to xy axes is

$$\begin{aligned} I_{xy} &= \int_A xy dx dy = \int_0^a \int_{-\sqrt{2px}}^{\sqrt{2px}} xy dx dy = \int_0^a x dx \int_{-\sqrt{2px}}^{\sqrt{2px}} y dy = \\ &= \int_0^a x dx \left\{ \frac{y^2}{2} \right\}_{-\sqrt{2px}}^{\sqrt{2px}} = 0. \end{aligned}$$

The products of inertia of the area with respect to xz and yz axes are $I_{xz} = I_{yz} = 0$.
The inertia matrix of the plane figure is

$$[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} \frac{b^2 A}{5} & 0 & 0 \\ 0 & \frac{3a^2 A}{7} & 0 \\ 0 & 0 & A \left(\frac{b^2}{5} + \frac{3a^2}{7} \right) \end{bmatrix}.$$

The first moment of the area A with respect to y axis is

$$\begin{aligned} M_y &= \int_A x dx dy = \int_0^a \int_{-\sqrt{2px}}^{\sqrt{2px}} x dx dy = \int_0^a x dx \int_{-\sqrt{2px}}^{\sqrt{2px}} dy = \\ &= \int_0^a x dx \{y\}_{-\sqrt{2px}}^{\sqrt{2px}} = 2 \int_0^a x \sqrt{2px} dx = \\ &= \frac{2b}{\sqrt{a}} \int_0^a x^{3/2} dx = \frac{2b}{\sqrt{a}} \left\{ \frac{x^{5/2}}{5/2} \right\}_0^a = \frac{2b}{\sqrt{a}} \frac{a^{5/2}}{5/2} = \frac{4ba^2}{5} \end{aligned}$$

The x coordinate of the mass center, Fig 4.17(b), is

$$x_C = \frac{M_y}{A} = \frac{4ba^2}{5} \frac{3}{4ab} = \frac{3a}{5}.$$

The first moment of the area A with respect to x axis is $M_x = 0$ and $y_C = M_x/A = 0$.

Using the parallel-axes theorem the moments of inertia of the area with respect to the centroidal axes $x'y'z'$, Fig 4.17(b), are

$$I_{x'x'} = I_{xx} - d^2 A = I_{xx} = \frac{b^2 A}{5},$$

$$I_{y'y'} = I_{yy} - (x_C)^2 A = I_{yy} - \left(\frac{3a}{5}\right)^2 A = \frac{3a^2 A}{7} - \frac{9a^2 A}{25} = \frac{12a^2 A}{175}$$

$$I_{z'z'} = I_{x'x'} + I_{y'y'} = \frac{b^2 A}{5} + \frac{12a^2 A}{175} = A \left(\frac{b^2}{5} + \frac{a^2}{175} \right),$$

$$I_{x'y'} = I_{xy} - (-x_C)(0)A = 0,$$

$$I_{x'z'} = I_{y'z'} = 0.$$

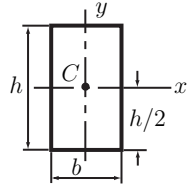
The centroidal inertia matrix of the plane figure is

$$[I'] = \begin{bmatrix} I_{x'x'} & -I_{x'y'} & -I_{x'z'} \\ -I_{y'x'} & I_{y'y'} & -I_{y'z'} \\ -I_{z'x'} & -I_{z'y'} & I_{z'z'} \end{bmatrix} = \begin{bmatrix} \frac{b^2 A}{5} & 0 & 0 \\ 0 & \frac{12a^2 A}{175} & 0 \\ 0 & 0 & A \left(\frac{b^2}{5} + \frac{a^2}{175} \right) \end{bmatrix}.$$

A = area, in^2 (m^2) C = location of the centroid

I_{xx}, I_{yy} = second moment of area about x, y axis, respectively, in^4 (m^4)

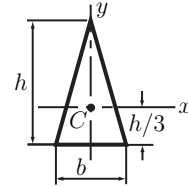
I_C = second polar moment of area about axis through C , in^4 (m^4)



$$A = bh$$

$$I_{xx} = \frac{bh^3}{12} \quad I_C = \frac{bh}{12}(b^2 + h^2)$$

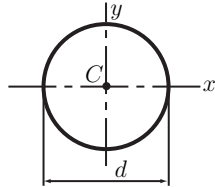
$$I_{yy} = \frac{b^3h}{12}$$



$$A = \frac{bh}{2}$$

$$I_{xx} = \frac{bh^3}{36} \quad I_C = \frac{bh}{36}(b^2 + h^2)$$

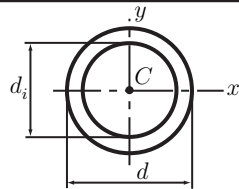
$$I_{yy} = \frac{b^3h}{36}$$



$$A = \frac{\pi d^2}{4}$$

$$I_{xx} = I_{yy} = \frac{\pi d^4}{64}$$

$$I_C = \frac{\pi d^4}{32}$$

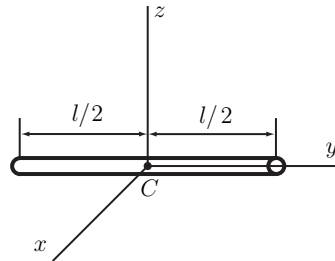


$$A = \frac{\pi}{4}(d^2 - d_i^2)$$

$$I_{xx} = I_{yy} = \frac{\pi}{64}(d^4 - d_i^4)$$

$$I_C = \frac{\pi}{32}(d^4 - d_i^4)$$

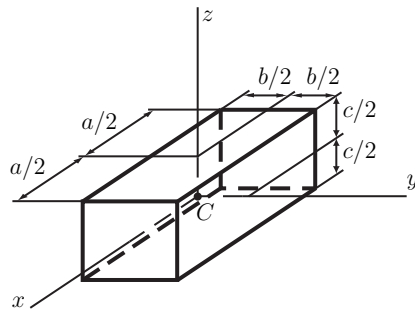
Inertia properties of some homogeneous bodies

 A = cross-sectional area ρ = mass density m = mass I_{xx}, I_{yy}, I_{zz} = moments of inertia C = location of the centroidwith respect to x, y, z axis

$$m = \rho l A$$

$$I_{xx} = I_{zz} = \frac{m}{12} l^2$$

$$I_{yy} = 0$$

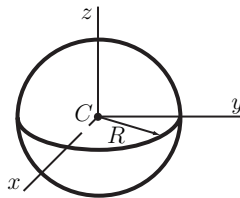


$$m = \rho abc$$

$$I_{xx} = \frac{1}{12} m (b^2 + c^2)$$

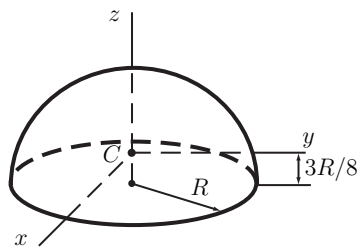
$$I_{yy} = \frac{1}{12} m (a^2 + c^2)$$

$$I_{zz} = \frac{1}{12} m (a^2 + b^2)$$



$$m = \frac{4}{3} \pi \rho R^3$$

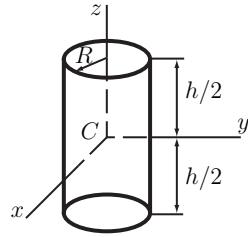
$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} m R^2$$



$$m = \frac{2}{3} \pi \rho R^3$$

$$I_{xx} = I_{yy} = \frac{83}{320} m R^2$$

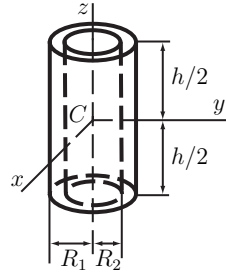
$$I_{zz} = \frac{2}{5} m R^2$$



$$m = \pi \rho R^2 h$$

$$I_{xx} = I_{yy} = \frac{1}{12} m (3R^2 + h^2)$$

$$I_{zz} = \frac{1}{2} m R^2$$



$$m = \pi \rho h (R_1^2 - R_2^2)$$

$$I_{xx} = I_{yy} = \frac{1}{12} m (3R_1^2 + 3R_2^2 + h^2)$$

$$I_{zz} = \frac{1}{2} m (R_1^2 + R_2^2)$$