

Chapter 3

Centroids

3.1 First Moment and Centroid of a Set of Points

The position vector of a point P relative to a point O is \mathbf{r}_P and a scalar associated with P is s , e.g., the mass m of a particle situated at P . The *first moment* of a point P with respect to a point O is the vector $\mathbf{M} = s\mathbf{r}_P$. The scalar s is called the *strength* of P .

The set of n points $P_i, i = 1, 2, \dots, n$, is $\{S\}$, Fig. 3.1(a)

$$\{S\} = \{P_1, P_2, \dots, P_n\} = \{P_i\}_{i=1,2,\dots,n}.$$

The strengths of the points P_i are $s_i, i = 1, 2, \dots, n$, i.e., n scalars, all having the same dimensions, and each associated with one of the points of $\{S\}$.

The *centroid* of the set $\{S\}$ is the point C with respect to which the sum of the first moments of the points of $\{S\}$ is equal to zero.

The position vector of C relative to an arbitrarily selected reference point O is \mathbf{r}_C , Fig. 3.1(b). The position vector of P_i relative to O is \mathbf{r}_i . The position vector of P_i relative to C is $\mathbf{r}_i - \mathbf{r}_C$. The sum of the first moments of the points P_i with respect to C is $\sum_{i=1}^n s_i(\mathbf{r}_i - \mathbf{r}_C)$. If C is to be centroid of $\{S\}$, this sum is equal to zero

$$\sum_{i=1}^n s_i(\mathbf{r}_i - \mathbf{r}_C) = \sum_{i=1}^n s_i\mathbf{r}_i - \mathbf{r}_C \sum_{i=1}^n s_i = 0.$$

The position vector \mathbf{r}_C of the centroid C , relative to an arbitrarily selected reference point O , is given by

$$\mathbf{r}_C = \frac{\sum_{i=1}^n s_i\mathbf{r}_i}{\sum_{i=1}^n s_i}.$$

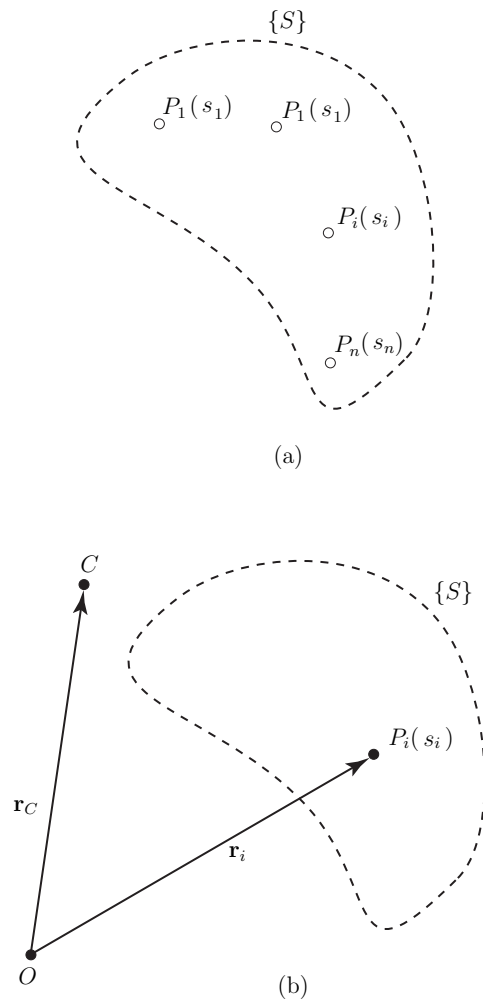


Fig. 3.1 (a) Set of points and (b) centroid of a set of points

If $\sum_{i=1}^n s_i = 0$ the centroid is not defined.

The centroid C of a set of points of given strength is a unique point, its location being independent of the choice of reference point O .

The cartesian coordinates of the centroid $C(x_C, y_C, z_C)$ of a set of points P_i , $i = 1, \dots, n$, of strengths s_i , $i = 1, \dots, n$, are given by the expressions

$$x_C = \frac{\sum_{i=1}^n s_i x_i}{\sum_{i=1}^n s_i}, \quad y_C = \frac{\sum_{i=1}^n s_i y_i}{\sum_{i=1}^n s_i}, \quad z_C = \frac{\sum_{i=1}^n s_i z_i}{\sum_{i=1}^n s_i}.$$

The *plane of symmetry* of a set is the plane where the centroid of the set lies, the points of the set being arranged in such a way that corresponding to every point on one side of the plane of symmetry there exists a point of equal strength on the other side, the two points being equidistant from the plane.

A set $\{S'\}$ of points is called a *subset* of a set $\{S\}$ if every point of $\{S'\}$ is a point of $\{S\}$. The centroid of a set $\{S\}$ may be located using the *method of decomposition*:

- divide the system $\{S\}$ into subsets;
- find the centroid of each subset;
- assign to each centroid of a subset a strength proportional to the sum of the strengths of the points of the corresponding subset;
- determine the centroid of this set of centroids.

3.2 Centroid of a Curve, Surface, or Solid

The position vector of the centroid C of a curve, surface, or solid relative to a point O is

$$\mathbf{r}_C = \frac{\int_D \mathbf{r} d\tau}{\int_D d\tau}, \quad (3.1)$$

where, D is a curve, surface, or solid, \mathbf{r} denotes the position vector of a typical point of D , relative to O , and $d\tau$ is the length, area, or volume of a differential element of D . Each of the two limits in this expression is called an “integral over the domain D (curve, surface, or solid).”

The integral $\int_D d\tau$ gives the total length, area, or volume of D , that is

$$\int_D d\tau = \tau.$$

The position vector of the centroid is

$$\mathbf{r}_C = \frac{1}{\tau} \int_D \mathbf{r} d\tau.$$

Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be mutually perpendicular unit vectors (cartesian reference frame) with the origin at O . The coordinates of C are x_C, y_C, z_C and

$$\mathbf{r}_C = x_C \mathbf{i} + y_C \mathbf{j} + z_C \mathbf{k}.$$

It results that

$$x_C = \frac{1}{\tau} \int_D x d\tau, \quad y_C = \frac{1}{\tau} \int_D y d\tau, \quad z_C = \frac{1}{\tau} \int_D z d\tau. \quad (3.2)$$

For the curved line shown in Fig. 3.2 the centroidal position is

$$x_C = \frac{\int x dl}{L}, \quad y_C = \frac{\int y dl}{L}, \quad (3.3)$$

where L is the length of the line. Note that the centroid C is not generally located along the line. A curve made up of simple curves is considered. For each simple

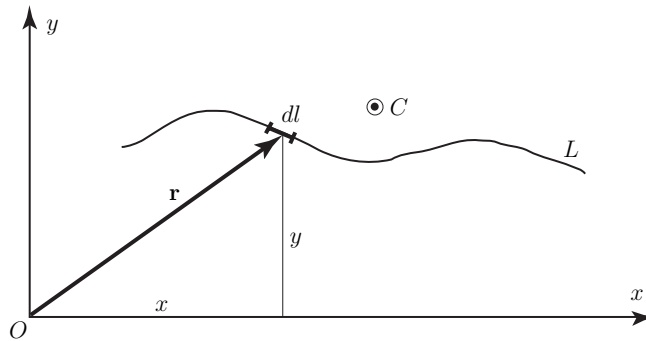


Fig. 3.2 Centroid of a planar curved line

curve the centroid is known. The line segment L_i , has the centroid C_i with coordinates $x_{C_i}, y_{C_i}, i = 1, \dots, n$. For the entire curve

$$x_C = \frac{\sum_{i=1}^n x_{C_i} L_i}{L}, \quad y_C = \frac{\sum_{i=1}^n y_{C_i} L_i}{L},$$

where $L = \sum_{i=1}^n L_i$.

3.3 Mass Center of a Set of Particles

The *mass center* of a set of particles $\{S\} = \{P_1, P_2, \dots, P_n\} = \{P_i\}_{i=1,2,\dots,n}$ is the centroid of the set of points at which the particles are situated with the strength of

each point being taken equal to the mass of the corresponding particle, $s_i = m_i$, $i = 1, 2, \dots, n$. For the system system of n particles in Fig. 3.3, one can write

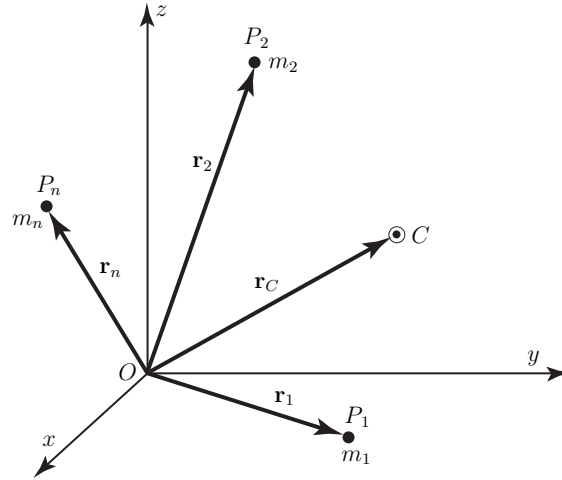


Fig. 3.3 Mass center position vector

$$\left(\sum_{i=1}^n m_i \right) \mathbf{r}_C = \sum_{i=1}^n m_i \mathbf{r}_i,$$

and the mass center position vector is

$$\mathbf{r}_C = \frac{\sum_{i=1}^n m_i \mathbf{r}_i}{M}, \quad (3.4)$$

where M is the total mass of the system.

3.4 Mass Center of a Curve, Surface, or Solid

The position vector of the mass center C of a continuous body D , curve, surface, or solid, relative to a point O is

$$\mathbf{r}_C = \frac{1}{m} \int_D \mathbf{r} \rho \, d\tau, \quad (3.5)$$

or using the orthogonal cartesian coordinates

$$x_C = \frac{1}{m} \int_D x\rho \, d\tau, \quad y_C = \frac{1}{m} \int_D y\rho \, d\tau, \quad z_C = \frac{1}{m} \int_D z\rho \, d\tau,$$

where, ρ is the mass density of the body: mass per unit of length if D is a curve, mass per unit area if D is a surface, and mass per unit of volume if D is a solid, \mathbf{r} is the position vector of a typical point of D , relative to O , $d\tau$ is the length, area, or volume of a differential element of D , $m = \int_D \rho \, d\tau$ is the total mass of the body, and x_C, y_C, z_C are the coordinates of C .

If the mass density ρ of a body is the same at all points of the body, $\rho = \text{constant}$, the density, as well as the body, are said to be *uniform*. The mass center of a uniform body coincides with the centroid of the figure occupied by the body.

The *method of decomposition* may be used to locate the mass center of a continuous body B :

- divide the body B into a number of bodies, which may be particles, curves, surfaces, or solids;
- locate the mass center of each body;
- assign to each mass center a strength proportional to the mass of the corresponding body (e.g., the weight of the body);
- locate the centroid of this set of mass centers.

3.5 First Moment of an Area

A planar surface of area A and a reference frame xOy in the plane of the surface are shown in Fig. 3.4. The first moment of area A about the x axis is

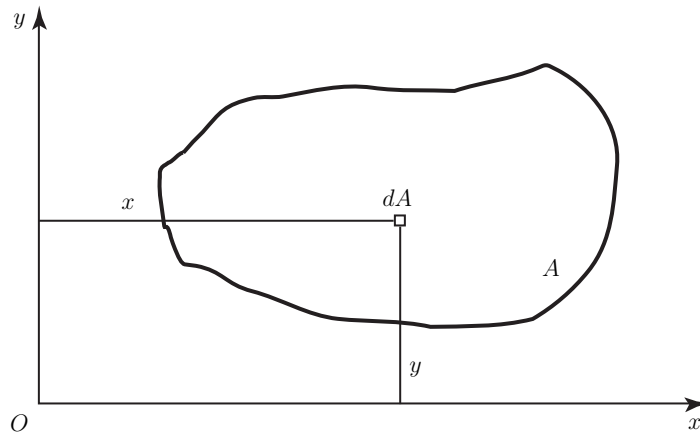


Fig. 3.4 Planar surface of area A

$$M_x = \int_A y dA, \quad (3.6)$$

and the first moment about the y axis is

$$M_y = \int_A x dA. \quad (3.7)$$

The first moment of area gives information of the shape, size, and orientation of the area.

The entire area A can be concentrated at a position $C(x_C, y_C)$, the centroid, Fig. 3.5. The coordinates x_C and y_C are the centroidal coordinates. To compute the centroidal coordinates one can equate the moments of the distributed area with that of the concentrated area about both axes

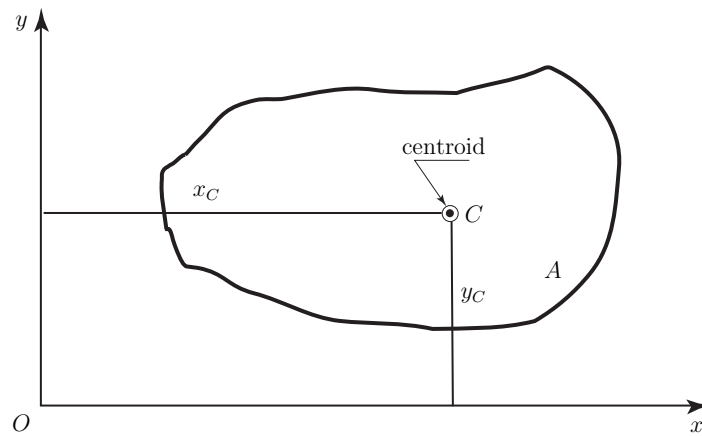


Fig. 3.5 Centroid and centroidal coordinates for a planar surface

$$A y_C = \int_A y dA, \quad \Rightarrow \quad y_C = \frac{\int_A y dA}{A} = \frac{M_x}{A}, \quad (3.8)$$

$$A x_C = \int_A x dA, \quad \Rightarrow \quad x_C = \frac{\int_A x dA}{A} = \frac{M_y}{A}. \quad (3.9)$$

The location of the centroid of an area is independent of the reference axes employed, i.e., the centroid is a property only of the area itself.

If the axes xy have their origin at the centroid, $O \equiv C$, then these axes are called *centroidal axes*. The first moments about centroidal axes are zero. All axes going through the centroid of an area are called centroidal axes for that area, and the first moments of an area about any of its centroidal axes are zero. The perpendicular distance from the centroid to the centroidal axis must be zero.

In Fig. 3.6 is shown a plane area with the axis of symmetry collinear with the axis y . The area A can be considered as composed of area elements in symmetric pairs such as shown in Fig. 3.6. The first moment of such a pair about the axis of symmetry y is zero. The entire area can be considered as composed of such symmetric pairs and the coordinate x_C is zero

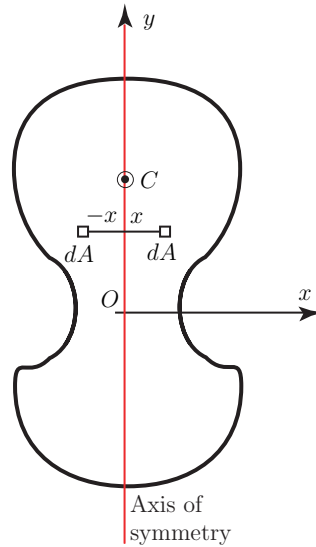


Fig. 3.6 Plane area with axis of symmetry

$$x_C = \frac{1}{A} \int_A x \, dA = 0.$$

Thus, *the centroid of an area with one axis of symmetry must lie along the axis of symmetry*. The axis of symmetry then is a centroidal axis, which is another indication that the first moment of area must be zero about the axis of symmetry. With two orthogonal axes of symmetry, the centroid must lie at the intersection of these axes. For such areas as circles and rectangles, the centroid is easily determined by inspection.

In many problems, the area of interest can be considered formed by the addition or subtraction of simple areas. For simple areas the centroids are known by inspection. The areas made up of such simple areas are *composite* areas. For composite areas

$$x_C = \frac{\sum_i A_i x_{Ci}}{A} \quad \text{and} \quad y_C = \frac{\sum_i A_i y_{Ci}}{A}, \quad (3.10)$$

where x_{Ci} and y_{Ci} (with proper signs) are the centroidal coordinates to simple area A_i , and where A is the total area.

The centroid concept can be used to determine the simplest resultant of a distributed loading. In Fig. 3.7 the distributed load $w(x)$ is considered. The resultant force F_R of the distributed load $w(x)$ loading is given as

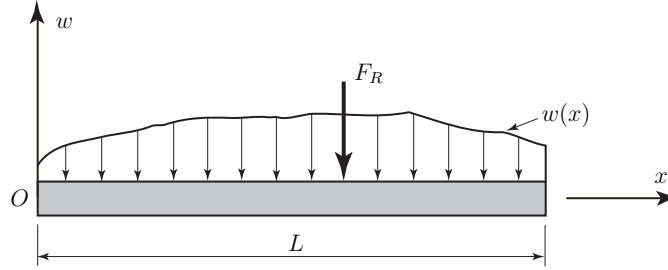


Fig. 3.7 Distributed load

$$F_R = \int_0^L w(x) dx. \quad (3.11)$$

From the equation above the *resultant force equals the area under the loading curve*. The position, \bar{x} , of the *simplest* resultant load can be calculated from the relation

$$F_R \bar{x} = \int_0^L xw(x) dx \implies \bar{x} = \frac{\int_0^L xw(x) dx}{F_R}. \quad (3.12)$$

The position \bar{x} is actually the centroidal coordinate of the loading curve area. Thus, the *simplest resultant force of a distributed load acts at the centroid of the area under the loading curve*. For the triangular distributed load shown in Fig. 3.8, one can replace the distributed loading by a force F equal to $(\frac{1}{2})(w_0)(b-a)$ at a position $\frac{1}{3}(b-a)$ from the right end of the distributed loading.

3.6 Theorems of Guldinus-Pappus

The theorems of Guldinus-Pappus are concerned with the relation of a surface of revolution to its generating curve, and the relation of a volume of revolution to its generating area.

Theorem. Consider a coplanar generating curve and an axis of revolution in the plane of this curve Fig. 3.9. The surface of revolution A developed by rotating the generating curve about the axis of revolution equals the product of the length of the

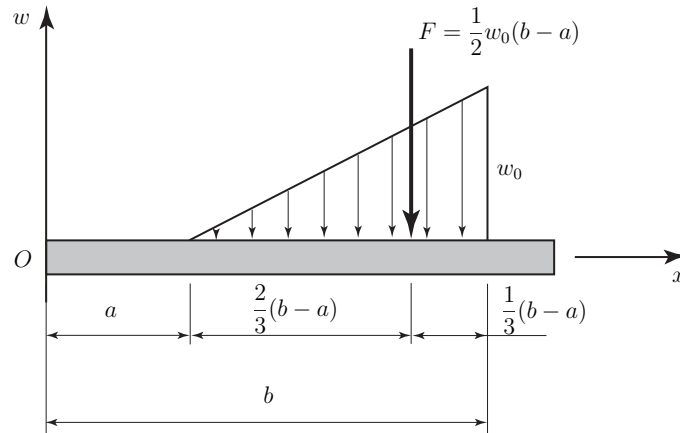


Fig. 3.8 Triangular distributed load

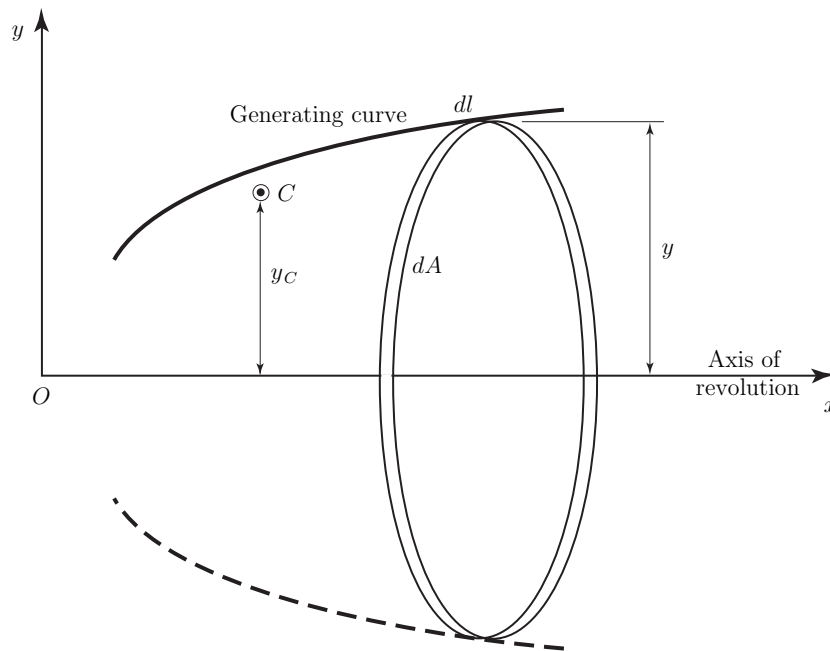


Fig. 3.9 Surface of revolution developed by rotating the generating curve about the axis of revolution

generating L curve times the circumference of the circle formed by the centroid of the generating curve y_C in the process of generating a surface of revolution

$$A = 2\pi y_C L. \tag{3.13}$$

The generating curve can touch but must not cross the axis of revolution.

Proof. An element dl of the generating curve is considered in Fig. 3.9. For a single revolution of the generating curve about the x axis, the line segment dl traces an area

$$dA = 2\pi y dl.$$

For the entire curve this area, dA , becomes the surface of revolution, A , given as

$$A = 2\pi \int y dl = 2\pi y_C L,$$

where L is the length of the curve and y_C is the centroidal coordinate of the curve. The circumferential length of the circle formed by having the centroid of the curve rotate about the x axis is $2\pi y_C$, q.e.d.

The surface of revolution A is equal to 2π times the first moment of the generating curve about the axis of revolution.

If the generating curve is composed of simple curves, L_i , whose centroids are known, Fig. 3.10, the surface of revolution developed by revolving the composed generating curve about the axis of revolution x is

$$A = 2\pi \left(\sum_{i=1}^4 L_i y_{Ci} \right), \quad (3.14)$$

where y_{Ci} is the centroidal coordinate to the i th line segment L_i .

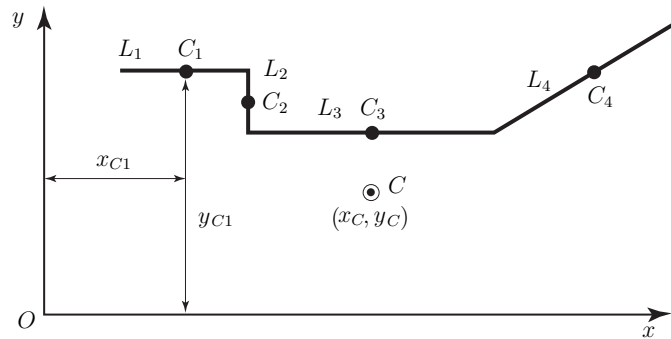


Fig. 3.10 Composed generating curve

Theorem. Consider a generating plane surface A and an axis of revolution coplanar with the surface Fig. 3.11. The volume of revolution V developed by rotating the generating plane surface about the axis of revolution equals the product of the area of the surface times the circumference of the circle formed by the centroid of the surface y_C in the process of generating the body of revolution

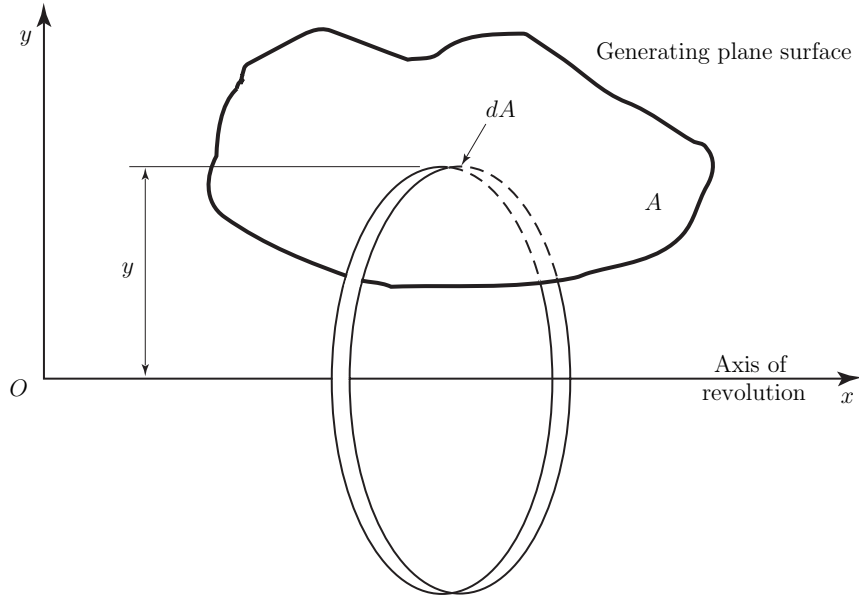


Fig. 3.11 Volume of revolution developed by rotating the generating plane surface about the axis of revolution

$$V = 2\pi y_C A. \quad (3.15)$$

The axis of revolution can intersect the generating plane surface only as a tangent at the boundary or have no intersection at all.

Proof. The plane surface A is shown in Fig. 3.11. The volume generated by rotating an element dA of this surface about the x axis is

$$dV = 2\pi y dA.$$

The volume of the body of revolution formed from A is then

$$V = 2\pi \int_A y dA = 2\pi y_C A.$$

Thus, the volume V equals the area of the generating surface A times the circumferential length of the circle of radius y_C , q.e.d.

The volume V equals 2π times the first moment of the generating area A about the axis of revolution.

3.7 Examples

Example 3.1

Find the position of the mass center for a non-homogeneous straight rod, with the length $OA = l$ (Fig. 3.12). The linear density ρ of the rod is a linear function with $\rho = \rho_0$ at O and $\rho = \rho_1$ at A .

Solution

A reference frame xOy is selected with the origin at O and the x -axis along the rectilinear rod (Fig. 3.12). Let $M(x, 0)$ be an arbitrarily given point on the rod, and let MM' be an element of the rod with the length dx and the mass $dm = \rho dx$. The density ρ is a linear function of x given by

$$\rho = \rho(x) = \rho_0 + \frac{\rho_1 - \rho_0}{l}x. \quad (3.16)$$

The center mass of the rod, C , has x_c as abscissa.

The mass center x_c , with respect to point O is

$$x_c \int_0^l \rho dx = \int_0^l x \rho dx. \quad (3.17)$$

From Eqs. (3.16) and (3.17) one can obtain

$$x_c \int_0^l \left(\rho_0 + \frac{\rho_1 - \rho_0}{l}x \right) dx = \int_0^l x \left(\rho_0 + \frac{\rho_1 - \rho_0}{l}x \right) dx. \quad (3.18)$$

From Eq. (3.18) after integration it results

$$x_c \frac{\rho_0 + \rho_1}{2} l = \frac{\rho_0 + 2\rho_1}{6} l^2.$$

From the previous equation the mass center x_c is

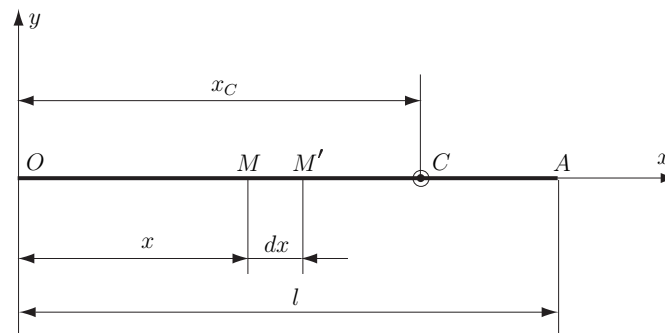


Fig. 3.12 Example 3.1

$$x_c = \frac{\rho_0 + 2\rho_1}{3(\rho_0 + \rho_1)} l.$$

In the special case of a homogeneous rod, the density and the position of the mass center are given by

$$\rho_0 = \rho_1 \quad \text{and} \quad x_c = \frac{l}{2}.$$

Example 3.2

Find the position of the centroid for a homogeneous circular arc. The radius of the arc is R and the center angle is 2α radians as shown in Fig. 3.13.

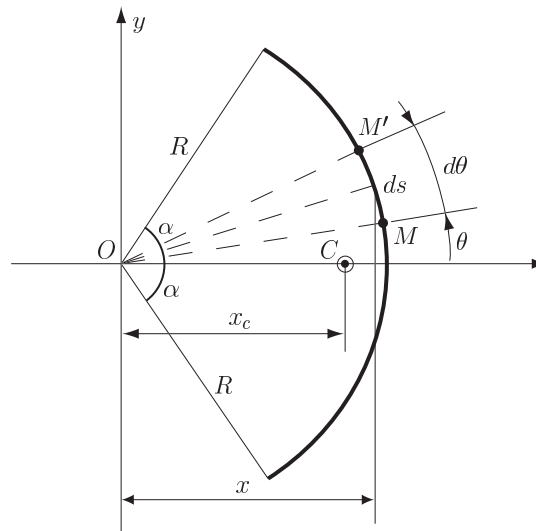


Fig. 3.13 Example 3.2

Solution

The axis of symmetry is selected as the x -axis ($y_C=0$). Let MM' be a differential element of arc with the length $ds = R d\theta$. The mass center of the differential element of arc MM' has the abscissa

$$x = R \cos \left(\theta + \frac{d\theta}{2} \right) \approx R \cos(\theta).$$

The abscissa x_C of the centroid for a homogeneous circular arc is calculated from Eq. (3.5)

$$x_c \int_{-\alpha}^{\alpha} R d\theta = \int_{-\alpha}^{\alpha} x R d\theta. \quad (3.19)$$

Because $x = R \cos(\theta)$ and with Eq. (3.19) one can write

$$x_c \int_{-\alpha}^{\alpha} R d\theta = \int_{-\alpha}^{\alpha} R^2 \cos(\theta) d\theta. \quad (3.20)$$

From Eq. (3.20) after integration it results

$$2x_c \alpha R = 2R^2 \sin(\alpha),$$

or

$$x_c = \frac{R \sin(\alpha)}{\alpha}.$$

For a semicircular arc when $\alpha = \frac{\pi}{2}$ the position of the centroid is

$$x_c = \frac{2R}{\pi},$$

and for the quarter-circular $\alpha = \frac{\pi}{4}$

$$x_c = \frac{2\sqrt{2}}{\pi} R.$$

Example 3.3

Find the position of the mass center for the area of a circular sector. The center angle is 2α radians and the radius is R as shown in Fig. 3.14.

Solution

The origin O is the vertex of the circular sector. The x -axis is chosen as the axis of symmetry and $y_C=0$. Let $MNPQ$ be a surface differential element with the area $dA = \rho d\rho d\theta$. The mass center of the surface differential element has the abscissa

$$x = \left(\rho + \frac{d\rho}{2} \right) \cos \left(\theta + \frac{d\theta}{2} \right) \approx \rho \cos(\theta). \quad (3.21)$$

Using the first moment of area formula with respect to Oy , Eq. (3.9), the mass center abscissa x_C is calculated as

$$x_C \iint \rho d\rho d\theta = \iint x \rho d\rho d\theta. \quad (3.22)$$

Equations (3.21) and (3.22) give

$$x_C \iint \rho d\rho d\theta = \iint \rho^2 \cos(\theta) d\rho d\theta,$$

or

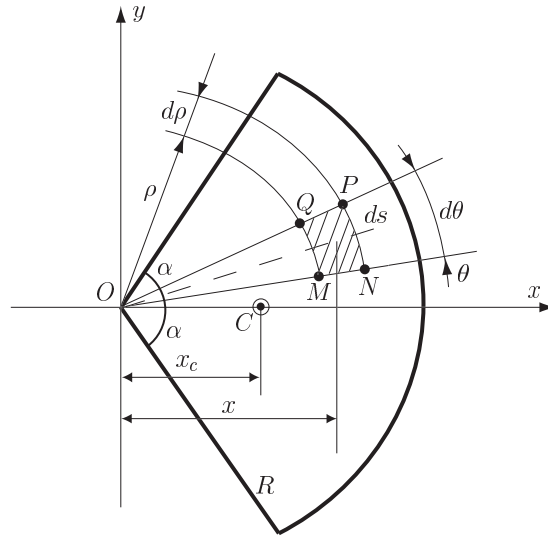


Fig. 3.14 Example 3.3

$$x_c \int_0^R \rho d\rho \int_{-\alpha}^{\alpha} d\theta = \int_0^R \rho^2 d\rho \int_{-\alpha}^{\alpha} \cos(\theta) d\theta. \quad (3.23)$$

From Eq.(3.23) after integration it results

$$x_c \left(\frac{1}{2} R^2 \right) (2\alpha) = \left(\frac{1}{3} R^3 \right) (2 \sin(\alpha)),$$

or

$$x_c = \frac{2R \sin(\alpha)}{3\alpha}. \quad (3.24)$$

For a semicircular area, $\alpha = \frac{\pi}{2}$, the x -coordinate to the centroid is

$$x_c = \frac{4R}{3\pi}.$$

For the quarter-circular area, $\alpha = \frac{\pi}{2}$, the x -coordinate to the centroid is

$$x_c = \frac{4R\sqrt{2}}{3\pi}.$$

Example 3.4

Find the position of the mass center for a homogeneous planar plate, with the shape and dimensions given in Fig. 3.15.

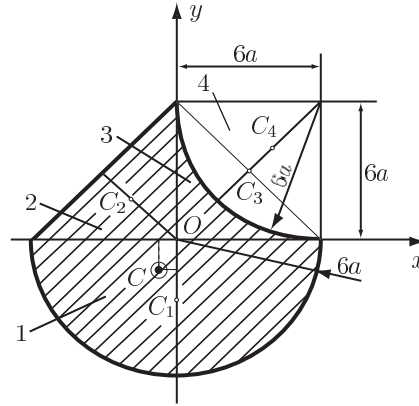


Fig. 3.15 Example 3.4

Solution

The plate is composed of four elements: the circular sector area 1, the triangle 2, the square 3, and the circular area 4 to be subtracted. Using the decomposition method, the positions of the mass center x_i , the areas A_i , and the first moments with respect to the axes of the reference frame M_{y_i} and M_{x_i} , for all four elements are calculated. The results are given in the following table

i	x_i	y_i	A_i	$M_{y_i} = x_i A_i$	$M_{x_i} = y_i A_i$
circular sector 1	0	$-\frac{8}{\pi}a$	$18\pi a^2$	0	$-144a^3$
triangle 2	$-2a$	$2a$	$18a^2$	$-36a^3$	$36a^3$
square 3	$3a$	$3a$	$36a^2$	$108a^3$	$108a^3$
circular sector 4	$6a - \frac{8a}{\pi}$	$6a - \frac{8a}{\pi}$	$-9\pi a^2$	$-54\pi a^3 + 72a^3$	$-54\pi a^3 + 72a^3$
Σ	-	-	$9(\pi + 6)a^2$	$18(8 - 3\pi)a^3$	$9(8 - 6\pi)a^3$

The x and y coordinates of the mass center C are

$$x_c = \frac{\sum x_i A_i}{\sum A_i} = \frac{2(8 - 3\pi)}{\pi + 6}a = -0.311a \quad \text{and} \quad y_c = \frac{\sum y_i A_i}{\sum A_i} = \frac{8 - 6\pi}{\pi + 6}a = -1.186a.$$

Example 3.5

Find the coordinates of the mass center for a homogeneous planar plate located under the curve of equation $y = \sin x$ from $x = 0$ to $x = a$.

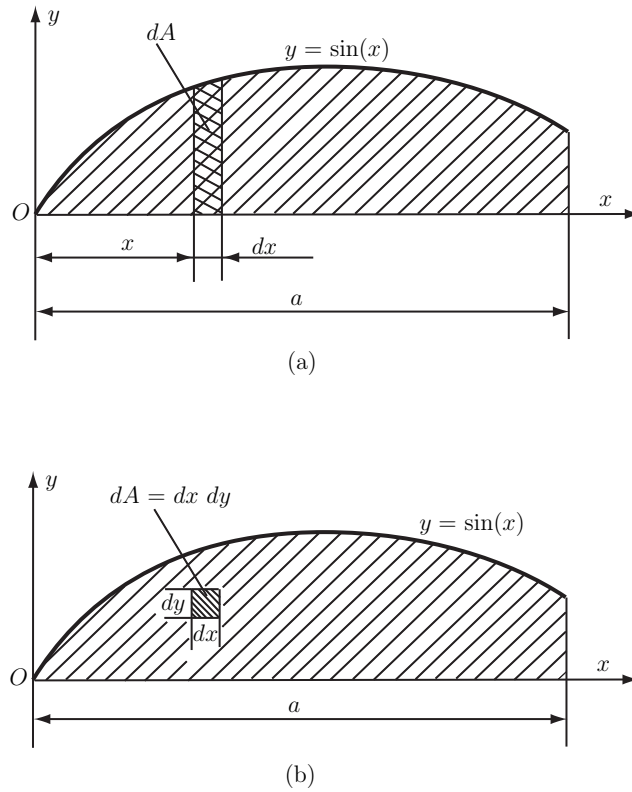


Fig. 3.16 Example 3.5

Solution

A vertical differential element of area $dA = y dx = (\sin x)(dx)$ is chosen as shown in Fig. 3.16(a). The x coordinate of the mass center is calculated from Eq. (3.9)

$$\begin{aligned}
 x_c \int_0^a (\sin x) dx &= \int_0^a x (\sin x) dx, \quad \text{or} \\
 x_c \{-\cos x\}_0^a &= \int_0^a x (\sin x) dx, \quad \text{or} \\
 x_c(1 - \cos a) &= \int_0^a x (\sin x) dx. \quad (3.25)
 \end{aligned}$$

The integral $\int_0^a x (\sin x) dx$ is calculated with

$$\begin{aligned}
 \int_0^a x (\sin x) dx &= \{x(-\cos x)\}_0^a - \int_0^a (-\cos x) dx = \{x(-\cos x)\}_0^a + \{\sin x\}_0^a \\
 &= \sin a - a \cos a. \quad (3.26)
 \end{aligned}$$

Using Eqs. (3.25) and (3.26) after integration it results

$$x_c = \frac{\sin a - a \cos a}{1 - \cos a}.$$

The x coordinate of the mass center, x_c , can be calculated using the differential element of area $dA = dx dy$, as shown in Fig. 3.16(b). The area of the figure is

$$\begin{aligned} A &= \int_A dx dy = \int_0^a \int_0^{\sin x} dx dy = \int_0^a dx \int_0^{\sin x} dy = \\ &= \int_0^a dx \{y\}_0^{\sin x} = \int_0^a (\sin x) dx = \{-\cos x\}_0^a = 1 - \cos a. \end{aligned}$$

The first moment of the area A about the y axis is

$$\begin{aligned} M_y &= \int_A x dA = \int_0^a \int_0^{\sin x} x dx dy = \int_0^a x dx \int_0^{\sin x} dy = \\ &= \int_0^a x dx \{y\}_0^{\sin x} = \int_0^a x (\sin x) dx = \sin a - a \cos a. \end{aligned}$$

The x coordinate of the mass center is $x_c = M_y/A$.

The y coordinate of the mass center is $y_c = M_x/A$, where the first moment of the area A about the x axis is

$$\begin{aligned} M_x &= \int_A y dA = \int_0^a \int_0^{\sin x} y dx dy = \int_0^a dx \int_0^{\sin x} y dy = \\ &= \int_0^a dx \left\{ \frac{y^2}{2} \right\}_0^{\sin x} = \int_0^a \frac{\sin^2 x}{2} dx = \frac{1}{2} \int_0^a \sin^2 x dx. \end{aligned}$$

The integral $\int_0^a \sin^2 x dx$ is calculated with

$$\begin{aligned} \int_0^a \sin^2 x dx &= \int_0^a \sin x d(-\cos x) = \{\sin x (-\cos x)\}_0^a + \int_0^a \cos^2 x dx = \\ &= -\sin a (\cos a) + \int_0^a (1 - \sin^2 x) dx = -\sin a (\cos a) + a - \int_0^a \sin^2 x dx, \end{aligned}$$

or

$$\int_0^a \sin^2 x dx = \frac{a - \sin a \cos a}{2}.$$

The coordinate y_c is

$$y_c = \frac{M_x}{A} = \frac{a - \sin a \cos a}{4(\sin a - a \cos a)}.$$

Example 3.6

The homogeneous plate shown in Fig. 3.17 is delimited by the hatched area. The circle, with the center at C_1 , has the unknown radius r . The circle, with the center at C_2 , has the given radius a ($r < a$). The position of the mass center of the hatched area C , is located at the intersection of the circle, with the radius r and the center at C_1 , and the positive x -axis. Find the radius r .

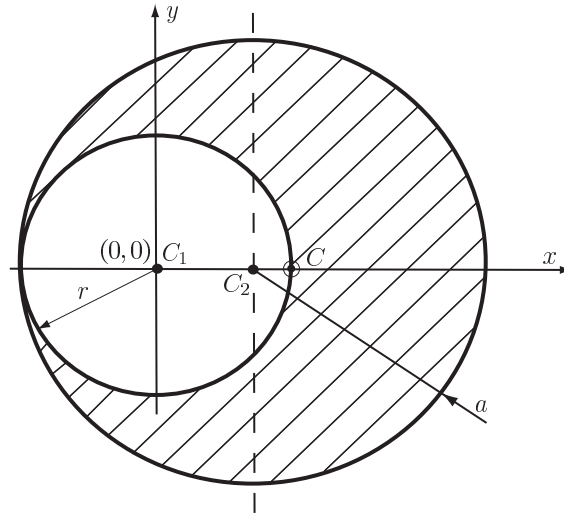


Fig. 3.17 Example 3.6

Solution

The x -axis is a symmetry axis for the planar plate and the origin of the reference frame is located at $C_1 = O(0, 0)$. The y coordinate of the mass center of the hatched area is $y_C = 0$.

The coordinates of the mass center of the circle with the center at C_2 and radius a are $C_2(a - r, 0)$. The area of the circle C_1 is $A_1 = \pi r^2$ and the area of the circle C_2 is $A_2 = \pi a^2$. The total area is given by

$$A = A_1 + A_2 = \pi (a^2 - r^2).$$

The coordinate x_C of the mass center is given by

$$x_C = \frac{x_{C_1} A_1 + x_{C_2} A_2}{A_1 + A_2} = \frac{x_{C_2} A_2}{A_1 + A_2} = \frac{x_{C_2} A_2}{A} = \frac{(a - r) \pi a^2}{\pi (a^2 - r^2)}$$

or

$$x_C (a^2 - r^2) = a^2 (a - r).$$

If $x_C = r$ the previous equation gives

$$r^2 + ar - a^2 = 0.$$

with the solutions

$$r = \frac{-a \pm a\sqrt{5}}{2}.$$

Because $r > 0$ the correct solution is

$$r = \frac{a(\sqrt{5} - 1)}{2} \approx 0.62a.$$

Example 3.7

Locate the position of the mass center of the homogeneous volume of a hemisphere of radius R with respect to its base, as shown in Fig. 3.18.

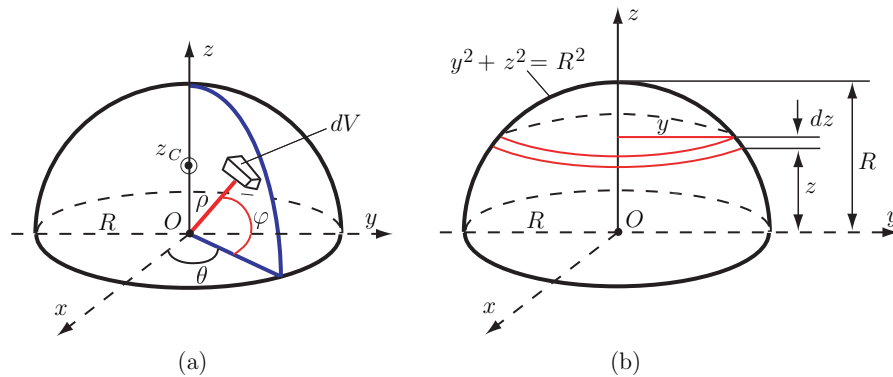


Fig. 3.18 Example 3.7

Solution

The reference frame selected as shown in Fig. 3.18(a) and the z -axis is the symmetry axis for the body: $x_C = 0$ and $y_C = 0$. Using the spherical coordinates $z = \rho \sin \varphi$ and the differential volume element is $dV = \rho^2 \cos \varphi d\rho d\theta d\varphi$. The z coordinate of the mass center is calculated from

$$z_C \iiint \rho^2 \cos \varphi d\rho d\theta d\varphi = \iiint \rho^3 \sin \varphi \cos \varphi d\rho d\theta d\varphi,$$

or

$$z_C \int_0^R \rho^2 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos \varphi d\varphi = \int_0^R \rho^3 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin \varphi \cos \varphi d\varphi,$$

or

$$z_C = \frac{\int_0^R \rho^3 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin \varphi \cos \varphi d\varphi}{\int_0^R \rho^2 d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos \varphi d\varphi}. \quad (3.27)$$

From Eq. (3.27) after integration it results

$$z_C = \frac{3R}{8}.$$

Another way of calculating the position of the mass center z_C is shown in Fig. 3.18(b). The differential volume element is

$$dV = \pi y^2 dz = \pi (R^2 - z^2) dz,$$

and the volume of the hemisphere of radius R is

$$V = \int_V dV = \int_0^R \pi (R^2 - z^2) dz = \pi \left(R^2 \int_0^R dz - \int_0^R z^2 dz \right) = \pi \left(R^3 - \frac{R^3}{3} \right) = \frac{2\pi R^3}{3}.$$

The coordinate z_C is calculated from the relation

$$\begin{aligned} z_C &= \frac{\int_V z dV}{V} = \frac{\pi \left(R^2 \int_0^R z dz - \int_0^R z^3 dz \right)}{V} = \frac{\pi}{V} \left(R^2 \frac{R^2}{2} - \frac{R^4}{4} \right) \\ &= \frac{\pi R^4}{4V} = \frac{\pi R^4}{4} \left(\frac{3}{2\pi R^3} \right) = \frac{3R}{8}. \end{aligned}$$

Example 3.8

Find the position of the mass center for a homogeneous right circular cone, with the base radius R and the height h , as shown in Fig. 3.19.

Solution

The reference frame is shown in Fig. 3.19. The z -axis is the symmetry axis for the right circular cone. By symmetry $x_C = 0$ and $y_C = 0$. The volume of the thin disk differential volume element is $dV = \pi r^2 dz$. From geometry:

$$\frac{r}{R} = \frac{h-z}{h},$$

or

$$r = R \left(1 - \frac{z}{h} \right).$$

The z coordinate of the centroid is calculated from

$$z_C \pi R^2 \int_0^h \left(1 - \frac{z}{h} \right)^2 dz = \pi R^2 \int_0^h z \left(1 - \frac{z}{h} \right)^2 dz,$$

or

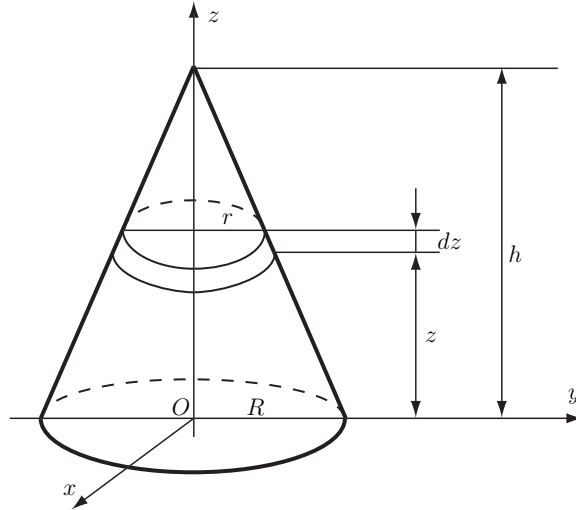


Fig. 3.19 Example 3.8

$$z_C = \frac{\int_0^h z \left(1 - \frac{z}{h}\right)^2 dz}{\int_0^h \left(1 - \frac{z}{h}\right)^2 dz} = \frac{h}{4}.$$

Example 3.9

The density of a square plate with the length a , is given as $\rho = kr$, where $k = \text{constant}$, and r is the distance from the origin O to a current point $P(x, y)$ on the plate as shown in Fig. 3.20. Find the mass M of the plate.

Solution

The mass of the plate is given by

$$M = \iint \rho dx dy.$$

The density is

$$\rho = k\sqrt{x^2 + y^2}, \quad (3.28)$$

where

$$\rho_0 = kr_0 = ka \frac{\sqrt{2}}{2} \Rightarrow k = \frac{\sqrt{2}}{a} \rho_0. \quad (3.29)$$

Using Eqs. (3.28) and (3.29) the density is

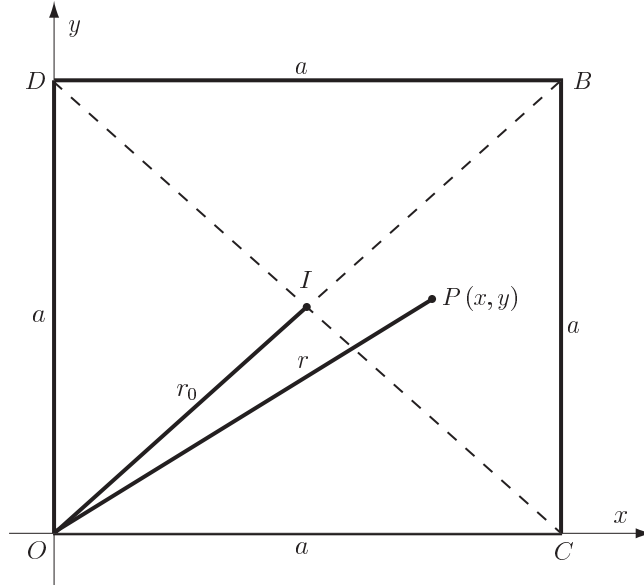


Fig. 3.20 Example 3.9

$$\rho = \frac{\sqrt{2}}{a} \rho_0 \sqrt{x^2 + y^2}. \quad (3.30)$$

Using Eq. (3.30) the mass of the plate is

$$\begin{aligned} M &= \frac{\sqrt{2}}{a} \rho_0 \int_0^a dx \int_0^a \sqrt{x^2 + y^2} dy \\ &= \frac{\sqrt{2}}{a} \rho_0 \int_0^a \left[\frac{y}{2} \sqrt{x^2 + y^2} + \frac{x^2}{2} \ln(y + \sqrt{x^2 + y^2}) \right] dx \\ &= \frac{\sqrt{2}}{a} \rho_0 \int_0^a \left[\frac{a}{2} \sqrt{x^2 + a^2} + \frac{x^2}{2} \ln \left(\frac{a + \sqrt{x^2 + a^2}}{x} \right) \right] dx \\ &= \frac{\sqrt{2}}{a} \rho_0 \left\{ \frac{a}{2} \int_0^a \sqrt{x^2 + a^2} dx + \frac{1}{2} \int_0^a x^2 \ln \left(\frac{a + \sqrt{x^2 + a^2}}{x} \right) dx \right\} \\ &= \left\{ \frac{\sqrt{2}}{a} \rho_0 \frac{a}{2} \left[\frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln(a + \sqrt{x^2 + a^2}) \right] \right\}_0^a \\ &\quad + \left\{ \frac{1}{2} \frac{\sqrt{2}}{a} \rho_0 \left[\ln \left(\frac{a + \sqrt{x^2 + a^2}}{x} \right) + \frac{a}{3} \frac{x}{2} \sqrt{x^2 + a^2} - \frac{a}{3} \frac{a^2}{2} \ln(a + \sqrt{x^2 + a^2}) \right] \right\}_0^a \\ &= \frac{1}{2} \frac{\sqrt{2}}{a} \rho_0 \left[\ln \left(\frac{a + \sqrt{x^2 + a^2}}{x} \right) + \frac{a}{3} \frac{x}{2} \sqrt{x^2 + a^2} - \frac{a}{3} \frac{a^2}{2} \ln(a + \sqrt{x^2 + a^2}) \right] \end{aligned}$$

$$= \frac{\rho_0 a^2}{3} \left[2 + \sqrt{2} \ln(1 + \sqrt{2}) \right].$$

Example 3.10

Revolving the circular area of radius R through 360° about the x -axis a complete torus is generated. The distance between the center of the circle and the x -axis is d , as shown in Fig. 3.21. Find the surface area and the volume of the obtained torus.

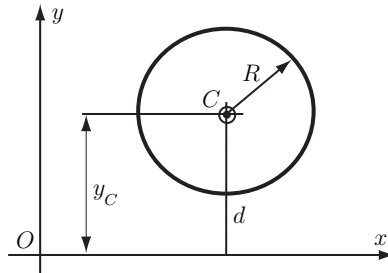


Fig. 3.21 Example 3.10

Solution

Using the Guldinus-Pappus formulas

$$A = 2\pi y_C L,$$

$$V = 2\pi y_C A,$$

and with $y_C = d$ the area and the volume are

$$A = (2\pi d)(2\pi R) = 4\pi^2 R d,$$

$$V = (2\pi d)(\pi R^2) = 2\pi^2 R^2 d.$$

Example 3.11

Find the position of the mass center for the semicircular area shown in Fig. 3.22.

Solution

Rotating the semicircular area with respect to x -axis a sphere is obtained. The volume of the sphere is given by

$$V = \frac{4\pi R^3}{3}.$$

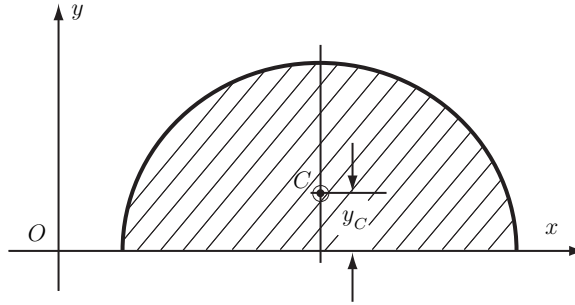


Fig. 3.22 Example 3.11

The area of the semicircular area is

$$A = \frac{\pi R^2}{2}.$$

Using the second Guldinus-Pappus theorem the position of the mass center is

$$y_C = \frac{V}{2\pi A} = \frac{\frac{4\pi R^3}{3}}{\frac{\pi R^2}{2}} = \frac{4R}{3\pi}.$$

Example 3.12

Find the first moment of the area with respect to the x -axis for the surface given in Fig. 3.23.

Solution

The composite area is considered to be composed of the semicircular area and the triangle. The mass center of the semicircular surface is given by

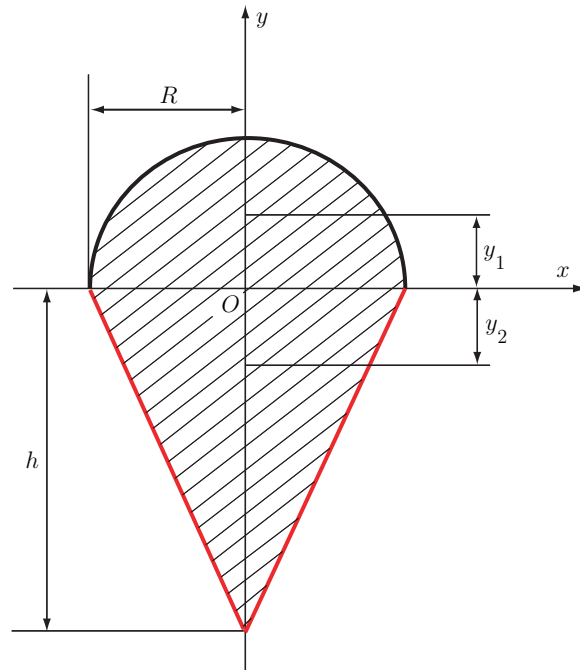
$$y_1 = \frac{2}{3} \frac{R \sin \frac{\pi}{2}}{\frac{\pi}{2}} = \frac{4R}{3\pi}.$$

The mass center of the triangle is given by

$$y_2 = -\frac{b}{3}.$$

The first moment of area with respect to x -axis for the total surface is given by

$$M_x = \sum_{i=1}^2 y_i A_i = \frac{4R}{3\pi} \frac{\pi R^2}{2} - \frac{h}{3} \frac{2rh}{2},$$

**Fig. 3.23** Example 3.12

or

$$M_x = \frac{R}{3} (2R^2 - h^2).$$