Chapter 1
Vector Algebra

1.1 Terminology and Notation

Scalars are mathematics quantities that can be fully defined by specifying their magnitude in suitable units of measure. The mass is a scalar and can be expressed in kilograms, the time is a scalar and can be expressed seconds, and the temperature can be expressed in degrees.

Vectors are quantities that require the specification of magnitude, orientation, and sense. The characteristics of a vector are the magnitude, the orientation, and the sense.

The *magnitude* of a vector is specified by a positive number and a unit having appropriate dimensions. No unit is stated if the dimensions are those of a pure number.

The *orientation* of a vector is specified by the relationship between the vector and given reference lines and/or planes.

The *sense* of a vector is specified by the order of two points on a line parallel to the vector.

Orientation and sense together determine the *direction* of a vector.

The *line of action* of a vector is a hypothetical infinite straight line collinear with the vector.

Displacement, velocity, and force are examples of vectors.

To distinguish vectors from scalars it is customary to denote vectors by boldface letters Thus, the vector shown in Fig. 1.1(a) is denoted by $\mathbf{r}$ or $\mathbf{r}_{AB}$. The symbol $|\mathbf{r}| = r$ represents the magnitude (or module, or absolute value) of the vector $\mathbf{r}$. In handwritten work a distinguishing mark is used for vectors, such as an arrow over the symbol, $\overrightarrow{r}$ or $\overrightarrow{AB}$, a line over the symbol, $\bar{r}$, or an underline, $\underline{r}$.

The vectors are depicted by either straight or curved arrows. A vector represented by a straight arrow has the direction indicated by the arrow. The direction of a vector represented by a curved arrow is the same as the direction in which a right-handed screw moves when the axis of the screw is normal to the plane in which the arrow is drawn and the screw is rotated as indicated by the arrow.

Figure 1.1 shows representations of vectors. Sometimes vectors are represented by means of a straight or curved arrow together with a measure number. In this case the vector is regarded as having the direction indicated by the arrow if the measure number is positive, and the opposite direction if it is negative. A *bound* vector is a

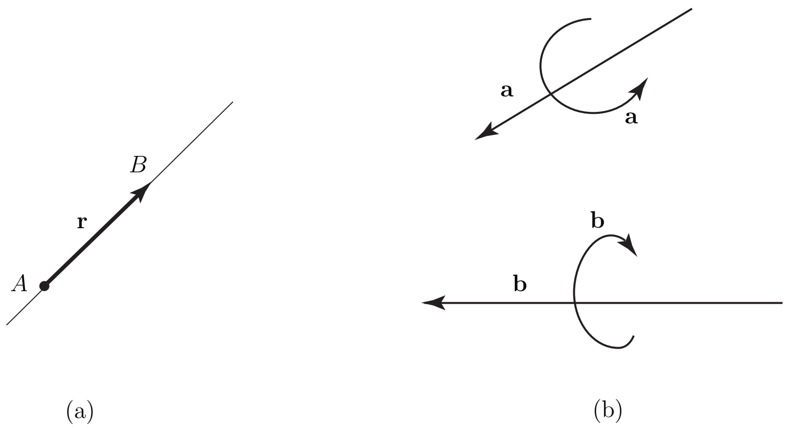


Fig. 1.1 Representations of vectors

vector associated with a particular point P in space (Fig. 1.2). The point P is the *point of application* of the vector, and the line passing through P and parallel to the vector is the line of action of the vector. The point of application may be represented as the tail, Fig. 1.2(a), or the head of the vector arrow, Fig. 1.2b). A *free* vector is not associated with a particular point P in space. A *transmissible* (or *sliding*) vector is a vector that can be moved along his line of action without change of meaning. To

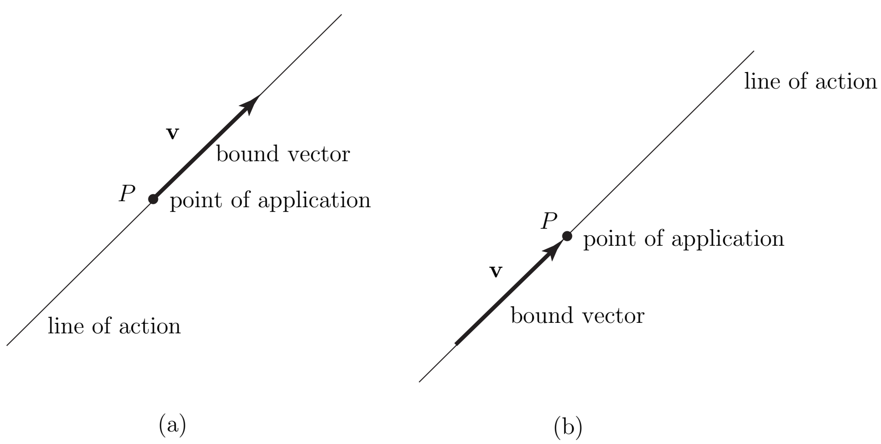


Fig. 1.2 Bound or fixed vector: (a) point of application represented as the tail of the vector arrow and (b) point of application represented as the head of the vector arrow

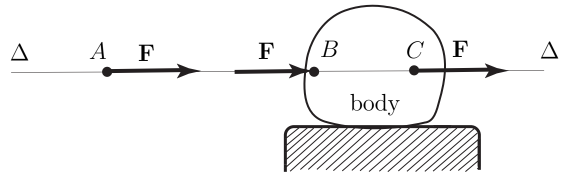


Fig. 1.3 Transmissible vector: the force vector **F** can be applied anywhere along the line Δ

move the body in Fig. 1.3 the force vector **F** can be applied anywhere along the line Δ or may be applied at specific points A, B, C. The force vector **F** is a transmissible vector because the resulting motion is the same in all cases.

The force **F** applied at B will cause a different deformation of the body than the same force **F** applied at a different point C. The points B and C are on the body. If one is interested in the deformation of the body, the force **F** positioned at C is a bound vector.

The operations of vector analysis deal only with the characteristics of vectors and apply, therefore, to both bound and free vectors.

Equality

Two vectors **a** and **b** are said to be equal to each other when they have the same characteristics. One then writes

$$\mathbf{a} = \mathbf{b}.$$

Equality does not imply physical equivalence. For instance, two forces represented by equal vectors do not necessarily cause identical motions of a body on which they act.

Product of a Vector and a Scalar

The product of a vector **v** and a scalar s, $s\mathbf{v}$ or $\mathbf{v}s$, is a vector having the following characteristics:

1. Magnitude.

$$|s\mathbf{v}| \equiv |\mathbf{v}s| = |s||\mathbf{v}|,$$

where $|s|$ denotes the absolute value (or magnitude, or module) of the scalar s.

2. Orientation. $s\mathbf{v}$ is parallel to **v**. If $s = 0$, no definite orientation is attributed to $s\mathbf{v}$.

3. Sense. If $s > 0$, the sense of $s\mathbf{v}$ is the same as that of **v**. If $s < 0$, the sense of $s\mathbf{v}$ is opposite to that of **v**. If $s = 0$, no definite sense is attributed to $s\mathbf{v}$.

Zero Vector

A *zero vector* is a vector that does not have a definite direction and whose magnitude is equal to zero. The symbol used to denote a zero vector is **0**.

Unit Vector

A *unit vector* is a vector with the magnitude equal to 1.

Given a vector **v**, a unit vector **u** having the same direction as **v** is obtained by forming the quotient of **v** and $|\mathbf{v}|$

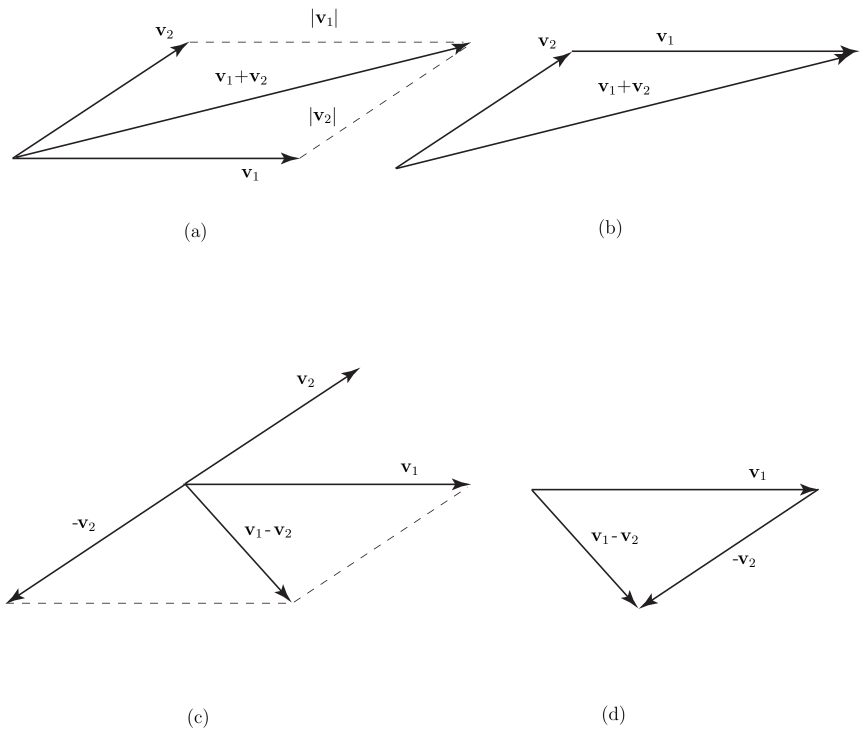


Fig. 1.4 Vector addition: (a) parallelogram law, (b) moving the vectors successively to parallel positions. Vector difference: (c) parallelogram law, (d) moving the vectors successively to parallel positions

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

Vector Addition

The sum of a vector $\mathbf{v}_1$ and a vector $\mathbf{v}_2$: $\mathbf{v}_1 + \mathbf{v}_2$ or $\mathbf{v}_2 + \mathbf{v}_1$ is a vector whose characteristics are found by either graphical or analytical processes. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ add according to the parallelogram law: $\mathbf{v}_1 + \mathbf{v}_2$ is equal to the diagonal of a parallelogram formed by the graphical representation of the vectors, see Fig. 1.4(a). The vector $\mathbf{v}_1 + \mathbf{v}_2$ is called the *resultant* of $\mathbf{v}_1$ and $\mathbf{v}_2$. The vectors can be added by moving them successively to parallel positions so that the head of one vector connects to the tail of the next vector. The resultant is the vector whose tail connects to the tail of the first vector, and whose head connects to the head of the last vector, see Fig. 1.4(b).

The sum $\mathbf{v}_1 + (-\mathbf{v}_2)$ is called the *difference* of $\mathbf{v}_1$ and $\mathbf{v}_2$ and is denoted by $\mathbf{v}_1 - \mathbf{v}_2$, see Figs. 1.4(c) and 1.4(d).

The sum of n vectors $\mathbf{v}_i$, $i = 1, \ldots, n$,

$$\sum_{i=1}^{n} \mathbf{v}_i \text{ or } \mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_n$$

is called the *resultant* of the vectors $\mathbf{v}_i$, $i = 1, \ldots n$.

The vector addition is:

1. commutative, that is, the characteristics of the resultant are independent of the order in which the vectors are added (commutativity)

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1.$$

2. associative, that is, the characteristics of the resultant are not affected by the manner in which the vectors are grouped (associativity)

$$\mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3.$$

3. distributive, that is, the vector addition obeys the following laws of distributivity

$$\mathbf{v} \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} (\mathbf{v} s_i), \text{ for } s_i \neq 0, s_i \in \mathscr{R},$$
$$s \sum_{i=1}^{n} \mathbf{v}_i = \sum_{i=1}^{n} (s \mathbf{v}_i), \text{ for } s \neq 0, s \in \mathscr{R},$$

where $\mathscr{R}$ is the set of real numbers.
Every vector can be regarded as the sum of n vectors ($n = 2, 3, \ldots$) of which all but one can be selected arbitrarily.

Resolution of Vectors and Components

Let $\mathbf{\imath}_1, \mathbf{\imath}_2, \mathbf{\imath}_3$ be any three unit vectors not parallel to the same plane (noncollinear vectors):

$$|\mathbf{\imath}_1| = |\mathbf{\imath}_2| = |\mathbf{\imath}_3| = 1.$$

For a given vector $\mathbf{v}$ (Fig. 1.5), there exist three unique scalars v_1, v_1, v_3, such that $\mathbf{v}$ can be expressed as

$$\mathbf{v} = v_1 \mathbf{\imath}_1 + v_2 \mathbf{\imath}_2 + v_3 \mathbf{\imath}_3.$$

The opposite action of addition of vectors is the *resolution* of vectors. Thus, for the given vector $\mathbf{v}$ the vectors $v_1 \mathbf{\imath}_1$, $v_2 \mathbf{\imath}_2$, and $v_3 \mathbf{\imath}_3$ sum to the original vector. The vector $v_k \mathbf{\imath}_k$ is called the $\mathbf{\imath}_k$ *component* of $\mathbf{v}$ and v_k is called the $\mathbf{\imath}_k$ *scalar component* of $\mathbf{v}$, where $k = 1, 2, 3$. A vector is often replaced by its components since the components are equivalent to the original vector.

Every vector equation $\mathbf{v} = \mathbf{0}$, where $\mathbf{v} = v_1 \mathbf{\imath}_1 + v_2 \mathbf{\imath}_2 + v_3 \mathbf{\imath}_3$, is equivalent to three scalar equations $v_1 = 0$, $v_2 = 0$, $v_3 = 0$.

If the unit vectors $\mathbf{\imath}_1, \mathbf{\imath}_2, \mathbf{\imath}_3$ are mutually perpendicular they form a *cartesian reference frame*. For a cartesian reference frame the following notation is used (Fig. 1.6):

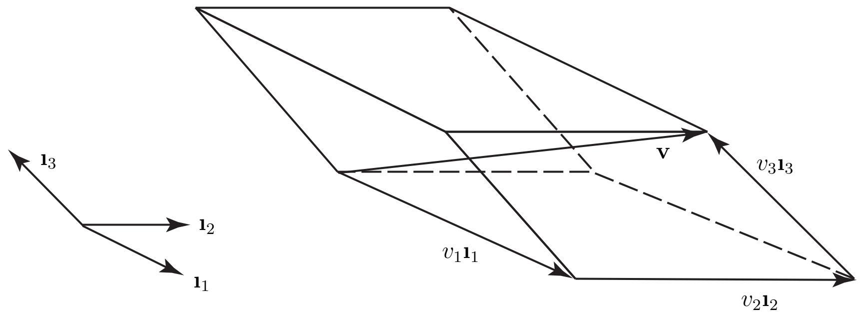


Fig. 1.5 Resolution of a vector **v** and components

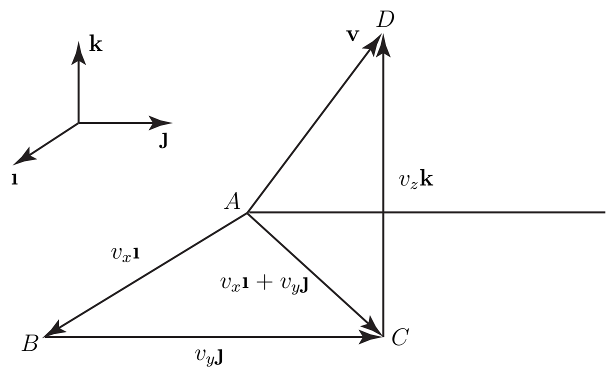


Fig. 1.6 Cartesian reference frame and the orthogonal scalar components v_x, v_y, v_z

$$\boldsymbol{\imath}_1 \equiv \boldsymbol{\imath},\ \boldsymbol{\imath}_2 \equiv \boldsymbol{\jmath},\ \boldsymbol{\imath}_3 \equiv \mathbf{k},$$

and

$$\boldsymbol{\imath} \perp \boldsymbol{\jmath},\ \boldsymbol{\imath} \perp \mathbf{k},\ \boldsymbol{\jmath} \perp \mathbf{k}.$$

The symbol $\perp$ denotes perpendicular.

When a vector **v** is expressed in the form $\mathbf{v} = v_x\boldsymbol{\imath} + v_y\boldsymbol{\jmath} + v_z\mathbf{k}$ where $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ are mutually perpendicular unit vectors (cartesian reference frame or orthogonal reference frame), the magnitude of **v** is given by

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

The vectors $\mathbf{v}_x = v_x\boldsymbol{\imath}$, $\mathbf{v}_y = v_y\boldsymbol{\jmath}$, and $\mathbf{v}_z = v_y\mathbf{k}$ are the *orthogonal* or *rectangular component vectors* of the vector **v**. The measures v_x, v_y, v_z are the *orthogonal* or *rectangular scalar components* of the vector **v**.

If $\mathbf{v}_1 = v_{1x}\boldsymbol{\imath} + v_{1y}\boldsymbol{\jmath} + v_{1z}\mathbf{k}$ and $\mathbf{v}_2 = v_{2x}\boldsymbol{\imath} + v_{2y}\boldsymbol{\jmath} + v_{2z}\mathbf{k}$, then the sum of the vectors is

$$\mathbf{v}_1 + \mathbf{v}_2 = (v_{1x} + v_{2x})\,\boldsymbol{\imath} + (v_{1y} + v_{2y})\,\boldsymbol{\jmath} + (v_{1z} + v_{2z})\,v_{1z}\mathbf{k}.$$

Angle between Two Vectors

Two vectors **a** and **b** are considered. One can move either vector parallel to itself (leaving its sense unaltered) until their initial points (tails) coincide.

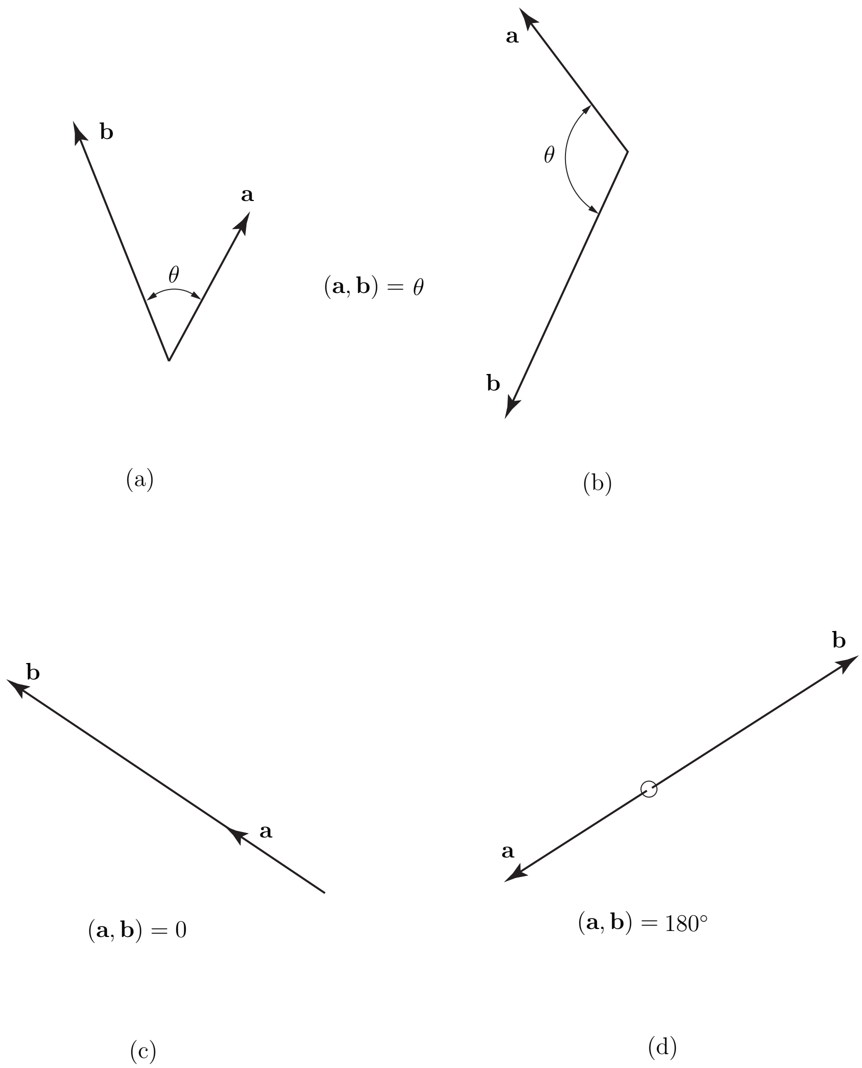


Fig. 1.7 The angle θ between the vectors **a** and **b**: (a) $0 < \theta < 90°$, (b) $90° < \theta < 180°$, and (c) $\theta = 0°$, and (d) $\theta = 180°$

The *angle* between **a** and **b** is the angle θ in Figs. 1.7(a) and 1.7(b). The angle between **a** and **b** is denoted by the symbols $(\mathbf{a}, \mathbf{b})$ or $(\mathbf{b}, \mathbf{a})$. Figure 1.7(c) represents the case $(\mathbf{a}, \mathbf{b}) = 0$, and Fig. 1.7(d) represents the case $(\mathbf{a}, \mathbf{b}) = 180°$.

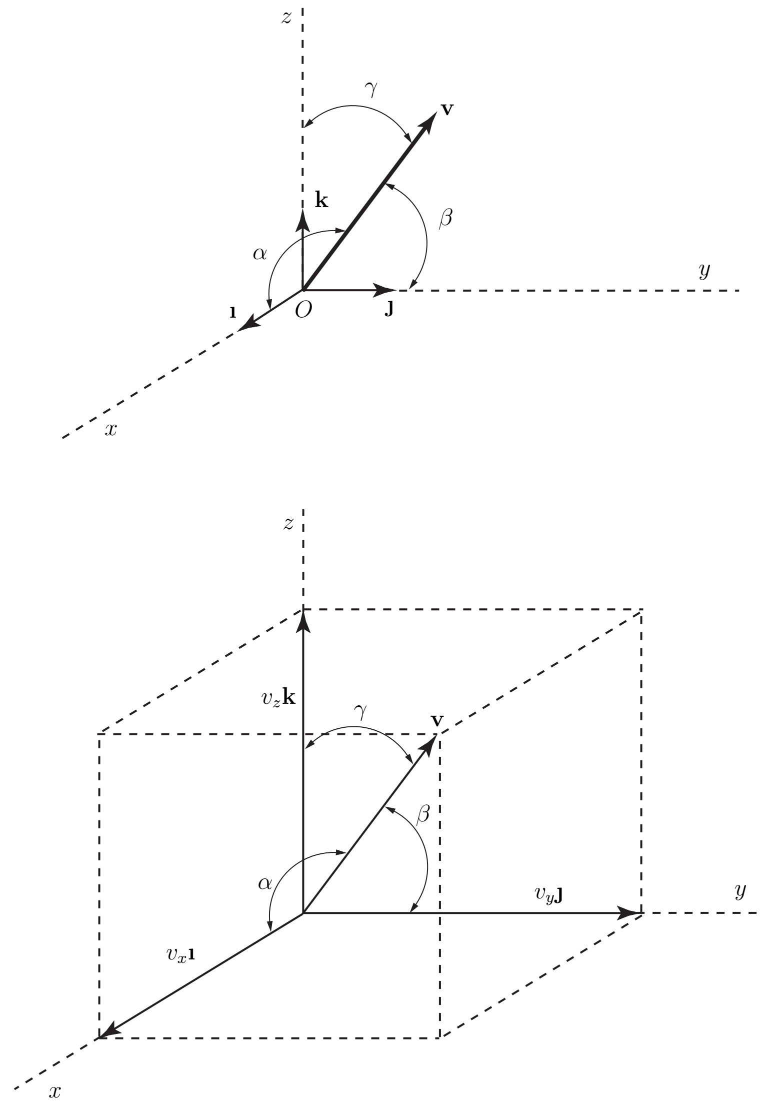


Fig. 1.8 Direction cosines

The direction of a vector $\mathbf{v} = v_x\boldsymbol{\imath} + v_y\boldsymbol{\jmath} + v_z\mathbf{k}$ and relative to a cartesian reference, $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$, is given by the cosines of the angles formed by the vector and the respective unit vectors. These are called *direction cosines* and are denoted as (Fig. 1.8)

$$\cos(\mathbf{v},\boldsymbol{\imath}) = \cos\alpha = l;\ \cos(\mathbf{v},\boldsymbol{\jmath}) = \cos\beta = m;\ \cos(\mathbf{v},\mathbf{k}) = \cos\gamma = n.$$

The following relations exist

$$v_x = |\mathbf{v}| \cos\alpha;\; v_y = |\mathbf{v}| \cos\beta;\; v_z = |\mathbf{v}| \cos\gamma.$$

1.2 Scalar (Dot) Product of Vectors

Definition. The scalar (dot) product of a vector **a** and a vector **b** is

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = |\mathbf{a}|\,|\mathbf{b}| \cos(\mathbf{a}, \mathbf{b}).$$

For any two vectors **a** and **b** and any scalar s

$$(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (s\mathbf{b}) = s\mathbf{a} \cdot \mathbf{b}.$$

If

$$\mathbf{a} = a_x \boldsymbol{\imath} + a_y \boldsymbol{\jmath} + a_z \mathbf{k},$$

and

$$\mathbf{b} = b_x \boldsymbol{\imath} + b_y \boldsymbol{\jmath} + b_z \mathbf{k},$$

where $\boldsymbol{\imath}$, $\boldsymbol{\jmath}$, $\mathbf{k}$ are mutually perpendicular unit vectors, then

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z\,.$$

The following relationships exist

$$\boldsymbol{\imath} \cdot \boldsymbol{\imath} = \boldsymbol{\jmath} \cdot \boldsymbol{\jmath} = \mathbf{k} \cdot \mathbf{k} = 1,$$
$$\boldsymbol{\imath} \cdot \boldsymbol{\jmath} = \boldsymbol{\jmath} \cdot \mathbf{k} = \mathbf{k} \cdot \boldsymbol{\imath} = 0.$$

Every vector **v** can be expressed in the form

$$\mathbf{v} = \boldsymbol{\imath} \cdot \mathbf{v} \boldsymbol{\imath} + \boldsymbol{\jmath} \cdot \mathbf{v} \boldsymbol{\jmath} + \mathbf{k} \cdot \mathbf{v} \mathbf{k}.$$

The vector **v** can always be expressed as

$$\mathbf{v} = v_x \boldsymbol{\imath} + v_y \boldsymbol{\jmath} + v_z \mathbf{k}.$$

Dot multiply both sides by $\boldsymbol{\imath}$

$$\boldsymbol{\imath} \cdot \mathbf{v} = v_x \boldsymbol{\imath} \cdot \boldsymbol{\imath} + v_y \boldsymbol{\imath} \cdot \boldsymbol{\jmath} + v_z \boldsymbol{\imath} \cdot \mathbf{k}.$$

But,

$$\boldsymbol{\imath} \cdot \boldsymbol{\imath} = 1, \text{ and } \boldsymbol{\imath} \cdot \boldsymbol{\jmath} = \boldsymbol{\imath} \cdot \mathbf{k} = 0.$$

Hence,

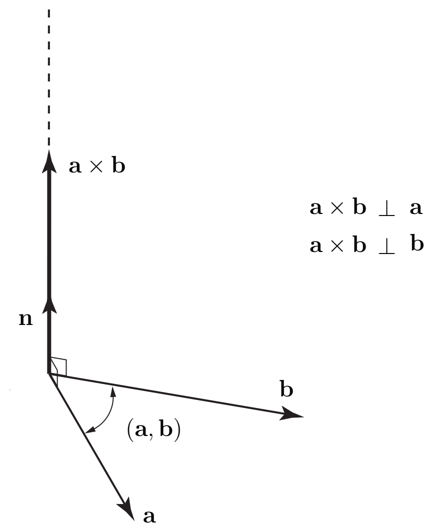


Fig. 1.9 Vector (cross) product of the vector **a** and the vector **b**

$$\mathbf{\imath}\cdot\mathbf{v} = v_x.$$

Similarly,

$$\mathbf{\jmath}\cdot\mathbf{v} = v_y \text{ and } \mathbf{k}\cdot\mathbf{v} = v_z.$$

1.3 Vector (Cross) Product of Vectors

Definition. The vector (cross) product of a vector **a** and a vector **b** is the vector (Fig. 1.9)

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin(\mathbf{a},\mathbf{b})\mathbf{n}$$

where **n** is a unit vector whose direction is the same as the direction of advance of a right-handed screw rotated from **a** toward **b**, through the angle $(\mathbf{a},\mathbf{b})$, when the axis of the screw is perpendicular to both **a** and **b**.
The magnitude of $\mathbf{a}\times\mathbf{b}$ is given by

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin(\mathbf{a},\mathbf{b}).$$

If **a** is parallel to **b**, $\mathbf{a}||\mathbf{b}$, then $\mathbf{a}\times\mathbf{b} = 0$. The symbol $||$ denotes parallel.
The relation $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ implies only that the product $|\mathbf{a}|\,|\mathbf{b}|\sin(\mathbf{a},\mathbf{b})$ is equal to zero, and this is the case whenever $|\mathbf{a}| = 0$, or $|\mathbf{b}| = 0$, or $\sin(\mathbf{a},\mathbf{b}) = 0$.
For any two vectors **a** and **b** and any real scalar s,

$$(s\mathbf{a}) \times \mathbf{b} = s(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (s\mathbf{b}) = s\mathbf{a} \times \mathbf{b}.$$

The sense of the unit vector $\mathbf{n}$ which appears in the definition of $\mathbf{a} \times \mathbf{b}$ depends on the order of the factors $\mathbf{a}$ and $\mathbf{b}$ in such a way that

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}.$$

Vector multiplication obeys the following law of distributivity (Varignon theorem)

$$\mathbf{a} \times \sum_{i=1}^{n} \mathbf{v}_i = \sum_{i=1}^{n} (\mathbf{a} \times \mathbf{v}_i).$$

A set of mutually perpendicular unit vectors $\imath, \jmath, \mathbf{k}$ is called *right-handed* if $\imath \times \jmath = \mathbf{k}$. A set of mutually perpendicular unit vectors $\imath, \jmath, \mathbf{k}$ is called *left-handed* if $\imath \times \jmath = -\mathbf{k}$.

If

$$\mathbf{a} = a_x \imath + a_y \jmath + a_z \mathbf{k},$$

and

$$\mathbf{b} = b_x \imath + b_y \jmath + b_z \mathbf{k},$$

where $\imath, \jmath, \mathbf{k}$ are mutually perpendicular unit vectors, then $\mathbf{a} \times \mathbf{b}$ can be expressed in the following determinant form:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

The determinant can be expanded by minors of the elements of the first row:

$$\begin{aligned} \begin{vmatrix} \imath & \jmath & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} &= \imath \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \jmath \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \\ &= \imath(a_y b_z - a_z b_y) - \jmath(a_x b_z - a_z b_x) + \mathbf{k}(a_x b_y - a_y b_x) \\ &= (a_y b_z - a_z b_y)\imath + (a_z b_x - a_x b_z)\jmath + (a_x b_y - a_y b_x)\mathbf{k}. \end{aligned}$$

1.4 Scalar Triple Product of Three Vectors

Definition. The scalar triple product of three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}.$$

It does not matter whether the dot is placed between **a** and **b**, and the cross between **b** and **c**, or vice versa, that is,

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = \mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}.$$

A change in the order of the factors appearing in a scalar triple product at most changes the sign of the product, that is,

$$[\mathbf{b},\mathbf{a},\mathbf{c}] = -[\mathbf{a},\mathbf{b},\mathbf{c}],$$

and

$$[\mathbf{b},\mathbf{c},\mathbf{a}] = [\mathbf{a},\mathbf{b},\mathbf{c}].$$

If **a**, **b**, **c** are parallel to the same plane, or if any two of the vectors **a**, **b**, **c** are parallel to each other, then $[\mathbf{a},\mathbf{b},\mathbf{c}] = 0$.
The scalar triple product $[\mathbf{a},\mathbf{b},\mathbf{c}]$ can be expressed in the following determinant form

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

1.5 Vector Triple Product of Three Vector

Definition. The vector triple product of three vectors **a**, **b**, **c** is the vector $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. The parentheses are essential because $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is not, in general, equal to $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.
For any three vectors **a**, **b**, and **c**,

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot\mathbf{c}\mathbf{b} - \mathbf{a}\cdot\mathbf{b}\mathbf{c}.$$

1.6 Derivative of a Vector

The derivative of a vector is defined in exactly the same way as is the derivative of a scalar function. The derivative of a vector has some of the properties of the derivative of a scalar function.
The derivative of the sum of two vector functions **a** and **b** is

$$\frac{d}{dt}(\mathbf{a}+\mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt},$$

The time derivative of the product of a scalar function f and a vector function **u** is

$$\frac{d(f\mathbf{a})}{dt} = \frac{df}{dt}\mathbf{a} + f\frac{d\mathbf{a}}{dt}.$$

1.7 Examples

Example 1.1

In Fig. 1.10 the rectangular component of the vector $\mathbf{F}$ on the OA direction is $\mathbf{f}$, with the magnitude $|\mathbf{f}| = f$. The vector $\mathbf{F}$ acts at an angle β with the positive direction of the x-axis. Find the magnitude of the vector $|\mathbf{F}| = F$.

Numerical application: $f = 20$, $\alpha = 30°$, $\beta = 60°$.

Solution

The component of $\mathbf{F}$ on the OA direction is $|\mathbf{F}|\cos\theta = f$. From Fig. 1.10 the angle θ of the vector $\mathbf{F}$ with the OA direction is $\theta = \beta - \alpha = 60° - 30° = 30°$. The magnitude F is calculated from the equation $|\mathbf{F}|\cos\theta = f \Leftrightarrow |\mathbf{F}|\cos 30° = 20 \Rightarrow F = |\mathbf{F}| = \dfrac{f}{\cos\theta} = \dfrac{20}{\cos 30°}$ or $F = 23.094$.

Example 1.2

Determine the unit vector of a line Δ that starts at point $A(x_A, y_A, z_A)$ and passes through a point $B(x_B, y_B, z_B)$. Determine the projection of the vector $\mathbf{P} = P_x\mathbf{\imath} + P_y\mathbf{\jmath} + P_z\mathbf{k}$ along the line Δ.

Numerical application: $A(1, 2, -3)$, $B(-3, 2, 0)$, $P_x = 2$, $P_y = 7$, and $P_z = 10$.

Solution

The unit vector is

$$\mathbf{u}_\Delta = \frac{\mathbf{r}_{AB}}{|\mathbf{r}_{AB}|} = \frac{(x_B - x_A)\mathbf{\imath} + (y_B - y_A)\mathbf{\jmath} + (z_B - z_A)\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}$$

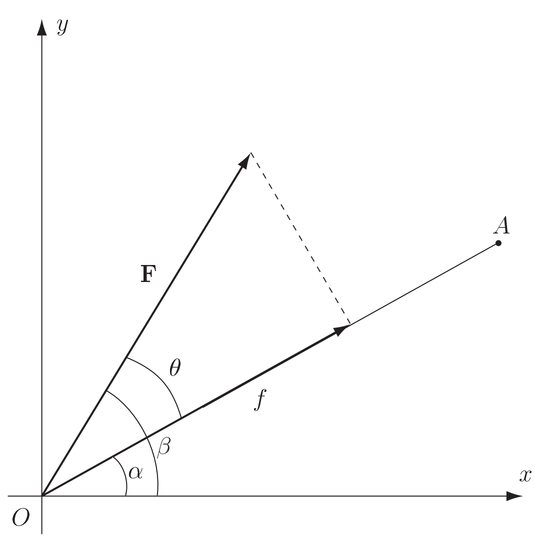


Fig. 1.10 Example 1.1

$$
\begin{aligned}
&= \frac{x_B - x_A}{\sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}}\mathbf{\imath} \\
&\quad + \frac{y_B - y_A}{\sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}}\mathbf{\jmath} \\
&\quad + \frac{z_B - z_A}{\sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}}\mathbf{k} \\
&= u_x\mathbf{\imath} + u_y\mathbf{\jmath} + u_z\mathbf{k}.
\end{aligned}
$$

The components are

$$
\begin{aligned}
u_x &= \frac{x_B - x_A}{\sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}} \\
&= \frac{-3-1}{\sqrt{(-3-1)^2+(2-2)^2+(0+3)^2}} = \frac{-4}{5} = -\frac{4}{5}, \\
u_y &= \frac{y_B - y_A}{\sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}} \\
&= \frac{2-2}{\sqrt{(-3-1)^2+(2-2)^2+(0+3)^2}} = \frac{0}{5} = 0, \\
u_z &= \frac{z_B - z_A}{\sqrt{(x_B-x_A)^2+(y_B-y_A)^2+(z_B-z_A)^2}} \\
&= \frac{0+3}{\sqrt{(-3-1)^2+(2-2)^2+(0+3)^2}} = \frac{3}{5},
\end{aligned}
$$

The projection of the vector $\mathbf{P}$ on the line Δ is

$$
\begin{aligned}
P &= \mathbf{P}\cdot\mathbf{u}_\Delta = (P_x\mathbf{\imath}+P_y\mathbf{\jmath}+P_z\mathbf{k})\cdot(u_x\mathbf{\imath}+u_y\mathbf{\jmath}+u_z\mathbf{k}) \\
&= P_xu_x + P_yu_y - P_zu_z \\
&= 2\frac{(-4)}{5} + 7\frac{0}{5} + 10\frac{3}{5} = \frac{(-8)}{5} + \frac{30}{5} = \frac{22}{5}.
\end{aligned}
$$

Example 1.3

The vectors $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, and $\mathbf{V}_4$ with the magnitude $|\mathbf{V}_1| = V_1$, $|\mathbf{V}_2| = V_2$, $|\mathbf{V}_3| = V_3$, and $|\mathbf{V}_4| = V_4$ are concurrent at the origin $O(0, 0, 0)$ and are directed through the points of coordinates $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$, $A_3(x_3, y_3, z_3)$, and $A_4(x_4, y_4, z_4)$, respectively. Determine the resultant vector of the system.

Numerical application: $V_1 = 10$, $V_2 = 25$, $V_3 = 15$, $V_4 = 40$, and $A_1(3, 1, 7)$, $A_2(5, -3, 4)$, $A_3(-4, -3, 1)$, $A_4(4, 2, -3)$.

Solution

The vector $\mathbf{V}_i$ can be written as

$$\mathbf{V}_i = V_{ix}\mathbf{\imath} + V_{iy}\mathbf{\jmath} + V_{iz}\mathbf{k}, \quad i = 1, 2, 3, 4,$$

where

$$V_{ix} = |\mathbf{V}_i| \cos\theta_{ix}, \ V_{iy} = |\mathbf{V}_i| \cos\theta_{iy}, \ V_{iz} = |\mathbf{V}_i| \cos\theta_{iz}.$$

The direction cosines of the vectors are

$$\cos\theta_{ix} = \frac{x_i}{\sqrt{x_i^2 + y_i^2 + z_i^2}},$$
$$\cos\theta_{iy} = \frac{y_i}{\sqrt{x_i^2 + y_i^2 + z_i^2}},$$
$$\cos\theta_{iz} = \frac{z_i}{\sqrt{x_i^2 + y_i^2 + z_i^2}}.$$

The resultant of the system is

$$R = \sqrt{(R_x)^2 + (R_y)^2 + (R_z)^2} = \sqrt{\left(\sum V_{ix}\right)^2 + \left(\sum V_{iy}\right)^2 + \left(\sum V_{iz}\right)^2}.$$

The direction cosines of the resultant are

$$\cos\theta_x = \frac{\sum V_{ix}}{R}, \ \cos\theta_y = \frac{\sum V_{iy}}{R}, \ \cos\theta_z = \frac{\sum V_{iz}}{R}.$$

For the given numerical data the vectors and the direction cosines are

i	V_i	A_i	$\cos\theta_{ix}$	$\cos\theta_{iy}$	$\cos\theta_{iz}$	V_{ix}	V_{iy}	V_{iz}
1	10	$(3,1,7)$	0.39	0.13	0.91	3.90	1.30	9.11
2	25	$(5,-3,4)$	0.70	−0.42	0.56	17.67	−10.60	14.14
3	15	$(-4,-3,1)$	−0.78	−0.58	0.19	−11.76	−8.82	2.94
4	40	$(4,2,-3)$	0.74	0.37	−0.55	29.71	14.85	−22.28

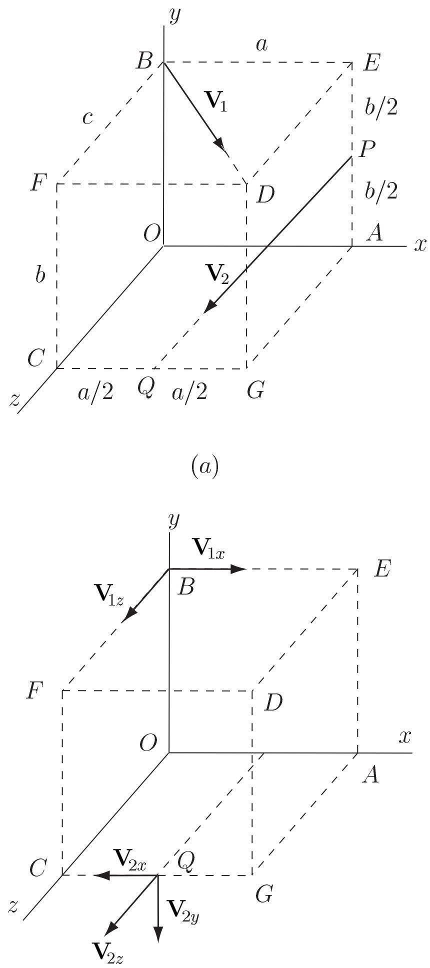
The numerical values for the resultant are

R	$\cos\theta_x$	$\cos\theta_y$	$\cos\theta_z$	R_x	R_y	R_z
39.855	0.991	-0.082	0.098	39.527	−3.274	3.913

The negative value of $\cos\theta_y$ signifies that the resultant has a negative component in the y direction.

Example 1.4
Find the resultant of the vector system $\mathbf{V}_1$ and $\mathbf{V}_2$, shown in Fig. 1.11(a).
Numerical application: $|\mathbf{V}_1| = V_1 = 5$, $|\mathbf{V}_2| = V_2 = 10$, $a = 4$, $b = 5$, and $c = 3$.
Solution
From Fig. 1.11(b) the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are

$$\mathbf{V}_1 = V_{1x}\mathbf{\imath} + V_{1y}\mathbf{\jmath} + V_{1z}\mathbf{k} = |\mathbf{V}_1| \frac{\mathbf{r}_{BD}}{|\mathbf{r}_{BD}|},$$



(b)

Fig. 1.11 Example 1.4

$$\mathbf{V}_2 = V_{2x}\mathbf{\imath} + V_{2y}\mathbf{\jmath} + V_{2z}\mathbf{k} = |\mathbf{V}_2| \frac{\mathbf{r}_{PQ}}{|\mathbf{r}_{PQ}|}.$$

The vectors $\mathbf{r}_{BD}$ and $\mathbf{r}_{PQ}$ are

$$\begin{aligned}
\mathbf{r}_{BD} &= (x_D - x_B)\,\mathbf{\imath} + (y_D - y_B)\,\mathbf{\jmath} + (z_D - z_B)\,\mathbf{k} \\
&= (a-0)\,\mathbf{\imath} + (b-b)\,\mathbf{\jmath} + (c-0)\,\mathbf{k} \\
&= a\mathbf{\imath} + c\mathbf{k} = 4\mathbf{\imath} + 3\mathbf{k}, \\
\mathbf{r}_{PQ} &= (x_Q - x_P)\,\mathbf{\imath} + (y_Q - y_P)\,\mathbf{\jmath} + (z_Q - z_P)\,\mathbf{k}
\end{aligned}$$

$$= \left(\frac{a}{2} - a\right)\mathbf{\imath} + \left(0 - \frac{b}{2}\right)\mathbf{\jmath} + (c - 0)\,\mathbf{k}$$
$$= -2\mathbf{\imath} - \frac{5}{2}\mathbf{\jmath} + 3\mathbf{k},$$

The magnitudes of the vectors $\mathbf{r}_{BD}$ and $\mathbf{r}_{PQ}$ are

$$|\mathbf{r}_{BD}| = \sqrt{(x_D - x_B)^2 + (y_D - y_B)^2 + (z_D - z_B)^2}$$
$$= \sqrt{(a - 0)^2 + (b - b)^2 + (c - 0)^2} = 5,$$
$$|\mathbf{r}_{PQ}| = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}$$
$$= \sqrt{\left(\frac{a}{2} - a\right)^2 + \left(0 - \frac{b}{2}\right)^2 + (c - 0)^2} = 4.38,$$

where $B = B(x_B, y_B, z_B) = B(0, b, 0) = B(0, 5, 0)$, $D = D(x_D, y_D, z_D) = D(a, b, c) = D(4, 5, 3)$, $P = P(x_P, y_P, z_P) = P(a, b/2, 0) = P(4, 5/2, 0)$ and $Q = Q(x_Q, y_Q, z_Q) = Q(a/2, 0, c) = Q(2, 0, 3)$.

The values of the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are

$$\mathbf{V}_1 = |\mathbf{V}_1| \frac{\mathbf{r}_{BD}}{|\mathbf{r}_{BD}|} = V_1 \frac{(x_D - x_B)\,\mathbf{\imath} + (y_D - y_B)\,\mathbf{\jmath} + (z_D - z_B)\,\mathbf{k}}{\sqrt{(x_D - x_B)^2 + (y_D - y_B)^2 + (z_D - z_B)^2}}$$
$$= 5\frac{4\mathbf{\imath} + 3\mathbf{k}}{5} = 4\mathbf{\imath} + 3\mathbf{k},$$
$$\mathbf{V}_2 = |\mathbf{V}_2| \frac{\mathbf{r}_{PQ}}{|\mathbf{r}_{PQ}|} = V_2 \frac{(x_Q - x_P)\,\mathbf{\imath} + (y_Q - y_P)\,\mathbf{\jmath} + (z_Q - z_P)\,\mathbf{k}}{\sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}}$$
$$= 10\frac{-2\mathbf{\imath} - \frac{5}{2}\mathbf{\jmath} + 3\mathbf{k}}{4.38} = -\frac{20}{4.38}\mathbf{\imath} - \frac{50}{8.76}\mathbf{\jmath} + \frac{30}{4.38}\mathbf{k}.$$

The cartesian components of the vector $\mathbf{V}_1$ are

$$V_{1x} = V_1 \frac{x_D - x_B}{\sqrt{(x_D - x_B)^2 + (y_D - y_B)^2 + (z_D - z_B)^2}} = 4,$$
$$V_{1y} = V_1 \frac{y_D - y_B}{\sqrt{(x_D - x_B)^2 + (y_D - y_B)^2 + (z_D - z_B)^2}} = 0,$$
$$V_{1z} = F_1 \frac{z_D - z_B}{\sqrt{(x_D - x_B)^2 + (y_D - y_B)^2 + (z_D - z_B)^2}} = 3.$$

The components of the vector $\mathbf{V}_2$ are

$$V_{2x} = V_2 \frac{x_Q - x_P}{\sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}} = -4.56,$$
$$V_{2y} = V_2 \frac{y_Q - y_P}{\sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}} = -5.7,$$
$$V_{2z} = V_2 \frac{z_Q - z_P}{\sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}} = 6.8.$$

The resultant vector has the components

$$R_x = \sum V_{ix} = V_{1x} + V_{2x} = -0.558,$$
$$R_y = \sum V_{iy} = V_{1y} + V_{2y} = -5.698,$$
$$R_z = \sum V_{iz} = V_{1z} + V_{2z} = 9.837,$$

and can be written in a vector form as

$$\mathbf{R} = R_x \mathbf{\imath} + R_y \mathbf{\jmath} + R_z \mathbf{k} = -0.558 \mathbf{\imath} - 5.698 \mathbf{\jmath} + 9.837 \mathbf{k}.$$

The magnitude of **R** is

$$|\mathbf{R}| = R = \sqrt{(R_x)^2 + (R_y)^2 + (R_z)^2} = 11.38.$$

The angles of the vector **R** with the cartesian axes are calculated from

$$\cos\theta_x = \frac{R_x}{|\mathbf{R}|} = \frac{R_x}{\sqrt{(R_x)^2 + (R_y)^2 + (R_z)^2}} = -0.049,$$
$$\cos\theta_y = \frac{R_y}{|\mathbf{R}|} = \frac{R_y}{\sqrt{(R_x)^2 + (R_y)^2 + (R_z)^2}} = -0.5,$$
$$\cos\theta_z = \frac{R_z}{|\mathbf{R}|} = \frac{R_z}{\sqrt{(R_x)^2 + (R_y)^2 + (R_z)^2}} = 0.864.$$

Example 1.5

The vector **p** of magnitude $|\mathbf{p}| = p$ is located in the $x - z$ plane and makes an angle θ with x-axis as shown in Fig. 1.12. The vector **q** of magnitude $|\mathbf{q}| = q$ is situated along the x-axis Compute the vector (cross) product $\mathbf{v} = \mathbf{p} \times \mathbf{q}$.

Numerical application: $|\mathbf{p}| = p = 5$, $|\mathbf{q}| = q = 4$, and $\theta = 30°$.

Solution

The vector product **v** is perpendicular to the vectors **p** and **q** and that is why the vector **v** is along the y-axis and with has the magnitude

$$|\mathbf{v}| = |\mathbf{p}|\,|\mathbf{q}| \sin\theta = pq \sin\theta = 5(4) \sin 30° = 10.$$

From Fig. 1.12 the direction of the vector **v** is upward.

The solution could also be obtained by expressing the vector product $\mathbf{v} = \mathbf{p} \times \mathbf{q}$ of the given vectors **p** and **q** in terms of the their rectangular components. Resolving **p** and **q** into components, one can write

$$\begin{aligned}\mathbf{v} = \mathbf{p} \times \mathbf{q} &= (p_x \mathbf{\imath} + p_y \mathbf{\jmath} + p_z \mathbf{k}) \times (q_x \mathbf{\imath} + q_y \mathbf{\jmath} + q_z \mathbf{k}) \\ &= \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{vmatrix} \\ &= (p_y q_z - p_z q_y)\,\mathbf{\imath} + (p_z q_x - p_x q_z)\,\mathbf{\jmath} + (p_x q_y - p_y q_x)\,\mathbf{k}.\end{aligned}$$

The components p_x, p_y, and p_z of the vector **p** are
$p_x = |\mathbf{p}|\cos\theta = p\cos\theta = 5\cos 30^\circ = 5\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$,
$p_y = 0$, and
$p_z = |\mathbf{p}|\sin\theta = p\sin\theta = 5\left(\frac{1}{2}\right) = \frac{5}{2}$.
The components q_x, q_y,and q_z of the vector **q** are $q_x = q = 4$, $q_y = 0$ and $q_z = 0$.
It results

$$\begin{aligned}\mathbf{v} = \mathbf{p} \times \mathbf{q} &= (p_y q_z - p_z q_y)\,\mathbf{\imath} + (p_z q_x - p_x q_z)\,\mathbf{\jmath} + (p_x q_y - p_y q_x)\,\mathbf{k} \\ &= \left(0(0) - \frac{5}{2}(0)\right)\mathbf{\imath} + \left(\frac{5}{2}(4) - \frac{5\sqrt{3}}{2}(0)\right)\mathbf{\jmath} + \left(\frac{5\sqrt{3}}{2}(0) - 0(4)\right)\mathbf{k} \\ &= \frac{5}{2}(4)\,\mathbf{\jmath} = 10\mathbf{\jmath}.\end{aligned}$$

Example 1.6

Figure 1.13 depicts three vectors **a**, **b**, and **c** that form a parallelepiped. Show that the scalar $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ represents the volume of the parallelepiped with the sides **a**, **b** and **c**.

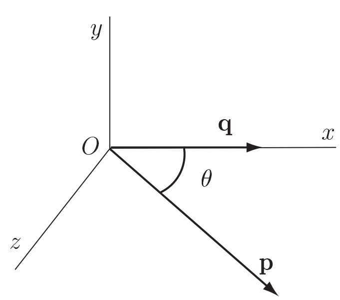


Fig. 1.12 Example 1.5

Solution
The scalar is

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = |\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|\sin\theta\cos\alpha = hA,$$

where $h = |\mathbf{a}|\cos\alpha$ represents the height of the parallelepiped and $A = |\mathbf{b}|\,|\mathbf{c}|\sin\theta$ represents the area of the parallelogram with the sides $\mathbf{b}$ and $\mathbf{c}$.
The product between h and A represents the volume of a parallelepiped, $v = hA$, so the scalar $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ represents the volume of the parallelepiped with the sides formed by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Example 1.7
Compute $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ and $(\mathbf{c}\times\mathbf{b})\cdot\mathbf{a}$ where $\mathbf{a} = a_x\mathbf{\imath}+a_y\mathbf{\jmath}+a_z\mathbf{k}$, $\mathbf{b} = b_x\mathbf{\imath}+b_y\mathbf{\jmath}+b_z\mathbf{k}$ and $\mathbf{c} = c_x\mathbf{\imath}+c_y\mathbf{\jmath}+c_z\mathbf{k}$.
Numerical application: $a_x = 2$, $a_y = 1$, $a_z = 3$, $b_x = 2$, $b_y = 1$, $b_z = 0$, $c_x = 2$, $c_y = 0$, and $c_z = 0$.
Solution
The scalar $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= (a_x\mathbf{\imath}+a_y\mathbf{\jmath}+a_z\mathbf{k})\cdot\begin{vmatrix}\mathbf{\imath} & \mathbf{\jmath} & \mathbf{k}\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}\\
&= \begin{vmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}\\
&= a_x(b_yc_z-b_zc_y)+a_y(b_zc_x-b_xc_z)+a_z(b_xc_y-b_yc_x)\\
&= 2(1(0)-0(0))+1(0(2)-2(0))+3(2(0)-1(2)) = -6
\end{aligned}$$

The scalar $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ is

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix}\mathbf{\imath} & \mathbf{\jmath} & \mathbf{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}\cdot(c_x\mathbf{\imath}+c_y\mathbf{\jmath}+c_z\mathbf{k})$$

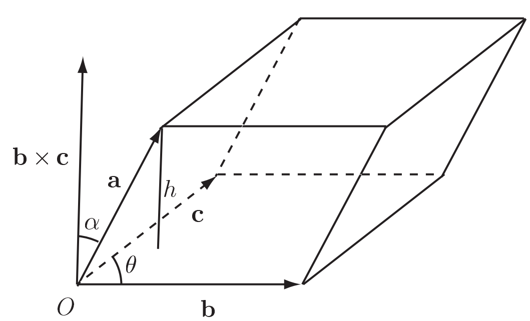


Fig. 1.13 Example 1.6

$$= (c_x\mathbf{\imath} + c_y\mathbf{\jmath} + c_z\mathbf{k}) \cdot \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = -\begin{vmatrix} a_x & a_y & a_z \\ c_x & c_y & c_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$= a_x (b_y c_z - b_z c_y) + a_y (b_z c_x - b_x c_z) + a_z (b_x c_y - b_y c_x)$$

$$= 2(1(0) - 0(0)) + 1(0(2) - 2(0)) + 3(2(0) - 1(2)) = -6$$

The scalar $(\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a}$ is

$$(\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ c_x & c_y & c_z \\ b_x & b_y & b_z \end{vmatrix} \cdot (a_x\mathbf{\imath} + a_y\mathbf{\jmath} + a_z\mathbf{k})$$

$$= (a_x\mathbf{\imath} + a_y\mathbf{\jmath} + a_z\mathbf{k}) \cdot \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ c_x & c_y & c_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ c_x & c_y & c_z \\ b_x & b_y & b_z \end{vmatrix} = -\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$= -[a_x (b_y c_z - b_z c_y) + a_y (b_z c_x - b_x c_z) + a_z (b_x c_y - b_y c_x)]$$

$$= -[2(1(0) - 0(0)) + 1(0(2) - 2(0)) + 3(2(0) - 1(2))] = 6$$

Remark: Note that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -(\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a}.$$

Example 1.7

Find the c_z component of the vector $\mathbf{c}$ such as the vectors $\mathbf{a} = a_x\mathbf{\imath} + a_y\mathbf{\jmath} + a_z\mathbf{k}$, $\mathbf{b} = b_x\mathbf{\imath} + b_y\mathbf{\jmath} + b_z\mathbf{k}$, and $\mathbf{c} = c_x\mathbf{\imath} + c_y\mathbf{\jmath} + c_z\mathbf{k}$ are coplanar.

Numerical application: $a_x = 2$, $a_y = 3$, $a_z = 0$, $b_x = 3$, $b_y = 2$, $b_z = -2$, $c_x = 2$, and $c_y = 3$.

Solution

The three vectors are coplanar if $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.
The scalar $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (a_x\mathbf{\imath} + a_y\mathbf{\jmath} + a_z\mathbf{k}) \cdot \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$
\begin{aligned}
&= a_x\left(b_y c_z - b_z c_y\right) + a_y\left(b_z c_x - b_x c_z\right) + a_z\left(b_x c_y - b_y c_x\right) \\
&= a_x b_y c_z - a_x b_z c_y + a_y b_z c_x - a_y b_x c_z + a_z b_x c_y - a_z b_y c_x \\
&= a_x b_y c_z - a_y b_x c_z - a_x b_z c_y + a_y b_z c_x + a_z b_x c_y - a_z b_y c_x \\
&= c_z\left(a_x b_y - a_y b_x\right) - a_x b_z c_y + a_y b_z c_x + a_z b_x c_y - a_z b_y c_x.
\end{aligned}
$$

The vectors **a**, **b**, and **c** are coplanar if

$$
\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0 \Leftrightarrow c_z\left(a_x b_y - a_y b_x\right) - a_x b_z c_y + a_y b_z c_x + a_z b_x c_y - a_z b_y c_x = 0,
$$

or

$$
c_z = \frac{a_x b_z c_y - a_y b_z c_x - a_z b_x c_y + a_z b_y c_x}{a_x b_y - a_y b_x}.
$$

It results

$$
\begin{aligned}
c_z &= \frac{2(-2)(3) - 3(-2)(2) - 0(3)(3) + 0(2)(2)}{2(2) - 3(3)} \\
&= \frac{-12 + 12 - 0 + 0}{4 - 9} = 0.
\end{aligned}
$$

Finally the vectors **a**, **b**, and **c** are coplanar if $c_z = 0$ or mathematically

$$
\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0 \Leftrightarrow c_z = 0.
$$

1.8 Problems

1.1 a) Find the angle made by the vector $\mathbf{v} = -10\boldsymbol{\imath} + 5\boldsymbol{\imath}$ with the positive x-axis and determine the unit vector in the direction of $\mathbf{v}$. b) Determine the magnitude of the resultant $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ and the angle which $\mathbf{v}$ makes with the positive x-axis, where the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are shown in Fig. 1.14. The magnitudes of the vectors are $|\mathbf{v}_1| = v_1 = 5$, $|\mathbf{v}_2| = v_2 = 10$, and the angles of the vectors with the positive x-axis are $\theta_1 = 30^\circ$, $\theta_2 = 60^\circ$.

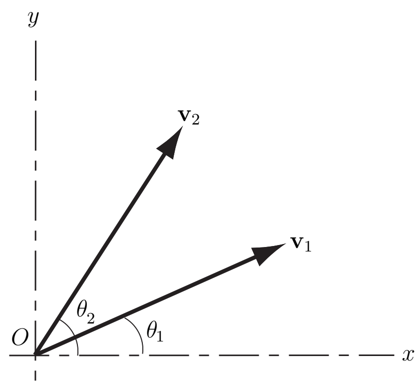


Fig. 1.14 Problem 1.1

1.2 The planar vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are given in xOy plane as shown in Fig. 1.15. The magnitude of the vectors are $a = P$, $b = 2P$, and $c = P\sqrt{2}$. The angles in the figure are $\alpha = 45^\circ$, $\beta = 120^\circ$, and $\gamma = 30^\circ$. Determine the magnitude of the resultant $\mathbf{v} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ and the angle that $\mathbf{v}$ makes with the positive x-axis.

1.3 The cube in Fig. 1.31 has the sides equal to l. Find the direction cosines of the resultant $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$.

1.4 The direction of the vectors $\mathbf{F}_i$, $i = 1,2,3,4$, are given by the lines O_iP_i, where $O_i = O_i(x_{O_i}, y_{O_i}, z_{O_i})$ and $P_i = P_i(x_{P_i}, y_{P_i}, z_{P_i})$. Find the resultant of the system shown in Fig. 1.17.
Numerical application: the magnitudes of the vectors are $|\mathbf{F}_1| = F_1 = 10$, $|\mathbf{F}_2| = F_2 = 15$, $|\mathbf{F}_3| = F_3 = 15$, $|\mathbf{F}_4| = F_4 = 20$, and the coordinates $O_1(0,2,0)$, $P_1(4,0,0)$, $O_2(0,0,5)$, $P_2(2,2,5)$, $O_3(2,0,3)$, $P_3(5,0,3)$, $O_4(4,0,3)$, and $P_4(7,5,5)$.

1.5 The following spatial vectors are given: $\mathbf{v}_1 = -3\,\boldsymbol{\imath} + 4\,\boldsymbol{\jmath} - 3\,\mathbf{k}$, $\mathbf{v}_2 = 3\,\boldsymbol{\imath} + 3\,\mathbf{k}$, and $\mathbf{v}_3 = 1\,\boldsymbol{\imath} + 2\,\boldsymbol{\jmath} + 3\,\mathbf{k}$. Find the expressions $\mathbf{E}_1 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{E}_2 = \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$, $\mathbf{E}_3 = (\mathbf{v}_1 \times \mathbf{v}_2) \times \mathbf{v}_3$, and $E_4 = (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3$.

1.6 Find the angle between the vectors $\mathbf{v}_1 = 2\,\boldsymbol{\imath} - 4\,\boldsymbol{\jmath} + 4\,\mathbf{k}$ and $\mathbf{v}_2 = 4\,\boldsymbol{\imath} + 2\,\boldsymbol{\jmath} + 4\,\mathbf{k}$. Find the expressions $\mathbf{v}_1 \times \mathbf{v}_2$ and $\mathbf{v}_1 \cdot \mathbf{v}_2$.

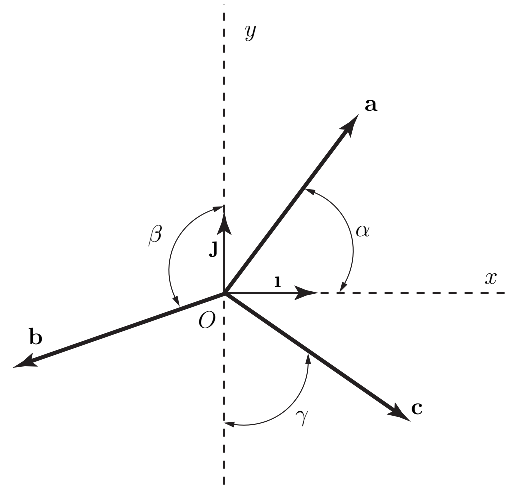


Fig. 1.15 Problem 1.2

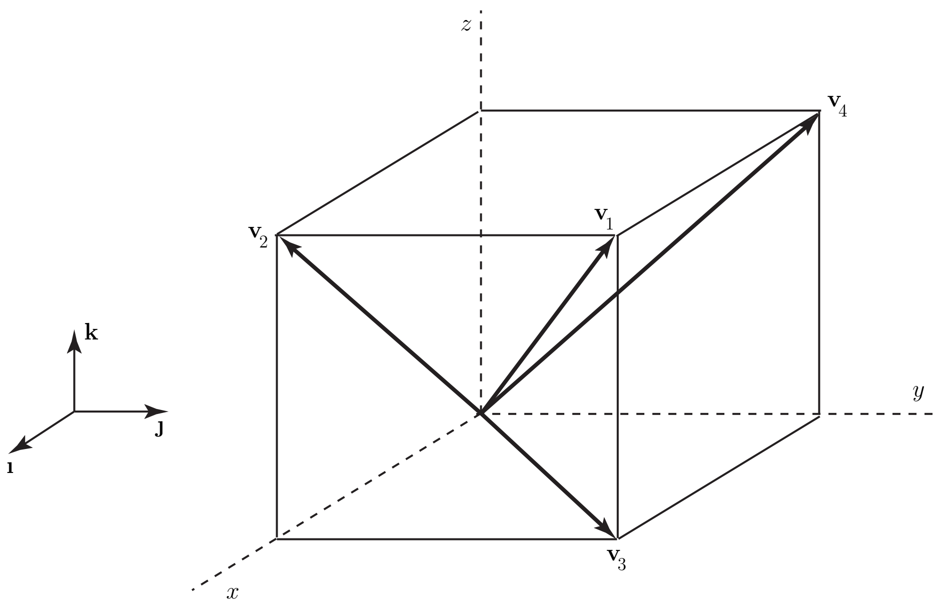


Fig. 1.16 Problem 1.3

1.7 The following vectors are given $\mathbf{v}_1 = 2\,\imath + 4\,\jmath + 6\,\mathbf{k}$, $\mathbf{v}_2 = 1\,\imath + 3\,\jmath + 5\,\mathbf{k}$, and $\mathbf{v}_3 = -2\,\imath + 2\,\mathbf{k}$. Find the vector triple product of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, and explain the result.

1.8 Solve the vectorial equation $\mathbf{x} \times \mathbf{a} = \mathbf{x} \times \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are two known given vectors.

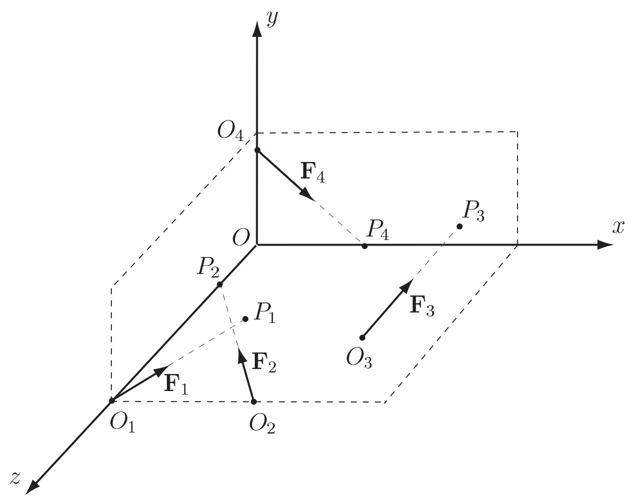


Fig. 1.17 Problem 1.4

1.9 Solve the vectorial equation $\mathbf{v} = \mathbf{a} \times \mathbf{x}$, where $\mathbf{v}$ and $\mathbf{a}$ are two known given vectors.

1.10 Solve the vectorial equation $\mathbf{a} \cdot \mathbf{x} = m$, where $\mathbf{a}$ is a known given vector and m is a known given scalar.