Problem Set 5

Problem 5.1 Product of Inertia of a Cross Section
Determine the product of inertia for the cross sectional area with respect to the $x$- and $y$-axes with origin located at the centroid $C$.

Figure 5.1: Problem 5.1
**Problem 5.2** Product of Inertia Transformations

Determine the moments of inertia $I_u$ and $I_v$ as well as the product of inertia $I_{uv}$ for the channel’s cross section. Use $\theta = 45^\circ$.

![Figure P5.2: Problem 5.2](image-url)
Problem 5.3

The polar moment of inertia of the area shown in Fig. P5.3, is $I_{Czz}$ about the $z$-axis passing through the centroid $C$. If the moment of inertia about the $y'$ axis is $I_{y'y'}$ and the moment of inertia about the $x$-axis is $I_{xx}$. Determine the area $A$. Numerical application: $I_{Czz} = 548 \times 10^6 \text{ mm}^4$, $I_{y'y'} = 383 \times 10^6 \text{ mm}^4$, $I_{xx} = 856 \times 10^6 \text{ mm}^4$, and $h = 250 \text{ mm}$.

Solution

\begin{align*}
I_{Czz} &= 548 \times 10^6; \% \text{ mm}^4 \\
I_{y'y'} &= 383 \times 10^6; \% \text{ mm}^4 \\
I_{xx} &= 856 \times 10^6; \% \text{ mm}^4 \\
h &= 250; \% \text{ mm} \\
I_{xxp} &= I_{xx} - A \times h^2; \\
I_{Czz} &= I_{xxp} + I_{y'y'} \\
\Rightarrow A &= (I_{xx} + I_{y'y'} - I_{Czz})/h^2; \\
A &= (I_{xx} + I_{y'y'} - I_{Czz})/h^2; \\
A &= 1.11 \times 10^4 \text{ (mm}^2) 
\end{align*}
Problem 5.4
Determine the product of inertia of the area with respect to the \( x \) and \( y \) axes.

![Figure P5.4: Problem 5.4](image)

Solution

\[ y = \frac{x}{4}(x-8); \]

\[
\begin{align*}
\text{\% using a thickness } dx \\
\text{\% area of the differential element parallel to } y\text{-axis} \\
\text{\% height of the differential element is } 0 - y = -y \\
\text{\% } dA = -y \ dx = -(x/4)*(x-8) \ dx \\
\text{\% coordinates of the centroid } C \text{ of the differential element} \\
\ x_\_ = x; \\
\ y_\_ = y/2; \ y_\_ = (x/4)*(x-8)/2; \\
\text{\% product of inertia of the differential element} \\
\text{\% with respect to } x \text{ and } y\text{-axes is} \\
\text{\% } dI_{xy} = dIC_{xy} + x_\_ \ast y_\_ \ast dA = 0 + x_\_ \ast y_\_ \ast dA \\
\ dI_{xy} = x_\_ \ast y_\_ \ast (-y); \\
\text{\% } dI_{xy} = x_\_ \ast y_\_ \ast dA = x_\_ \ast y_\_ \ast (-y) \ast dx \\
\text{\% } dI_{xy} = [-(x^3*(x - 8)^2)/32] \ dx \\
\text{\% } I_{xy} = \int(dI_{xy}) \text{ where } 0<x<4 \\
I_{xy} = \int(dI_{xy}, x, 0, 4); \\
\text{\% } I_{xy} = -46.933 \ (in^4) 
\end{align*}
\]
Problem 5.5

Determine the orientation of the principal axes, which have their origin at centroid $C$ of the beam’s cross-sectional area. Also, find the principal moments of inertia.

\[ \text{Solution} \]

% moment and product of inertia with respect to x and y-axes
% the perpendicular distances measured from each subdivided
% segment to the x and y-axes are indicated in figure (a)
% applying the parallel axis theorem

\[
\begin{align*}
I_{xx} &= 2 \times \left( \frac{1}{12} \times 80 \times 20^3 + 80 \times 20 \times 140^2 \right) + \frac{1}{12} \times 20 \times 300^3; \\
&= 1.078 \times 10^8 \text{ (mm}^4) \\
I_{yy} &= 2 \times \left( \frac{1}{12} \times 20 \times 80^3 + 20 \times 80 \times 50^2 \right) + \frac{1}{12} \times 300 \times 20^3; \\
&= 9.907 \times 10^6 \text{ (mm}^4) \\
I_{xy} &= 80 \times 20 \times (-50) \times 140 + 80 \times 20 \times 50 \times (-140); \\
&= -2.24 \times 10^7 \text{ (mm}^4) \\
I_{\text{max}} &= \frac{I_{xx} + I_{yy}}{2} + \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}; \\
&= \text{principal moment of inertia} \\
I_{\text{min}} &= \frac{I_{xx} + I_{yy}}{2} - \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}; \\
&= \text{principal moment of inertia}
\end{align*}
\]
\% Imax = 1.13e+08 (mm^4) \\
\% Imin = 5.03e+06 (mm^4)

\% orientation of principal axes \\
\% tan(2*theta)) = -Ixy/((Ixx-Iyy)/2) \\
theta = atand(-Ixy/((Ixx-Iyy)/2))/2; \\
\% tan(2*theta)) = 0.458 \\
\% theta = 12.3 (deg) and theta = -77.7 (deg)
**Problem 5.6**

Determine the mass moment of inertia $I_{yy}$ of the slender rod. The rod is made of material having a variable density $\rho = \rho_0(1 + x/l)$, where $\rho_0$ is constant. The cross-sectional area of the rod is $A$. Express the result in terms of the mass $m$ of the rod.

![Figure P5.6: Problem 5.6](image)

**Solution**

```matlab
rho = rho0*(1+x/l);
% mass of the differential element parallel to x-axis
% dm = rho dV = rho (A dx) = rho0 A (1+x/l) dx;
% A is the cross-sectional area of the rod
% mass of the is determined by integrating dm
% m = int(dm) = int (rho0 A (1+x/l) dx) where 0<x<l
m = int(rho*A, x, 0, l);
% m = (3*A*l*rho0)/2
% mass moment of inertia of the differential element about y-axis
% dIyy = r^2 dm = x^2 dm where r = x
% dIyy = x^2*rho*A;
% mass moment of inertia can be determined by integrating dIyy
% Iyy = int(dIyy) where 0<x<l
Iyy = int(x^2*rho*A, x, 0, l);
% Iyy = (7*A*l^3*rho0)/12

% m = (3*A*l*rho0)/2 => A*l*rho0 = 2*m/3
% => Iyy = Izz = 7 m l^2/18
```
Problem 5.7

The thin plate has a mass per unit area of 10 kg/m\(^2\). Determine its mass moment of inertia about the \(z\) axis.

![Figure P5.7: Problem 5.7](image)

Solution

% composite parts: the thin plate can be subdivided into four segments
% segments 3 and 4 are holes and should be considered as negative parts
% mass for the segments
m1 = 0.4*0.4*10; % kg
m2 = m1;
% m1 = m2 = 1.6 kg
m3 = pi*0.1^2*10; % kg
m4 = m3;
% m3 = m4 = 0.1 pi kg
% mass moment of inertia of each segment about \(z\)-axis can be determined
% using parallel-axis theorem
% \(I_{zz} = I_{Czz} + m d^2\)

\[
I_{zz} = \frac{1}{12} \times 1.6 \times 0.4^2 + \ldots
\]
\[
= \frac{1}{12} \times 1.6 \times (0.4^2 + 0.4^2) + 1.6 \times 0.2^2 - \ldots
\]
\[
= \frac{1}{4} \times (0.1 \times \pi) \times 0.1^2 - \ldots
\]
\[
= \frac{(1/2) \times (0.1 \times \pi)}{0.1^2 + 0.1 \times \pi \times 0.2^2}
\]
% \(I_{zz} = 0.1131\) (kg m\(^2\))