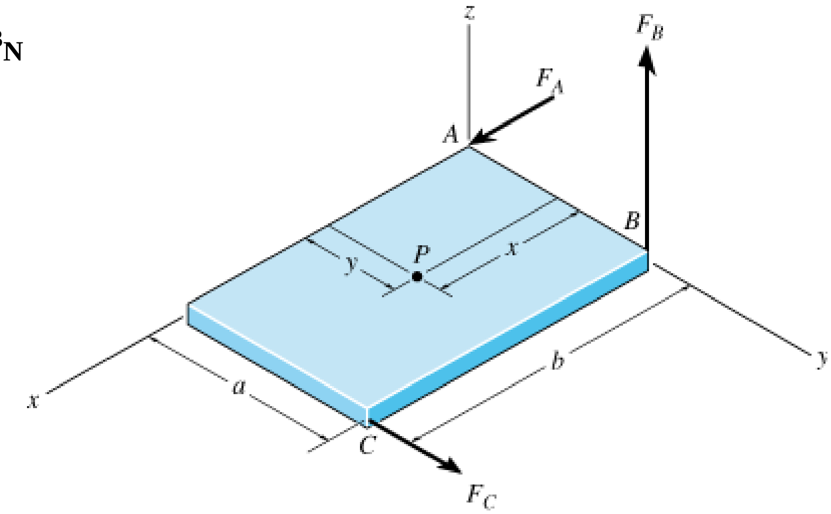
Replace the three forces acting on the plate by a wrench. Specify the magnitude of the force and couple moment for the wrench and the point *P(x, y)* where its line of action intersects the plate.



Units Used: $kN := 10^3 N$

Given :

$F_A := 500N$ $\quad a := 4m$

$F_B := 800N$ $\quad b := 6m$

$F_C := 300N$

Solution:

$$F_R := \begin{pmatrix} F_A \\ F_C \\ F_B \end{pmatrix}$$

Guesses $\quad x := 1m \quad y := 1m$

$M := 100N \cdot m$

Given

$$M \cdot \frac{F_R}{|F_R|} + \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \times F_R = \begin{pmatrix} b \\ a \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ F_C \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ F_B \end{pmatrix}$$

$$\begin{pmatrix} M \\ x \\ y \end{pmatrix} := \text{Find}(M, x, y) \qquad M = 3.07\, kN \cdot m \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.163 \\ 2.061 \end{pmatrix} m$$

1. Find the x-coordinate of the centroid of the indicated region.

A = 2 m

$k = p/8\ m^{-1}$

Curve intersects with the X-axis at $X = X_0$ when $k X_0 = p/2$, at:

$X_0 = p/(2 k) = p/(2\ p/8\ m^{-1}) = 4\ m$

Performing the required integrations using vertical strips of width dX and height Y(X) = A cos(kX):

$$\text{Area} = \int_0^{X0} A \cos(k X)\, dX$$
$$= (A/k) \sin(k X) \Big|_0^{X0}$$
$$= (A/k) \sin(k X_0)$$
$$= (A/k) = 2\ m / (p/8\ m^{-1}) = 16/p\ m^2$$

The first moment about the y-axis:

$$M_y = \int_0^{X0} X A \cos(k X)\, dX$$

Integrating by parts:

u = X du = dX dv = A cos(k X) dX v = (A/k) sin(k X)

$$M_y = X (A/k) \sin(k X) \Big|_0^{X0} - \int_0^{X0} (A/k) \sin(k X)\, dX$$
$$= X (A/k) \sin(k X) + (A/k^2) \cos(k X) \Big|_0^{X0}$$
$$= X_0 (A/k) \sin(k X_0) + (A/k^2) \{ \cos(k X_0) - 1 \}$$

Recalling that $k X_0$ is equal to p/2:

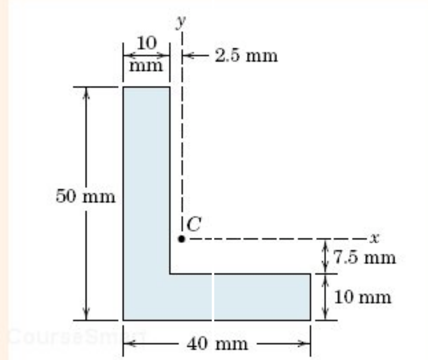
$$M_y = X_0 (A/k) - (A/k^2)$$
$$= 4\ m\ \ 16/p\ m^2 - 2\ m / (p/8\ m^{-1})^2$$
$$= [\ 64/p - 128/p^2\ ]\ m^3$$

The X-coordinate of the centroid:

$$X_C = M_y/\text{Area}$$
$$= [\ 64/p - 128/p^2\ ] / 16/p \quad m$$
$$= [\ 4 - 8/p\ ]\ m$$
$$= 1.454\ m$$

Sample Problem A/11

Determine the orientation of the principal axes of inertia through the centroid of the angle section and determine the corresponding maximum and minimum moments of inertia.



Solution. The location of the centroid C is easily calculated, and its position is shown on the diagram.

Products of Inertia. The product of inertia for each rectangle about its centroidal axes parallel to the x-y axes is zero by symmetry. Thus, the product of inertia about the x-y axes for part I is

$[I_{xy} = \bar{I}_{xy} + d_x d_y A] \qquad I_{xy} = 0 + (-12.5)(+7.5)(400) = -3.75(10^4)\ \text{mm}^4$

where $\qquad d_x = -(7.5 + 5) = -12.5$ mm

and $\qquad d_y = +(20 - 10 - 2.5) = 7.5$ mm

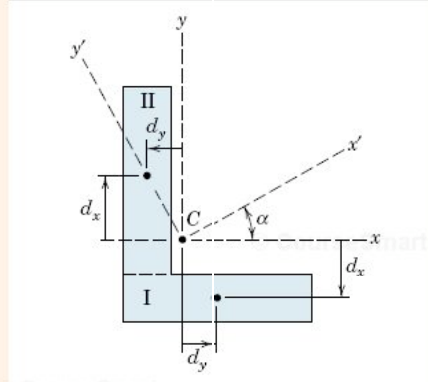
Likewise for part II,

$[I_{xy} = \bar{I}_{xy} + d_x d_y A] \qquad I_{xy} = 0 + (12.5)(-7.5)(400) = -3.75(10^4)\ \text{mm}^4$

where $\quad d_x = +(20 - 7.5) = 12.5$ mm, $\qquad d_y = -(5 + 2.5) = -7.5$ mm

For the complete angle

$$I_{xy} = -3.75(10^4) - 3.75(10^4) = -7.50(10^4)\ \text{mm}^4$$



Moments of Inertia. The moments of inertia about the x- and y-axes for part I are

$$[I = \bar{I} + Ad^2] \qquad I_x = \tfrac{1}{12}(40)(10)^3 + (400)(12.5)^2 = 6.58(10^4)\ \text{mm}^4$$
$$I_y = \tfrac{1}{12}(10)(40)^3 + (400)(7.5)^2 = 7.58(10^4)\ \text{mm}^4$$

and the moments of inertia for part II about these same axes are

$$[I = \bar{I} + Ad^2] \qquad I_x = \tfrac{1}{12}(10)(40)^3 + (400)(12.5)^2 = 11.58(10^4)\ \text{mm}^4$$
$$I_y = \tfrac{1}{12}(40)(10)^3 + (400)(7.5)^2 = 2.58(10^4)\ \text{mm}^4$$

Thus, for the entire section we have

$$I_x = 6.58(10)^4 + 11.58(10)^4 = 18.17(10^4)\ \text{mm}^4$$
$$I_y = 7.58(10^4) + 2.58(10^4) = 10.17(10^4)\ \text{mm}^4$$

Helpful Hint

Mohr's circle. Alternatively we could use Eqs. A/11 to obtain the results for I_{max} and I_{min}, or we could construct the Mohr circle from the calculated values of I_x, I_y, and I_{xy}. These values are spotted on the diagram to locate points A and B, which are the extremities of the diameter of the circle. The angle 2α and I_{max} and I_{min} are obtained from the figure, as shown.

Principal Axes. The inclination of the principal axes of inertia is given by Eq. A/10, so we have

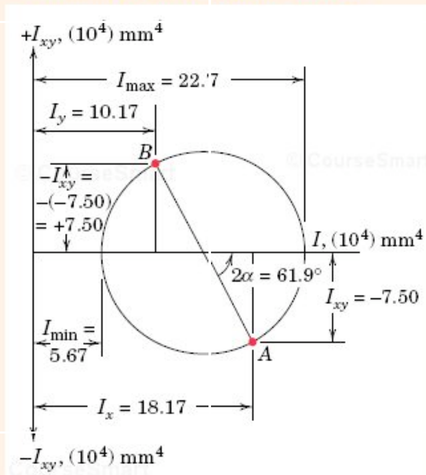
$$\left[\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}\right] \qquad \tan 2\alpha = \frac{2(-7.50)}{10.17 - 18.17} = 1.875$$

$$2\alpha = 61.9^\circ \qquad \alpha = 31.0^\circ \qquad \textit{Ans.}$$

We now compute the principal moments of inertia from Eqs. A/9 using α for θ and get I_{max} from $I_{x'}$ and I_{min} from $I_{y'}$. Thus,

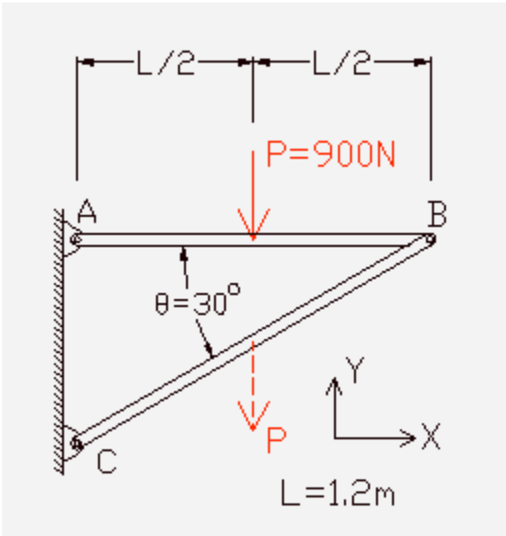
$$I_{max} = \left[\frac{18.17 + 10.17}{2} + \frac{18.17 - 10.17}{2}(0.471) + (7.50)(0.882)\right](10^4)$$
$$= 22.7(10^4)\ \text{mm}^4 \qquad \textit{Ans.}$$

$$I_{min} = \left[\frac{18.17 + 10.17}{2} - \frac{18.17 - 10.17}{2}(0.471) - (7.50)(0.882)\right](10^4)$$
$$= 5.67(10^4)\ \text{mm}^4 \qquad \textit{Ans.}$$



Given: Two members, connected as shown below, with two possible loading conditions. In case (a), a load of magnitude P equal to 900 N is applied vertically downward to the midpoint of the upper member. In case (b), a load of that same magnitude, P, is applied vertically downward to the midpoint of the lower member.

Find: The force exerted by the lower member on the upper member in both cases.



0. Observations:

1. The weights of the two members are negligible. The two members are connected to each other and the ground by pins. In case (a), the only loads applied to the lower member are at its two pin connection points. In case (a) the lower member is a two force member. Any force exerted on or by that member in case (a) must be parallel to the line between its two connection points. In case (b), the only loads applied to the upper member at its two pin connection points. In case (b) the upper member is a two force member. Any force exerted on or by that member in case (b) must be parallel to the line between the its two connection points.

2. The upper member is horizontal.

3. The lower member makes an angle of 30 degrees with the horizontal.

4. The horizontal length of the upper member, L, is equal to 1.2 m. This must also be the horizontal span of the lower member. The vertical span of the lower member must be L tan(30^o).

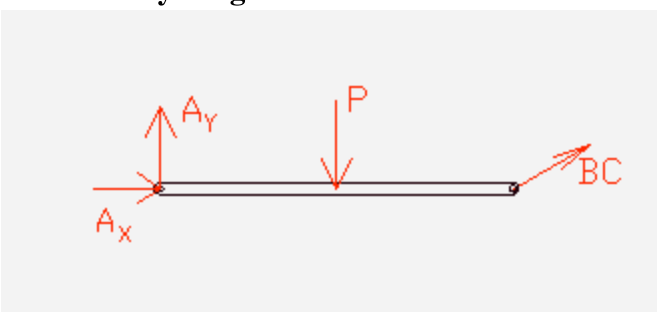
5. In case (a) we know the magnitude and direction of the load P applied to the upper member. We also know the direction of the force applied by the lower member onto the upper member (lower member is a two force member). In order to determine the magnitude of that force, it seems most

appropriate to consider the upper member as the mechanical system. As we are not interested in the forces at the pin A at the left end of the upper member, the equation obtained by summing moments about that pin is most useful. We note that pushing down on the upper member should tend to compress the lower member. We expect the force from the lower member onto the upper member to resist this tendency and exhibit an upward vertical component.

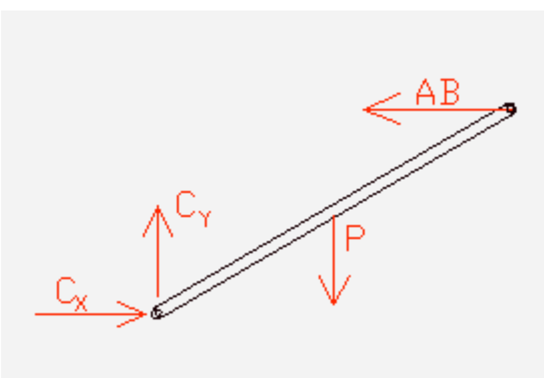
6. In case (b) we know the magnitude and direction of the load P applied to the lower member. We also know the direction of the force applied by the upper member onto the lower member (upper member is a two force member). In order to determine the magnitude of that force, it seems most appropriate to consider the lower member as the mechanical system. As we are not interested in the forces at the pin C at the left end of the lower member, the equation obtained by summing moments about that pin is most useful. Once we determine the force exerted by the upper member on the lower member, the law of action and reaction tells us that the force exerted by the lower member onto to the upper member is equal in magnitude, opposite in direction, and collinear. We note that pushing down on the middle of the lower member would give it a tendency to rotate clockwise and in doing so, tend to stretch the upper member. We expect the force from the upper member onto the lower member to resist this tendency and exhibit a horizontal component to the left.

1. Mechanical System - Upper member (AB) in case (a). Lower member (BC) in case (b).

2. Free Body Diagram



This is the free body diagram of member AB for case (a) where the force is applied to this member. The supporting pin forces at A are shown. The diagonal member, BC, is a two force member believed to be in compression. Its resistance to this compression is reflected in a force acting onto member AB with an upward trend. For this reason the force exerted by that member on to AB (force labeled BC in free body) is shown acting diagonally upward along the 30^o angle made by that member. The coordinate axes are defined in the system definition figure provided above.



This figure shows the free body diagram of member BC for case (b) where the force is applied to this member. The supporting pin forces at C are shown. The horizontal member, AB, is a two force member believed to be in tension. Its resistance to this tension is reflected in a force acting on member BC with a leftward trend. For this reason the force exerted by that member on to BC (force labeled AB in free body) is shown acting horizontally to the left. The coordinate axes are defined in the system definition figure provided above.

3. Equations

Case (a) - Upper member

S $F_X = A_X$ + BC cos(30^o) = 0 { Included for completeness, not used }

S $F_Y = A_Y$ -P + BC sin(30^o) = 0 { Included for completeness, not used }

Summing moments about the left end of the upper member (A):

S M_A = -P*Ł2 + L* BC sin(30^o) = 0 { vertical load P is a horizontal (perpendicular) distance of half the length of the member from A. The direction of rotational tendency about point A associated with this force is clockwise or negative. The horizontal component of BC passes through point A. The vertical component of BC is a horizontal (perpendicular) distance of the length of the member from A. The direction of rotational tendency about the point A associated with this force component is counter-clockwise or positive. The forces at A produce no moment about point A }

Case (b) - Lower member

S $F_X = C_X$ - AB = 0 { Included for completeness, not used }

S $F_Y = C_Y$ -P = 0 { Included for completeness, not used }

Summing moments about the left end of the lower member (C):

S M_C = -P*Ł2 + L tan(30^o) AB = 0 { vertical load P is a horizontal (perpendicular) distance of half the horizontal span of the member from C. The direction of rotational tendency about point C associated with this force is clockwise or negative. The horizontal force AB is a vertical (perpendicular) distance equal to the vertical span of the lower member (L tan(30^o)) from C. The direction of rotational tendency about point C associated with this force is counter-clockwise or positive. The forces at C produce no moment about the point C }

4. Solve

Case (a) - From the moment equation:

L* BC sin(30^o) = P * L /2

BC = P / (2 sin(30^o))
 = P / (2*1/2)
 = P = 900 N

Given the fact that the force makes an angle of 30^o with the horizontal, and noting that our positive result implies that the direction shown on the free body diagram is correct (thus member BC is in compression in case (a) as we speculated above):

Force vector by lower member on upper member = 900 N { cos(30^o) **i** + sin(30^o) **j** } N
 = 779 N **i** + 450 N **j**

Case (b) - From the moment equation

L tan(30^o) AB = P Ł2

AB = P / (2 tan(30^o))
 = P / (2 / $3^{1/2}$)
 = 900 N / (2 / $3^{1/2}$)
 = 779 N

The positive sign of our result indicates that the direction shown on the free body diagram is correct (thus member AB is in tension in case (b) as we speculated above). Thus the force exerted by the upper member on the lower member is of magnitude 779 N and is in the negative X-direction. From the law of action and reaction, the force of the lower member onto the upper member (the force of interest), is equal in magnitude, opposite in direction, and collinear. Thus:
Force vector by lower member on upper member = 779 N **i**

Results
Force exerted by lower member (BC) onto the upper member (AB):

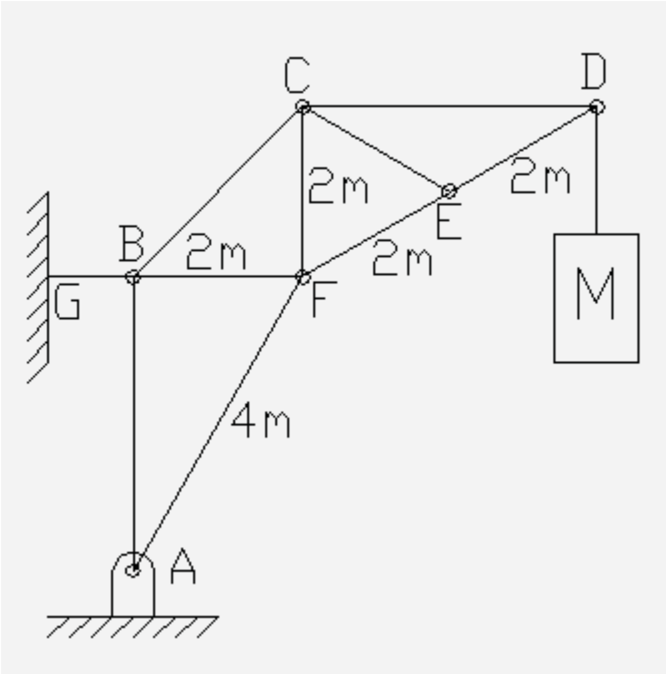
Case (a) Load applied to upper member by lower member = 779 N **i** + 450 N **j**

Case (b) Load applied to upper member by lower member = 779 N **i**

This problem is an excellent introduction to structural engineering and mechanics of materials. Case (b) offers several advantages. The load transmitted between the members in this case is smaller than it is in case (a). Most metallic structural members tend to perform better under tension than they do under compression. In case (a), the lower member is in compression while the upper member will exhibit a combination of bending and tension (wait for your mechanics of materials class to get the whole story here). In case (b), the upper member is in tension, while the lower member will exhibit a combination of bending and compression (again you must wait for your mechanics of materials course for the “rest of the story”. Make sure you bring this problem to the attention of Paul Harvey, or whomever teaches you mechanics of materials).

Given: The truss shown below supporting a mass M equal to 500 kg.

Find: The force in each member of the truss.



0. Observations:

A. All connections between the members are pins. Each member is connected at two points. All external loads are applied at connection points. The weights of the members are small compared to the external loads. The system may be classifed as a truss. All members are two force members, transmitting force along the line between the member connection points.

B. In order to determine the directions of all of the members (and hence all of the forces), we need to determine the coordinates of each of the pins. Setting our origin at point A, X positive to the right, Y positive upward, all coordinates in meters:

A (0,0)

B $(0,(4^2-2^2)^{1/2}) = (0, 12^{1/2})$

C $(2,12^{1/2}+2)$

D $(2+12^{1/2},12^{1/2}+2)$

F $(2,12^{1/2})$

E $(2+3^{1/2},12^{1/2}+1)$ { midway between D and F, average of those two points)

G $(X_G, 12^{1/2})$ { where X_G is some unspecified negative number. }

C. From the above coordinates we can determine the direction of each member (and hence each force). This can be done by subtracting the coordinates of two points along the line of action of the force and then evaluating the unit vector parallel to that direction.
X-Direction: BF, BG, and CD
Y-Direction: AB and CF

$\mathbf{e}_{EF}$ and $\mathbf{e}_{DE}$ = +- { $3^{1/2}/2\ \mathbf{i} + 0.5\ \mathbf{j}$ } { 30 degree angle }

$\mathbf{e}_{BC}$ = +- { $1/2^{1/2}\ \mathbf{i} + 1/2^{1/2}\mathbf{j}$ } { 45 degree angle }

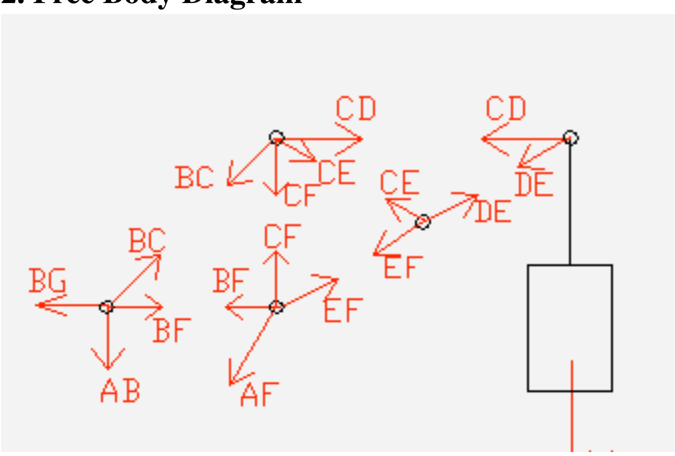
$\mathbf{e}_{AF}$ = +- { $0.5\ \mathbf{i} + 3^{1/2}/2\ \mathbf{j}$ } { 60 degree angle }

$\mathbf{e}_{CE}$ = +- { $3^{1/2}/2\ \mathbf{i} - 0.5\ \mathbf{j}$ } { -30 degree angle }

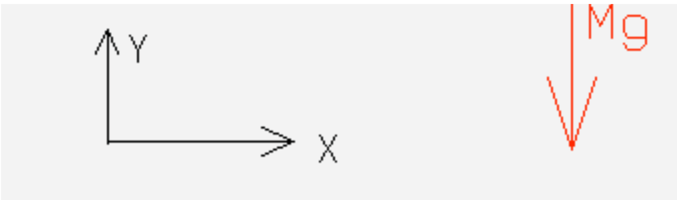
D. We are asked to determine the force in each and every member of the truss. This will require the consideration of several mechanical systems. One approach would be to successively consider different connecting pins. If we can sequentially identify pins that are connected to no more than two members transmitting loads that have not yet been determined, we can readily determine the forces in those members (method of joints). In so doing, we must exploit the fact that the force transmitted by a two force member is along the line between the two connection points. We must further exploit the fact that the forces acting at the two ends of the member are oppositely directed. We can see that considering the pins in the following sequence will enable us to determine the member forces as follows:
Pin D (Members CD and DE)
Pin E (Members CE and EF)
Pin C (members CF and BC)
Pin F (Members BF and AF)
Pin B (Members BG and AF)
For convenience we will assume all members to be in tension. In this way any negative results will indicate a member transmitting a compressive force.

1. Mechanical System - Successive consideration of 5 mechanical systems, beginning with pin D and following sequentially with pins E, C, F and B (each considered individually). Note that the mechanical system including pin D also includes the supported mass and the cable that connects pin D to the mass. The other mechanical system include only the pins.

2. Free Body Diagram



The figure provides the free body diagrams of the required five mechanical systems. The coordinate axes used are shown. Note that as every member of a truss is a two force member, each of the forces exerted by a truss member on a pin, is parallel to the member. Further note that the forces exerted by either end of a two force member must be equal in magnitude and opposite in direction. Both of these facts are reflected in the free body diagrams. Additionally, all forces are shown pulling the



pins closer together. This reflects an assumption that each and every member of the truss is in tension (being stretched) and is exerting a resisting force pulling the attached pins toward one another. This is convenient as a positive result for any member force indicates tension, while a negative result indicates compression in that member. This approach is reflected in the signs of the various terms appearing in the equilibrium equations for each of the pins. As noted above, by sequentially considering pins D, E, C, F, and B, all member forces can be evaluated.

3. Equations

Note the use of the various unit vectors in expressing the force components. The force components are obtained by multiplying the magnitude of the force by the corresponding component of the associated unit vector. The choice of the positive or negative sign is made based upon the free body diagrams of the pins.

Pin D:

$S\ F_X = -CD - 3^{1/2}/2\ DE = 0$

$S\ F_Y = -\ 1/2\ DE - M\ g = 0$

Pin E

$S\ F_X = 3^{1/2}/2\ DE - 3^{1/2}/2\ EF - 3^{1/2}/2\ CE = 0$

$S\ F_Y = 1/2\ DE - 1/2\ EF + 1/2\ CE = 0$

Pin C

$S\ F_X = CD + 3^{1/2}/2\ CE - 1/2^{1/2}\ BC = 0$

$S\ F_Y = -1/2\ CE - CF - 1/2^{1/2}\ BC = 0$

Pin F

$S\ F_X = -BF + 3^{1/2}/2\ EF - 1/2\ AF = 0$

$S\ F_Y = CF + 1/2\ EF - 3^{1/2}/2\ AF = 0$

Pin B

$S\ F_X = BF - BG + 1/2^{1/2}\ BC = 0$

$S\ F_Y = -AB + 1/2^{1/2}\ BC = 0$

4. Solve

From the Y equation for pin D:

$DE = -\ 2\ M\ g = -\ 2 * 500\ kg\ 9.81\ m/s^2 = -9810\ N$

DE = 9.81 kN compression

From the X equation for pin D:
$CD = - 3^{1/2}/2\ DE = 8500$ N
CD = 8.50 kN tension.

Multiplying the Y equation for pin E by the square root of 3, and then subtracting the X equation for pin E from that result, we observe that all of the terms except the one involving CE vanish:
$3^{1/2}\ CE = 0$
CE = 0

Using this result in the y equation for pin E:
EF = DE = -9810 N
EF = 9.81 kN compression

From the X equation for pin C:
$BC = 2^{1/2}\ CD + (3/2)^{1/2}\ CE = 12010$ N
BC = 12.01 kN tension

From the Y equation for pin C:
$CF = -1/2\ CE - 1/2^{1/2}\ BC = -8500$ N
CF = 8.50 kN compression

From the Y equation for pin F:
$AF = 1/3^{1/2}\ EF + 2/3^{1/2}\ CF = -15470$ N
AF = 15.47 kN compression

From the X equation for pin F:
$BF = 3^{1/2}/2\ EF - 1/2\ AF = -759$ N
BF = 0.759 kN compression

From the X equation for pin B
$BG = BF + 1/2^{1/2}\ BC = 7740$ N
BG = 7.74 kN tension

From the Y equation for pin B:
$AB = 1/2^{1/2}\ BC = 8500$ N
AB = 8.50 kN tension

Results
DE = 9810 N compression
CD = 8.50 kN tension
CE = 0
EF = 9.81 kN compression
BC = 12.01 kN tension
CF = 8.50 kN compression
AF = 15.47 kN compression

BF = 0.759 kN compression
BG = 7.74 kN tension
AB = 8.50 kN tension