Chapter 1
Vector Algebra

1.1 Terminology and Notation

Scalars are mathematics quantities that can be fully defined by specifying their magnitude in suitable units of measure. Mass is a scalar quantity and can be expressed in kilograms, time is a scalar and can be expressed in seconds, and temperature is a scalar quantity that can be expressed in degrees Celsius.

Vectors are quantities that require the specification of magnitude, orientation, and sense. The characteristics of a vector are the magnitude, the orientation, and the sense.

The magnitude of a vector is specified by a positive number and a unit having appropriate dimensions. No unit is stated if the dimensions are those of a pure number.

The orientation of a vector is specified by the relationship between the vector and given reference lines and/or planes.

The sense of a vector is specified by the order of two points on a line parallel to the vector.

Orientation and sense together determine the direction of a vector.

The line of action of a vector is a hypothetical infinite straight line collinear with the vector.

Displacement, velocity, and force are examples of vectors quantities.

To distinguish vectors from scalars it is customary to denote vectors by boldface letters. Thus, the displacement vector from point \( A \) to point \( B \) could be denoted as \( \mathbf{r} \) or \( \mathbf{r}_{AB} \). The symbol \( |\mathbf{r}| = r \) represents the magnitude (or module, norm, or absolute value) of the vector \( \mathbf{r} \). In handwritten work a distinguishing mark is used for vectors, such as an arrow over the symbol, \( \vec{r} \) or \( \vec{AB} \), a line over the symbol, \( \overline{r} \), or an underline, \( \underline{\mathbf{r}} \).

The vectors are most frequently depicted by straight arrows. A vector represented by a straight arrow has the direction indicated by the arrow. The displacement vector from point \( A \) to point \( B \) is depicted in Fig. 1.1(a) as a straight arrow. In some cases it is necessary to depict a vector whose direction is perpendicular to the surface.
in which the representation will be drawn. Under this circumstance the use of a portion of a circle with a direction arrow is useful. The orientation of the vector is perpendicular to the plane containing the circle and the sense of the vector is the same as the direction in which a right-handed screw moves when the axis of the screw is normal to the plane in which the arrow is drawn and the screw is rotated as indicated by the arrow. Figure 1.1(b) uses this representation to depict a vector directed out of the reading surface toward the reader.

![Fig. 1.1 Representations of vectors](image1)

A *bound* vector is a vector associated with a particular point $P$ in space (Fig. 1.2). The point $P$ is the *point of application* of the vector, and the line passing through $P$ and parallel to the vector is the line of action of the vector. The point of application may be represented as the tail, Fig. 1.2(a), or the head of the vector arrow, Fig. 1.2(b). A *free* vector is not associated with any particular point in space. A *transmissible* (or *sliding*) vector is a vector that can be moved along its line of action without change of meaning.

![Fig. 1.2 Bound or fixed vector: (a) point of application represented as the tail of the vector arrow and (b) point of application represented as the head of the vector arrow](image2)

To move the rigid body in Fig. 1.3 the force vector $\mathbf{F}$ can be applied anywhere along the line $\Delta$ or may be applied at specific points $A$, $B$ and $C$. The force vector $\mathbf{F}$ is a transmissible vector because the resulting motion is the same in all cases.
If the body is not rigid, the force $F$ applied at $A$ will cause a different deformation of the body than $F$ applied at a different point $B$. If one is interested in the deformation of the body, the force $F$ positioned at $C$ is a bound vector.

The operations of vector analysis deal only with the characteristics of vectors and apply, therefore, to bound, free, and transmissible vectors.

**Equality**

Two vectors $a$ and $b$ are said to be equal to each other when they have the same characteristics. One then writes

$$a = b.$$  \hfill (1.1)

Equality does not imply physical equivalence. For instance, two forces represented by equal vectors do not necessarily cause identical motions of a body on which they act.

**Product of a Vector and a Scalar**

The product of a vector $v$ and a scalar $s$, $sv$ or $vs$, is a vector having the following characteristics:

1. Magnitude. $|sv| = |v||s|$, where $|s| = s$ denotes the absolute value (or magnitude, or module) of the scalar $s$.
2. Orientation. $sv$ is parallel to $v$. If $s = 0$, no definite orientation is attributed to $sv$.
3. Sense. If $s > 0$, the sense of $sv$ is the same as that of $v$. If $s < 0$, the sense of $sv$ is opposite to that of $v$. If $s = 0$, no definite sense is attributed to $sv$.

**Zero Vector**

A zero vector is a vector that does not have a definite direction and whose magnitude is equal to zero. The symbol used to denote a zero vector is $\mathbf{0}$.

**Unit Vector**

A unit vector is a vector with magnitude equal to 1. Given a vector $v$, a unit vector $u$ having the same direction as $v$ is obtained by forming the product of $v$ with the reciprocal of the magnitude of $v$

$$u = v \frac{1}{|v|} = \frac{v}{|v|}. \hfill (1.2)$$
**Vector Addition**

The sum of a vector \( \mathbf{v}_1 \) and a vector \( \mathbf{v}_2 \): \( \mathbf{v}_1 + \mathbf{v}_2 \) or \( \mathbf{v}_2 + \mathbf{v}_1 \) is a vector whose characteristics can be found by either graphical or analytical processes. The vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) add according to the parallelogram law: the vector \( \mathbf{v}_1 + \mathbf{v}_2 \) is represented by the diagonal of a parallelogram formed by the graphical representation of the vectors, see Fig. 1.4(a).

![Fig. 1.4](image-url)

**Fig. 1.4** Vector addition: (a) parallelogram law, (b) moving the vectors successively to parallel positions. Vector difference: (c) parallelogram law, (d) moving the vectors successively to parallel positions

The vector \( \mathbf{v}_1 + \mathbf{v}_2 \) is called the **resultant** of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). The vectors can be added by moving them successively to parallel positions so that the head of one vector connects to the tail of the next vector. The resultant is the vector whose tail connects to the tail of the first vector, and whose head connects to the head of the last vector, see Fig. 1.4(b).

The sum \( \mathbf{v}_1 + (-\mathbf{v}_2) \) is called the **difference** of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) and is denoted by \( \mathbf{v}_1 - \mathbf{v}_2 \), see Figs. 1.4(c) and 1.4(d). The sum of \( n \) vectors \( \mathbf{v}_i, i = 1, \ldots, n \),

\[
\sum_{i=1}^{n} \mathbf{v}_i \text{ or } \mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_n
\]

is called the **resultant** of the vectors \( \mathbf{v}_i, i = 1, \ldots, n \).

Vector addition is:

1. commutative, that is, the characteristics of the resultant are independent of the order in which the vectors are added (commutativity law for addition)

\[
\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1.
\]

2. associative, that is, the characteristics of the resultant are not affected by the manner in which the vectors are grouped (associativity law for addition)

\[
\mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3) = (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3.
\]
3. distributive, that is, the vector addition obeys the following laws of distributivity

\[(s_1 + s_2)\mathbf{v} = s_1 \mathbf{v} + s_2 \mathbf{v}\quad \text{and}\quad s(\mathbf{v}_1 + \mathbf{v}_2) = s\mathbf{v}_1 + s\mathbf{v}_2,\]

or equivalent (for the general case)

\[\sum_{i=1}^{n} s_i \mathbf{v} = \sum_{i=1}^{n} (s_i \mathbf{v}) \quad \text{and} \quad s \sum_{i=1}^{n} \mathbf{v}_i = \sum_{i=1}^{n} (s \mathbf{v}_i).\]

Moreover, the characteristics of the resultant is not affected by the manner in which the vector is multiplied with scalars (associativity law for multiplication)

\[s_1 (s_2 \mathbf{v}) = (s_1 s_2) \mathbf{v}.\]

Every vector can be regarded as the sum of \(n\) vectors \((n = 2, 3, \ldots)\) of which all but one can be selected arbitrarily.

**Linear Independence**

If \(\mathbf{v}_i, i = 1, \ldots, n\) are vectors and \(s_i, i = 1, \ldots, n\) are scalars, then a linear combination of the vectors with the scalars as coefficients is defined as \(\sum_{i=1}^{n} s_i \mathbf{v}_i = s_1 \mathbf{v}_1 + \ldots + s_n \mathbf{v}_n.\)

A collection of non-zero vectors is said to be linearly independent if no vector in the set can be written as a linear combination of the remaining vectors in the set. The dimension of the space is equal to the maximum number of non-zero vectors that can be included in a linearly independent set of vectors. Thus for a three-dimensional space the maximum number of non-zero vectors in a linearly independent collection is three. Given a set of three linearly independent vectors, any other vector can be constructed as a resultant of scalar multiplication of the three vectors. Such a set of vectors is called a basis set. A set of vectors which is not linearly independent is called linearly dependent.

**Resolution of Vectors and Components**

Let \(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\) be three linearly independent unit vectors as a basis set:

\[|\mathbf{i}_1| = |\mathbf{i}_2| = |\mathbf{i}_3| = 1.\]

For a given vector \(\mathbf{v}\) (Fig. 1.5), there exist three unique scalars \(v_1, v_1, v_3\), such that \(\mathbf{v}\) can be expressed as

![Fig. 1.5 Resolution of a vector v and components](image-url)
\[ \mathbf{v} = v_1 \mathbf{i}_1 + v_2 \mathbf{i}_2 + v_3 \mathbf{i}_3. \]  

(1.3)

The opposite action of addition of vectors is the \textit{resolution} of vectors. Thus, for the given vector \( \mathbf{v} \) the vectors \( v_1 \mathbf{i}_1, v_2 \mathbf{i}_2, \) and \( v_3 \mathbf{i}_3 \) sum to the original vector. The vector \( v_k \mathbf{i}_k \) is called the \textit{k} \textit{component} of \( \mathbf{v} \) relative to the given basis set and \( v_k \) is called the \textit{k} \textit{scalar component} of \( \mathbf{v} \) relative to the given basis set, where \( k = 1, 2, 3 \). A vector is often replaced by its components since the components are equivalent to the original vector.

Frequently a vector will be given and its components relative to a particular basis set need to be calculated. A trivial example of this situation occurs when the vector to be resolved is the zero vector. Then each of its components are zero. Thus, under these circumstances every vector equation \( \mathbf{v} = 0 \), where \( \mathbf{v} = v_1 \mathbf{i}_1 + v_2 \mathbf{i}_2 + v_3 \mathbf{i}_3 \), is equivalent to three scalar equations \( v_1 = 0, \ v_2 = 0, \ v_3 = 0 \). Note that the zero vector \( \mathbf{0} \) is not the number zero.

If the unit vectors \( \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3 \) are mutually perpendicular they form a \textit{cartesian basis} or a \textit{cartesian reference frame}. For a cartesian reference frame the following notation is used (Fig. 1.6)

\[ \mathbf{i}_1 \equiv \mathbf{i}, \ \mathbf{i}_2 \equiv \mathbf{j}, \ \mathbf{i}_3 \equiv \mathbf{k} \ \text{and} \ \mathbf{i} \perp \mathbf{j}, \ \mathbf{i} \perp \mathbf{k}, \ \mathbf{j} \perp \mathbf{k}. \]

The symbol \( \perp \) denotes perpendicular. When a vector \( \mathbf{v} \) is expressed in the form

\[ \mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k} \]

where \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are mutually perpendicular unit vectors (cartesian reference frame or orthogonal reference frame), the magnitude of \( \mathbf{v} \) is given by

\[ |\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}. \]

(1.4)

The vectors \( \mathbf{v}_x = v_x \mathbf{i}, \ \mathbf{v}_y = v_y \mathbf{j}, \) and \( \mathbf{v}_z = v_z \mathbf{k} \) are the \textit{orthogonal or rectangular component vectors} of the vector \( \mathbf{v} \). The measures \( v_x, v_y, v_z \) are the \textit{orthogonal or rectangular scalar components} of the vector \( \mathbf{v} \).

The resolution of a vector into components frequently facilitate the valuation of a vector equation. If \( \mathbf{v}_1 = v_{1x} \mathbf{i} + v_{1y} \mathbf{j} + v_{1z} \mathbf{k} \) and \( \mathbf{v}_2 = v_{2x} \mathbf{i} + v_{2y} \mathbf{j} + v_{2z} \mathbf{k} \), then the sum of the vectors is
1.1 Terminology and Notation

\[ \mathbf{v}_1 + \mathbf{v}_2 = (v_{1x} + v_{2x}) \mathbf{i} + (v_{1y} + v_{2y}) \mathbf{j} + (v_{1z} + v_{2z}) \mathbf{k} \]

Similarly,
\[ \mathbf{v}_1 - \mathbf{v}_2 = (v_{1x} - v_{2x}) \mathbf{i} + (v_{1y} - v_{2y}) \mathbf{j} + (v_{1z} - v_{2z}) \mathbf{k} \]

In the MATLAB® environment, a three-dimensional row vector \( \mathbf{v} \) is written as a list of variables \( \mathbf{v} = [ v_x, v_y, v_z ] \) or \( \mathbf{v} = [ v_x, \ v_y, \ v_z ] \) where \( v_x, v_y, \) and \( v_z \) are the spatial coordinates of the vector \( \mathbf{v} \). The elements of a row are separated with blanks or commas. The list of elements are surrounded with square brackets, [ ]. The first component of the vector \( \mathbf{v} \) is \( v_x = v(1) \), the second component is \( v_y = v(2) \), and the third component is \( v_z = v(3) \). The colon ; is used to separate the end of each row for a column vector. To create a numerical vector the following statement is used:

\[ \mathbf{p} = [ \ 1 \ 2 \ 3 \ ] \]

where 1, 2, and 3 are the numerical components of the row vector \( \mathbf{p} \). When a variable name is assigned to data, the data is immediately displayed, along with its name. The display of the data can be suppressed by using the semicolon, ;, at the end of a statement.

Based on Maple kernel, symbolic MATLAB Toolbox can perform symbolical calculation and a vector \( \mathbf{v} \) can be expressed in MATLAB in a symbolical fashion. In MATLAB the \texttt{sym} command constructs symbolic variables and expressions. The commands:

\[ \mathbf{v}_x = \text{sym}(\text{'v}_x', 'real'); \]
\[ \mathbf{v}_y = \text{sym}(\text{'v}_y', 'real'); \]
\[ \mathbf{v}_z = \text{sym}(\text{'v}_z', 'real'); \]

create a symbolic variables \( \mathbf{v}_x, \mathbf{v}_y, \) and \( \mathbf{v}_z \) and also assume that the variables are real numbers. The symbolic variables can then be treated as mathematical variables. One can use the statement \texttt{syms} for generating a shortcut for constructing symbolic objects:

\[ \text{syms} \ \mathbf{v}_x \ \mathbf{v}_y \ \mathbf{v}_z \ \text{real} \]
\[ \mathbf{v} = [ \mathbf{v}_x \ \mathbf{v}_y \ \mathbf{v}_z ]; \]

where \( \mathbf{v} \) is a symbolic vector. The same symbolic vector can be created with:

\[ \mathbf{v} = \text{sym}(\text{['v}_x \ \text{'v}_y \ \text{'v}_z']); \]

In MATLAB a vector is defined as a matrix with either one row or one column. To make distinction between row vectors and column vectors is essential, especially when operations with vectors are required. Many errors are caused by using a row vector instead a column vector, or vice versa. The command \texttt{zeros}([m,n]) or \texttt{zeros}([m n]) returns an \( m \)-by-\( n \) matrix of zeros. A zero row vector \[ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \] is generated with \texttt{zeros(1,3)} and a zero column vector is generated with \texttt{zeros(3,1)}. The command \texttt{ones}([m,n]) or \texttt{ones}([m n]) returns an \( m \)-by-\( n \) matrix of ones. In MATLAB two vectors \( \mathbf{u} \) and \( \mathbf{v} \) of the same size (defined either as column vectors or row vectors) can be added together using the next command:
Vectors addition in MATLAB must follow strict rules. The vectors should be either column vectors or row vectors in order to be added and should have the same dimension. It is not possible to add a row vector to a column vector. To subtract one vector from another of the same size, use a minus (-) sign. The subtraction applied to the vectors $u$ and $v$ can be written in MATLAB as:

$$ u - v $$

or

$$ v - u $$

The magnitude of the vector $p$ can be found using the next MATLAB command:

$$ \text{norm}(p) $$

The MATLAB command $\text{norm}(p)$ does not work if the components of the vector $p$ are given symbolically. Thus, a more general MATLAB function is created for the magnitude of the vector, $v$, with the components $v(1), v(2), \text{ and } v(3)$. A MATLAB function is a program that performs an action and returns a result. The MATLAB function $\text{magn}$ calculates the magnitude of the vector, $v$, in a symbolical or numerical fashion:

```matlab
function val = magn(v)
% The symbolic magnitude function of a vector
% $v = [v(1) \ v(2) \ v(3)]$
% The function accepts sym as the input argument
val = sqrt(v(1)*v(1)+v(2)*v(2)+v(3)*v(3));
```

The MATLAB statement $\text{sqrt}(x)$ is the square root of the elements of $x$. The power of MATLAB comes into play when one can add new functions to enhance the language. The m-file function file starts with a line declaring the function, the arguments and the outputs. Next the statements required to produce the outputs from the inputs (arguments) are presented. It is important to note that the argument and output names used in a function file are strictly local variables that exist only within the function itself. The function returns information via the output. To calculate the magnitude of the vector $v = [v_x \ v_y \ v_z]$ using the $\text{magn}$ function the following MATLAB command is used:

$$ \text{mv} = \text{magn}(v) $$

and the output is:

$$ \text{mv} = \frac{(v_x^2+v_y^2+v_z^2)^{1/2}}{\text{norm}(p)} $$

or

$$ \text{mv} = \frac{\text{magn}(v)}{\sqrt{v_x^2+v_y^2+v_z^2}} $$

To create a unit vector in the direction of the vector $v$ the following command is used $p/\text{norm}(p)$ or $v/\text{magn}(v)$ where the division symbol (/) divides all the elements in the vector by the magnitude of the vector, producing a vector of the same size and direction.
Vector transposition is as easy as adding an apostrophe, \', (prime) to the name of the vector. Thus if \[\mathbf{v} = [v_x \ v_y \ v_z]\] then \[\mathbf{v}'\] is:

\[v_x' \ v_y' \ v_z'\]

The mutually perpendicular unit vectors \(\mathbf{i}, \mathbf{j},\) and \(\mathbf{k}\) are defined in MATLAB by:

\[
i=[1 \ 0 \ 0]; \quad j=[0 \ 1 \ 0]; \quad k=[0 \ 0 \ 1];
\]

**Angle Between Two Vectors**

The angle between two vectors can be determined by moving either or both vectors parallel to themselves (leaving the sense unaltered) until their initial points (tails) coincide. This angle will always be in the range between \(0^\circ\) and \(180^\circ\) inclusive. Four possible situations are shown in Fig. 1.7 where the two vectors are denoted \(\mathbf{a}\) and \(\mathbf{b}\). The angle between \(\mathbf{a}\) and \(\mathbf{b}\) is the angle \(\theta\) in Figs. 1.7(a) and 1.7(b).

![Fig. 1.7](image)

*Fig. 1.7* The angle \(\theta\) between the vectors \(\mathbf{a}\) and \(\mathbf{b}\): (a) \(0^\circ < \theta < 90^\circ\), (b) \(90^\circ < \theta < 180^\circ\), and (c) \(\theta = 0^\circ\), and (d) \(\theta = 180^\circ\)

Between \(\mathbf{a}\) and \(\mathbf{b}\) is denoted by the symbols \((\mathbf{a}, \mathbf{b})\) or \((\mathbf{b}, \mathbf{a})\). Figure 1.7(c) represents the case \((\mathbf{a}, \mathbf{b}) = 0\), and Fig. 1.7(d) represents the case \((\mathbf{a}, \mathbf{b}) = 180^\circ\).

The direction of a vector \(\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}\) and relative to a cartesian reference, \(\mathbf{i}, \mathbf{j}, \mathbf{k}\), is given by the cosines of the angles formed by the vector and the respective unit vectors. These are called *direction cosines* and are denoted as (Fig. 1.8)

\[
\cos(\mathbf{v}, \mathbf{i}) = \cos \alpha = \cos \theta_x = l, \quad \cos(\mathbf{v}, \mathbf{j}) = \cos \beta = \cos \theta_y = m, \quad \text{and} \quad \cos(\mathbf{v}, \mathbf{k}) = \cos \gamma = \cos \theta_z = n.
\]

(1.5)

The following relations exist:

\[
v_x = |v| \cos \alpha; \quad v_y = |v| \cos \beta; \quad v_z = |v| \cos \gamma.
\]

From these definitions, it follows that

\[
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \quad \text{or} \quad l^2 + m^2 + n^2 = 1.
\]

(1.6)

Equation (1.6) is proved using the MATLAB commands:

```matlab
syms v_x v_y v_z
v = [v_x v_y v_z];
```

```
```

```
```
\[ mv = \text{magn}(v); \]
\[ l = v_x/mv; \]
\[ m = v_y/mv; \]
\[ n = v_z/mv; \]
\[ \text{simplify}(l^2+m^2+n^2) \]

The MATLAB statement `\text{simplify}(x)` simplifies each element of the symbolic matrix \( x \).

Recall, the formula for the unit vector of the vector \( v \) is

\[ \mathbf{u}_v = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{v}}{v} = \frac{v_x}{v} \mathbf{i} + \frac{v_y}{v} \mathbf{j} + \frac{v_z}{v} \mathbf{k}, \]

or written another way

\[ \mathbf{u}_v = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}. \quad (1.7) \]

### 1.2 Position Vector

The position vector of a point \( P \) relative to a point \( O \) is a vector \( \mathbf{r}_{OP} = \overrightarrow{OP} \) having the following characteristics:

1. magnitude the length of line \( OP \);
2. orientation parallel to line \( OP \);
3. sense \( OP \) (from point \( O \) to point \( P \)).

The vector \( \mathbf{r}_{OP} \) is shown as an arrow connecting \( O \) to \( P \), as depicted in Fig. 1.9(a). The position of a point \( P \) relative to \( P \) is a zero vector.

Let \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) be mutually perpendicular unit vectors (cartesian reference frame) with the origin at \( O \), as shown in Fig. 1.9(b). The axes of the cartesian reference frame are \( x, y, z \). The unit vectors \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are parallel to \( x, y, z \), and they have the senses

![Fig. 1.8 Direction cosines](image-url)
1.3 Scalar (Dot) Product of Vectors

Definition. The scalar (dot) product of a vector \( \mathbf{a} \) and a vector \( \mathbf{b} \) is

\[
\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\mathbf{a}, \mathbf{b}).
\]

(1.11)

For the scalar (dot) product the following rules apply:

1. for any vectors \( \mathbf{a} \) and \( \mathbf{b} \) one can write the commutative law for scalar product

\[
\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.
\]

2. for any two vectors \( \mathbf{a} \) and \( \mathbf{b} \) and any scalar \( s \) the following relation is written

\[
(s \mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (s \mathbf{b}) = s \mathbf{a} \cdot \mathbf{b}.
\]
3. For any vectors \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{c} \) the distributive law in the first argument is

\[
(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c},
\]

and the distributive law in the second argument is

\[
\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.
\]

It can be shown that the dot product is distributive and the following relation can be written

\[
s_a \mathbf{a} \cdot (s_b \mathbf{b} + s_c \mathbf{c}) = s_a s_b \mathbf{a} \cdot \mathbf{b} + s_a s_c \mathbf{a} \cdot \mathbf{c}.
\]

If

\[
\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \quad \text{and} \quad \mathbf{b} = b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k},
\]

where \( \mathbf{i} \), \( \mathbf{j} \), \( \mathbf{k} \) are mutually perpendicular unit vectors, then

\[
\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z. \tag{1.12}
\]

The following relationships exist

\[
\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1,
\]

\[
\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} = 0.
\]

Every vector \( \mathbf{v} \) can be expressed in the form

\[
\mathbf{v} = \mathbf{i} \cdot \mathbf{v} \mathbf{i} + \mathbf{j} \cdot \mathbf{v} \mathbf{j} + \mathbf{k} \cdot \mathbf{v} \mathbf{k}. \tag{1.13}
\]

**Proof.** The vector \( \mathbf{v} \) can always be expressed as

\[
\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}.
\]

Dot multiply both sides by \( \mathbf{i} \)

\[
\mathbf{i} \cdot \mathbf{v} = v_x \mathbf{i} \cdot \mathbf{i} + v_y \mathbf{i} \cdot \mathbf{j} + v_z \mathbf{i} \cdot \mathbf{k}.
\]

But,

\[
\mathbf{i} \cdot \mathbf{i} = 1, \quad \text{and} \quad \mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = 0.
\]

Hence, \( \mathbf{i} \cdot \mathbf{v} = v_x \). Similarly, \( \mathbf{j} \cdot \mathbf{v} = v_y \) and \( \mathbf{k} \cdot \mathbf{v} = v_z \).

The MATLAB command `dot(v, u)` calculates the scalar product (or vector dot product) of the vectors \( \mathbf{v} \) and \( \mathbf{u} \). The dot product of two vectors \( \mathbf{v} \) and \( \mathbf{u} \) can be expressed as:

\[
\text{sum}(\mathbf{v} \cdot \mathbf{u})
\]

The command `sum(x)` with \( x \) defined as a vector, returns the sum of its elements. The MATLAB command `.*`, named *array multiplication* is the element-by-element
1.4 Vector (Cross) Product of Vectors

Definition. The vector (cross) product of a vector \( \mathbf{a} \) and a vector \( \mathbf{b} \) is the vector (Fig. 1.10)

\[
\mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin(\mathbf{a}, \mathbf{b}) \mathbf{n}
\]  

where \( \mathbf{n} \) is a unit vector whose direction is the same as the direction of advance of a right-handed screw rotated from \( \mathbf{a} \) toward \( \mathbf{b} \), through the angle \( \langle \mathbf{a}, \mathbf{b} \rangle \), when the axis of the screw is perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \). The magnitude of \( \mathbf{a} \times \mathbf{b} \) is given by

\[
|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin(\mathbf{a}, \mathbf{b}).
\]

If \( \mathbf{a} \) is parallel to \( \mathbf{b} \), \( \mathbf{a} || \mathbf{b} \), then \( \mathbf{a} \times \mathbf{b} = 0 \). The symbol \( || \) denotes parallel. The relation \( \mathbf{a} \times \mathbf{b} = 0 \) implies only that the product \( |\mathbf{a}| |\mathbf{b}| \sin(\mathbf{a}, \mathbf{b}) \) is equal to zero, and this is the case whenever \( |\mathbf{a}| = 0 \), or \( |\mathbf{b}| = 0 \), or \( \sin(\mathbf{a}, \mathbf{b}) = 0 \).

For any two vectors \( \mathbf{a} \) and \( \mathbf{b} \) and any real scalar \( s \) the following relation can be written

\[
(s \mathbf{a}) \times \mathbf{b} = s(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (s \mathbf{b}) = s \mathbf{a} \times \mathbf{b}.
\]

The sense of the unit vector \( \mathbf{n} \) which appears in the definition of \( \mathbf{a} \times \mathbf{b} \) depends on the order of the factors \( \mathbf{a} \) and \( \mathbf{b} \) in such a way that (cross product is not commutative)

\[
\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}.
\]
The cross product distributive law for the first argument can be written as

\[(a + b) \times c = a \times c + b \times c,\]

while the distributive law for the second argument is

\[a \times (b + c) = a \times b + a \times c.\]

Vector multiplication obeys the following law of distributivity (Varignon theorem)

\[a \times \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} (a \times v_i).\]

A set of mutually perpendicular unit vectors \(\hat{i}, \hat{j}, \hat{k}\) is called right-handed if \(\hat{i} \times \hat{j} = \hat{k}\) (Fig. 1.11). A set of mutually perpendicular unit vectors \(\hat{i}, \hat{j}, \hat{k}\) is called left-handed if \(\hat{i} \times \hat{j} = -\hat{k}\).

If \(a = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}\), and \(b = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}\), where \(\hat{i}, \hat{j}, \hat{k}\) are right-handed mutually perpendicular unit vectors, then \(a \times b\) can be expressed in the following determinant form

\[a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \quad (1.16)\]

The determinant can be expanded by minors of the elements of the first row

\[\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{i} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{j} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} = (a_x b_z - a_z b_x) \hat{i} + (a_y b_z - a_z b_y) \hat{j} + (a_x b_y - a_y b_x) \hat{k}. \quad (1.17)\]

As a general rule a third order determinant can be expanded by diagonal multiplication, i.e., repeating the first two columns on the right side of the determinant, and adding the signed diagonal products of the diagonal elements as
The determinant in Eq. (1.16) can be expanded using the general rule as

\[
\begin{vmatrix}
1 & j & k \\
a_x & a_y & a_z \\
b_x & b_y & b_z
\end{vmatrix}
= -k a_y b_z - j a_z b_x + i a_x b_y + k a_y b_z - j a_z b_x + i a_x b_y
\]

Next a MATLAB function that calculates the cross product of two vectors is presented:

```matlab
function val = crossproduct(a,b)
    % symbolic cross product function of a vector a x b
    a = a(:);
    % a(:) represents all elements of a,
    % regarded as a single column
    b = b(:);
    % b(:) represents all elements of b,
    % regarded as a single column
    val = [a(2,:).*b(3,:)-a(3,:).*b(2,:) ...
           a(3,:).*b(1,:)-a(1,:).*b(3,:) ...
           a(1,:).*b(2,:)-a(2,:).*b(1,:)];
```

In the previous MATLAB function, the general MATLAB command colon (:), i.e., `a(:)`, has been used. The colon (:) is one of the most useful operators in MATLAB. It can create vectors, subscript arrays, and specify for iterations. The ellipses (...) after the command are used to execute the commands together.

### 1.5 Scalar Triple Product of Three Vectors

**Definition.** The scalar triple product of three vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) is defined as

\[
[a, b, c] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}.
\]  

(1.18)

The MATLAB commands for the scalar triple product of three vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) is:

```matlab
syms a_x a_y a_z b_x b_y b_z c_x c_y c_z real
a=[a_x a_y a_z]; b=[b_x b_y b_z]; c=[c_x c_y c_z];
```
\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \]

It does not matter whether the dot is placed between \( \mathbf{a} \) and \( \mathbf{b} \), and the cross between \( \mathbf{b} \) and \( \mathbf{c} \), or vice versa, that is,

\[ [\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}. \tag{1.19} \]

The relation given by Eq. (1.19) is demonstrated using the MATLAB commands:

\begin{verbatim}
% [a,b,c] = a.(b x c)
abc = dot(a, cross(b, c));
% [a,b,c] = (a x b).c
axbc = simplify(dot(cross(a, b), c));
% a.(b x c)==(a x b).c
abxc == axbc
\end{verbatim}

The MATLAB relational operator == or eq is used to compare each element of array for equality. The statement \( \text{LHS} == \text{RHS} \) or \( \text{eq(LHS, RHS)} \) compares each element of the array \( \text{LHS} \) for equality with the corresponding element of the array \( \text{RHS} \), and returns an array with elements set to logical 1 (true) if \( \text{LHS} \) and \( \text{RHS} \) are equal, or logical 0 (false) where they are not equal.

A change in the order of the factors appearing in a scalar triple product at most changes the sign of the product, that is,

\[ [\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{a}, \mathbf{b}, \mathbf{c}] \quad \text{and} \quad [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]. \]

If \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) are parallel to the same plane, or if any two of the vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) are parallel to each other, then \( [\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0 \).

The scalar triple product \( [\mathbf{a}, \mathbf{b}, \mathbf{c}] \) can be expressed in the following determinant form

\[ [\mathbf{a}, \mathbf{b}, \mathbf{c}] = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}. \tag{1.20} \]

In MATLAB the scalar triple product of three vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) is expressed as:

\[
det([[a; b; c]])
\]

where \( \text{det}(x) \) is the determinant of the square matrix \( x \). To verify Eq. (1.20) the following MATLAB command is used:

\[
det([[a; b; c]]) == \text{simplify}(\text{dot}([a, \text{cross(b, c)}]))
\]

**Exercise: Volume of a Parallelepiped**

Figure 1.12 depicts three vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) that form a parallelepiped. Show that the scalar \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \) represents the volume of the parallelepiped with the sides \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \).
1.5 Scalar Triple Product of Three Vectors

Solution
The scalar scalar triple product is \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = |\mathbf{a}| |\mathbf{b}| |\mathbf{c}| \sin \theta \cos \alpha = hA \), where \( h = |\mathbf{a}| \cos \alpha \) represents the height of the parallelepiped and \( A = |\mathbf{b}| |\mathbf{c}| \sin \theta \) represents the area of the parallelogram with the sides \( \mathbf{b} \) and \( \mathbf{c} \). The product between \( h \) and \( A \) represents the volume of a parallelepiped, \( V = hA \), so the scalar \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \) represents the volume of the parallelepiped with the sides formed by the vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \).

Exercise: Vector Expressed in a Base
Let \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \) and \( \mathbf{w} \) be non-zero vectors and \( [\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0 \). The vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \) and \( \mathbf{w} \) are given vectors. The vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) are free vectors and can be moved in a given point. The vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) form the edges of a parallelepiped of non-zero volume. Then the scalars \( s_a, s_b, \) and \( s_c \) exist such as the vector \( \mathbf{w} \) can be represented as a linear combination of the vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \): \( \mathbf{w} = s_a \mathbf{a} + s_b \mathbf{b} + s_c \mathbf{c} \). Show that the scalars \( s_a, s_b, \) and \( s_c \) are given by

\[
\begin{align*}
  s_a &= \frac{[\mathbf{w}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \\
  s_b &= \frac{[\mathbf{a}, \mathbf{w}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \\
  s_c &= \frac{[\mathbf{a}, \mathbf{b}, \mathbf{w}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} 
\end{align*}
\]

Solution
The components of the vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) and the scalars \( s_a, s_b, \) and \( s_c \) are introduced as symbolic variables using MATLAB:

\[
\begin{align*}
  \text{syms} &\ a_x \ a_y \ a_z \ b_x \ b_y \ b_z \ c_x \ c_y \ c_z \ \text{real} \\
  \text{syms} &\ s_a \ s_b \ s_c \ \text{real}
\end{align*}
\]

The vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) are:

\[
\begin{align*}
  \mathbf{a} &= [ a_x \ a_y \ a_z ]; \\
  \mathbf{b} &= [ b_x \ b_y \ b_z ]; \\
  \mathbf{c} &= [ c_x \ c_y \ c_z ];
\end{align*}
\]

and the vector \( \mathbf{w} \) is:

\[
\mathbf{w} = s_a \mathbf{a} + s_b \mathbf{b} + s_c \mathbf{c};
\]

The scalar triple products \( [\mathbf{a}, \mathbf{b}, \mathbf{c}], [\mathbf{w}, \mathbf{b}, \mathbf{c}], [\mathbf{a}, \mathbf{w}, \mathbf{c}], \) and \( [\mathbf{a}, \mathbf{b}, \mathbf{w}] \) are:

\[
\text{abc} = \text{det}([\mathbf{a}; \mathbf{b}; \mathbf{c}]);
\]
wbc = det([w; b; c]);
awc = det([a; w; c]);
abw = det([a; b; w]);

The scalars \( s_a, s_b, \) and \( s_c \) are obtained from: 
\[
\begin{align*}
\frac{\text{w}}{\text{a}} & = \begin{vmatrix} \text{b} & \text{c} \\ \text{a} & \text{c} \end{vmatrix}, & \frac{\text{a}}{\text{w}} & = \begin{vmatrix} \text{b} & \text{c} \\ \text{a} & \text{b} \end{vmatrix}, & \frac{\text{b}}{\text{c}} & = \begin{vmatrix} \text{a} & \text{c} \\ \text{a} & \text{b} \end{vmatrix}
\end{align*}
\]
or:
\[
\begin{align*}
simplify(wbc/abc) & = s_a, \\
simplify(awc/abc) & = s_b, \\
simplify(abw/abc) & = s_c.
\end{align*}
\]

### 1.6 Vector Triple Product of Three Vector

*Definition.* The vector triple product of three vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) is the vector \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \).

The parentheses are essential because \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) is not, in general, equal to \( (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \). For any three vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \)
\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. \tag{1.21}
\]

The previous relation given by Eq. (1.21) can be explained using the MATLAB statements:
\[
\begin{align*}
\% \text{ a x (b x c)} \\
\text{axbxc} = \text{cross(a, cross(b, c))}; \\
\% (a.c)b - (a.b)c \\
\text{RHS} = \text{dot(a, c)*b - dot(a, b)*c}; \\
\% a x (b x c) - (a.c)b + (a.b)c = [0, 0, 0] \\
\text{simplify(axbxc-RHS)}
\end{align*}
\]

### 1.7 Derivative of a Vector Function

The derivative of a vector function is defined in exactly the same way as is the derivative of a scalar function. Thus
\[
\frac{d}{dt} \mathbf{a} = \lim_{\Delta t \to 0} \frac{\mathbf{a}(t + \Delta t) - \mathbf{a}(t)}{\Delta t}.
\]

The derivative of a vector has some of the properties of the derivative of a scalar function. The derivative of the sum of two vector functions \( \mathbf{a} \) and \( \mathbf{b} \) is
\[
\frac{d}{dt} (\mathbf{a} + \mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}. \tag{1.22}
\]
The components of the vectors \( \mathbf{a} \) and \( \mathbf{b} \) are functions of time, \( t \), and are introduced in MATLAB with:

\[
\begin{align*}
\text{syms} & \ t \ \text{real} \\
\mathbf{a}_x & = \text{sym}(\text{\texttt{a_x(t)}}); \\
\mathbf{a}_y & = \text{sym}(\text{\texttt{a_y(t)}}); \\
\mathbf{a}_z & = \text{sym}(\text{\texttt{a_z(t)}}); \\
\mathbf{b}_x & = \text{sym}(\text{\texttt{b_x(t)}}); \\
\mathbf{b}_y & = \text{sym}(\text{\texttt{b_y(t)}}); \\
\mathbf{b}_z & = \text{sym}(\text{\texttt{b_z(t)}}); \\
\mathbf{a} & = [\mathbf{a}_x \ \mathbf{a}_y \ \mathbf{a}_z]; \\
\mathbf{b} & = [\mathbf{b}_x \ \mathbf{b}_y \ \mathbf{b}_z];
\end{align*}
\]

To calculate symbolically the derivative of a vector using the MATLAB the command \( \text{diff}(p, t) \) is used, which gives the derivative of \( p \) with respect to \( t \). The relation given by Eq. (1.22) can be demonstrated using the MATLAB command:

\[
\text{diff}(\mathbf{a}+\mathbf{b}, t) = \text{diff}(\mathbf{a}, t) + \text{diff}(\mathbf{b}, t)
\]

The time derivative of the product of a scalar function \( f \) and a vector function \( \mathbf{a} \) is

\[
\frac{d(f \mathbf{a})}{dt} = \frac{df}{dt} \mathbf{a} + f \frac{d\mathbf{a}}{dt}.
\]  

Equation (1.23) is verified using the MATLAB command:

\[
\begin{align*}
\text{syms} & \ f \ \text{real} \\
\text{diff}(f \ast \mathbf{a}, t) & = \text{diff}(f, t) \ast \mathbf{a} + f \ast \text{diff}(\mathbf{a}, t)
\end{align*}
\]

Combining the previous results one can conclude

\[
\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \frac{d\mathbf{a}}{dt} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dt} \quad \text{and} \quad \frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}.
\]  

Equation (1.24) is demonstrated with the MATLAB commands:

\[
\begin{align*}
\text{diff} & (\mathbf{a} \ast \mathbf{b}, t) \Rightarrow \text{diff}(\mathbf{a}, t) \ast \mathbf{b}, \mathbf{a} \ast \text{diff}(\mathbf{b}, t) \ast \\
\text{diff} (\text{\texttt{cross(a, b)}}, t) & = \text{\texttt{cross(diff(a, t), b)}} \ldots \\
& + \text{\texttt{cross(a, diff(b, t))}}
\end{align*}
\]

where \( \mathbf{a} \ast \mathbf{b} \) is the array transpose of \( \mathbf{a} \).

The general derivative a vector is

\[
\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}) = \frac{dv_x}{dt} \mathbf{i} + v_x \frac{d\mathbf{i}}{dt} + \frac{dv_y}{dt} \mathbf{j} + v_y \frac{d\mathbf{j}}{dt} + \frac{dv_z}{dt} \mathbf{k} + v_z \frac{d\mathbf{k}}{dt},
\]

and if the reference basis or reference frame \( [\mathbf{i}, \mathbf{j}, \mathbf{k}] \) is unchanging then

\[
\frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt} \mathbf{i} + \frac{dv_y}{dt} \mathbf{j} + \frac{dv_z}{dt} \mathbf{k}.
\]
1.8 Cauchy’s Inequality, Lagrange’s Identity, and Triangle Inequality

The vectors \( \mathbf{a} \) and \( \mathbf{b} \) are non-zero vectors. The \textit{Cauchy’s inequality} can be written in vector form as

\[
(a \cdot b)^2 \leq a^2 b^2, \tag{1.25}
\]

where \( a^2 = |a|^2 = \mathbf{a} \cdot \mathbf{a} \) and \( b^2 = |b|^2 = \mathbf{b} \cdot \mathbf{b} \). If \( \mathbf{a} \) and \( \mathbf{b} \) are parallel vectors then

\[
(a \cdot b)^2 = a^2 b^2.
\]

The vector derivation of the inequality is

\[
(a \cdot b)^2 = a^2 b^2 \cos^2 \theta \leq a^2 b^2.
\]

The \textit{Lagrange’s identity} in vector form is

\[
(a \cdot b)^2 = a^2 b^2 - (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}). \tag{1.26}
\]

The vectors \( \mathbf{a} \) and \( \mathbf{b} \) are non-zero vectors and the vectorial product between \( \mathbf{a} \) and \( \mathbf{b} \) is

\[
\mathbf{a} \times \mathbf{b} = (a_x b_z - a_z b_y) \mathbf{i} + (a_z b_x - a_x b_z) \mathbf{j} + (a_y b_x - a_x b_y) \mathbf{k},
\]

where \( a_x, a_y, a_z \) and \( b_x, b_y, b_z \) are the Cartesian components of the vectors \( \mathbf{a} \) and \( \mathbf{b} \). One can compute

\[
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = (a_x b_z - a_z b_y)^2 + (a_z b_x - a_x b_z)^2 + (a_y b_x - a_x b_y)^2. \tag{1.27}
\]

The scalar product definition gives

\[
(a \cdot b)^2 = |(a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \cdot (b_x \mathbf{i} - b_y \mathbf{j} + b_z \mathbf{k})|^2 = (a_x b_x + a_y b_y + a_z b_z)^2, \tag{1.28}
\]

and

\[
a^2 b^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 = \left| a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \right|^2 \left| b_x \mathbf{i} - b_y \mathbf{j} + b_z \mathbf{k} \right|^2 = (a_x^2 + a_y^2 + a_z^2) \left( b_x^2 + b_y^2 + b_z^2 \right). \tag{1.29}
\]

Using Eqs. (1.27), (1.28) and (1.29) it results

\[
(a_x^2 + a_y^2 + a_z^2) \left( b_x^2 + b_y^2 + b_z^2 \right) - (a_x b_x + a_y b_y + a_z b_z)^2
\]

\[
= (a_x b_z - a_z b_y)^2 + (a_y b_x - a_x b_y)^2 + (a_z b_y - a_y b_z)^2,
\]

or

\[
a^2 b^2 - (a \cdot b)^2 = (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}).
\]

The previous equation can be written as the identity given by Eq. (1.26).
The MATLAB proof for Lagrange’s identity is

```matlab
syms a_x a_y a_z b_x b_y b_z real
a = [ a_x a_y a_z ];
b = [ b_x b_y b_z ];
% LHS = (a.b)^2
% RHS = (a.a)*(b.b) - (a x b).(a x b)
LHS = (dot(a,b))^2;
RHS = dot(a,a)*dot(b,b)-dot(cross(a,b),cross(a,b));
expand(LHS)==expand(RHS)
```

If \( \mathbf{a} \) and \( \mathbf{b} \) are non-zero vectors the following relation can be obtained

\[
|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|. \tag{1.30}
\]

The inequality given by Eq. (1.30) is known as triangle inequality.

*Proof*: It is obvious that

\[
(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = |\mathbf{a}|^2 + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + |\mathbf{b}|^2. \tag{1.31}
\]

The following relation exists

\[
\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} \leq 2|\mathbf{a} \cdot \mathbf{b}| \leq 2|\mathbf{a}| |\mathbf{b}|. \tag{1.32}
\]

Equations (1.31) and (1.32) give

\[
|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) \leq |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}| |\mathbf{b}| = (|\mathbf{a}| + |\mathbf{b}|)^2.
\]

Moreover one can prove that

\[
|\mathbf{a}| - |\mathbf{b}| \leq |\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|, \text{ for } |\mathbf{a}| > |\mathbf{b}|,
\]

\[
|\mathbf{b}| - |\mathbf{a}| \leq |\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|, \text{ for } |\mathbf{a}| < |\mathbf{b}|.
\]

For this let \( \mathbf{a} = (\mathbf{a} + \mathbf{b}) - \mathbf{b} \) and applying the inequality given by Eq. (1.30) for \( \mathbf{a} + \mathbf{b} \) and \( -\mathbf{b} \), it results

\[
|\mathbf{a}| = |(\mathbf{a} + \mathbf{b}) + (-\mathbf{b})| \leq |\mathbf{a} + \mathbf{b}| + |-\mathbf{b}|,
\]

or

\[
|\mathbf{a} + \mathbf{b}| \geq |\mathbf{a}| - |-\mathbf{b}| = |\mathbf{a}| - |\mathbf{b}|. \tag{1.33}
\]

Using Eqs. (1.30) and (1.33) the following relations can be written

\[
|\mathbf{a}| - |\mathbf{b}| \leq |\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|, \text{ for } |\mathbf{a}| > |\mathbf{b}|,
\]

\[
|\mathbf{b}| - |\mathbf{a}| \leq |\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|, \text{ for } |\mathbf{a}| < |\mathbf{b}|.
\]
1.9 Examples

Example 1.1
In Fig. E1.1(a) the rectangular component of the vector $\mathbf{F}$ on the $OA$ direction is $f$, with the magnitude $|f| = f$. The vector $\mathbf{F}$ acts at an angle $\beta$ with the positive direction of the $x$-axis. Find the magnitude $|\mathbf{F}| = F$ of the vector $\mathbf{F}$. Numerical application: $f = 20$, $\alpha = 30^\circ$, and $\beta = 60^\circ$.

![Diagram](a)

![Diagram](b)

Fig. E1.1 Example 1.1

Solution
The component of $\mathbf{F}$ on the $OA$ direction is $|\mathbf{F}| \cos \theta = f$. From Fig. E1.1 the angle $\theta$ of the vector $\mathbf{F}$ with the $OA$ direction is $\theta = \beta - \alpha = 60^\circ - 30^\circ = 30^\circ$. The magnitude $F$ is calculated from the equation

$$|\mathbf{F}| \cos \theta = f \iff |\mathbf{F}| \cos 30^\circ = 20 \iff F = \frac{f}{\cos \theta} = \frac{20}{\cos 30^\circ} \text{ or } F = 23.094.$$

The MATLAB program starts with the statements:

```matlab
clear all
% clears all the objects in the MATLAB workspace and
% resets the default MuPAD symbolic engine
clc % clears the command window and homes the cursor
close all % closes all the open figure windows
```
The MATLAB commands for the input data are:

\[
\begin{align*}
    f &= 20; \\
    \alpha &= \pi/6; \\
    \beta &= \pi/3;
\end{align*}
\]

The angle $\theta$ and the magnitude of the vector $F$ are calculated with:

\[
\begin{align*}
    \theta &= \beta - \alpha; \\
    F &= f / \cos(\theta);
\end{align*}
\]

The statement $\cos(s)$ is the cosine of the argument $s$ in radians. The numerical solution for $F$ is printed using the statement:

\[
\text{fprintf(''}F = %f\text{'',F)}
\]

The statement $\text{fprintf(f,format,s)}$ writes data in the real part of array $s$ to the file $f$. The data is formatted under control of the specified format string and contains ordinary characters and/or C language conversion specifications. The conversion character $\%f$ specifies the output as fixed-point notation. For more details, about $\text{fprintf}$ see online help.

Next the two vectors $f$ and $F$ will be plotted. The $x$ and $y$ components of the vectors $f$ and $F$ are:

\[
\begin{align*}
    \% \text{ components of vector } f \\
    f_x &= f \times \cos(\alpha); \\
    f_y &= f \times \sin(\alpha); \\
\end{align*}
\]

\[
\begin{align*}
    \% \text{ components of vector } F \\
    F_x &= F \times \cos(\beta); \\
    F_y &= F \times \sin(\beta); \\
\end{align*}
\]

The following MATLAB commands are used to introduce the plotting of the vectors:

\[
\begin{align*}
    \text{hold on} \\
    s &= 1.5; % \text{ scale factor} \\
    \text{axis([0 } f_x+s 0 F_y+s]) \\
    \text{axis square}
\end{align*}
\]

The MATLAB command $\text{hold on}$ retains the current graph and all axis properties so that succeeding plot commands add to the existing graph. The MATLAB command $\text{axis([xMIN xMAX yMIN yMAX])}$ sets scaling for the $x$ and $y$ axes on the current plot and the statement $\text{axis square}$ makes the current axis box square in size. The direction of vector $f$ is represented with:

\[
\text{line([0 } s+f_x],[0 s+f_y],'\text{LineStyle}', '--')}
\]

where the command $\text{line(x,y)}$ creates the line in vectors $x$ and $y$ to the current axes. The $\text{LineStyle}$ specifies the line style: ‘-‘ solid line (default), ‘--‘ dashed line, ‘:‘ dotted line, and ‘?.‘ dash-dot line.
The vector $\mathbf{f}$ is represented with:

$$\text{quiver}(0,0,f_x,f_y,0,'Color','k','LineWidth',1.5)$$

The MATLAB command $\text{quiver}(x,y,u,v,s,\text{LineSpec})$ draws vectors specified by $u$ and $v$ at the coordinates $x$ and $y$. The parameter $s$ automatically scales the arrows to fit within the grid: $s = 2$ doubles the relative length, $s = 0.5$ halves the length, and $s = 0$ plots the vectors without automatic scaling. The parameter LineSpec specifies line style, marker symbol, and the 'Color' specifiers are 'r' red, 'g' green, 'b' blue, 'y' yellow, and 'k' black. The 'LineWidth' creates the width of the line in points (1 point = 1/72 inch) and by default the line width is 0.5 point. The vector $\mathbf{f}$ is denoted with the MATLAB command:

$$\text{text}(f_x/s+s,f_y/s+s,'f',... 'fontsize',14,'fontweight','b')$$

where $\text{text}(x,y,\text{''text''})$ adds the text in the quotes to location $(x,y)$. The fontsize for the vector is 14 and the font is bold, 'fontweight','b'. The ellipses (...) after the command was used to execute the statements together. The vector $\mathbf{F}$ is plotted and denoted with the MATLAB commands:

$$\text{quiver}(0,0,F_x,F_y,0,'Color','r','LineWidth',2.5)$$
$$\text{text}(F_x/s-s,F_y/s-s,'F',... 'fontsize',14,'fontweight','b')$$

The line that connects the end of the vector $\mathbf{F}$ with the end of the vector $\mathbf{f}$ is represented in MATLAB with:

$$\text{line}([F_x \ f_x], [F_y \ f_y], 'LineStyle','--')$$

The labels for the $x$ and $y$ axes are added with:

$$\text{xlabel('x')}$$
$$\text{ylabel('y')}$$

The MATLAB figure of the vectors is shown in Fig. E1.1(b).

**Example 1.2**

The coordinates of two points $A$ and $B$ relative to the origin $O(0,0,0)$ are given by: $A(x_A = 1, y_A = 2, z_A = 3)$ and $B(x_B = 3, y_B = 3, z_B = 3)$. Determine the unit vector of the line $\Delta$ that starts at point $A(x_A, y_A, z_A)$ and passes through the point $B(x_B, y_B, z_B)$.

**Solution**

The position vectors of the points $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ with respect to the origin $O(0,0,0)$ are

$$\overrightarrow{OA} = x_A \mathbf{i} + y_A \mathbf{j} + z_A \mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = x_B \mathbf{i} + y_B \mathbf{j} + z_B \mathbf{k}.$$  

The symbolic expressions of the vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ are introduced in MATLAB as:
The vector $\vec{AB} = \vec{r}_{AB}$ is defined as

$$\vec{r}_{AB} = r_B - r_A = (x_B - x_A)\hat{i} + (y_B - y_A)\hat{j} + (z_B - z_A)\hat{k},$$

or in MATLAB:

```matlab
r_AB = r_B - r_A;
```

The magnitude of the vector $\vec{r}_{AB}$ is

$$|\vec{r}_{AB}| = r_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

or in MATLAB:

```matlab
mr_AB = sqrt(dot(r_AB, r_AB));
```

The unit vector, $\mathbf{u}_\Delta$, of the line $\Delta$ (line $AB$) is

$$\mathbf{u}_\Delta = \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|} = \frac{(x_B - x_A)\hat{i} + (y_B - y_A)\hat{j} + (z_B - z_A)\hat{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}} = \frac{x_B - x_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\hat{i} + \frac{y_B - y_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\hat{j} + \frac{z_B - z_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\hat{k}.$$ 

Using MATLAB the unit vector is:

```matlab
u_AB = r_AB/mr_AB;
```

The numerical values of the components of the unit vector $\mathbf{u}_\Delta = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$ are obtained replacing the symbolic expressions with their numerical values

- \(u_x = \frac{x_B - x_A}{r_{AB}} = \frac{3 - 1}{\sqrt{(3 - 1)^2 + (3 - 2)^2 + (3 - 3)^2}} = \frac{2}{2.2361} = 0.894,\)
- \(u_y = \frac{y_B - y_A}{r_{AB}} = \frac{3 - 2}{\sqrt{(3 - 2)^2 + (3 - 2)^2 + (3 - 3)^2}} = \frac{1}{2.2361} = 0.447,\)
- \(u_z = \frac{z_B - z_A}{r_{AB}} = \frac{3 - 3}{\sqrt{(3 - 3)^2 + (3 - 2)^2 + (3 - 3)^2}} = \frac{0}{2.2361} = 0,\)
where the magnitude of the vector \( \mathbf{r}_{AB} \) is

\[
\mathbf{r}_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} = \sqrt{(3 - 3)^2 + (3 - 2)^2 + (3 - 3)^2} = \sqrt{5} = 2.2361.
\]

To obtain the numerical values in MATLAB, \( x_A, y_A, z_A \) are replaced with 1, 2, 3 and \( x_B, y_B, z_B \) are replaced with 3, 3, 3. For this purpose two lists are created: a list with the symbolical variables \( \{x_A, y_A, z_A, x_B, y_B, z_B\} \) and a list with the corresponding numeric values \( \{1, 2, 3, 3, 3, 3\} \):

\[
\text{slist} = \{x_A, y_A, z_A, x_B, y_B, z_B\};
\]
\[
\text{nlist} = \{1, 2, 3, 3, 3, 3\};
\]

To obtain the numerical value for the symbolic unit vector \( \mathbf{u}_{AB} \) the following statement is introduced:

\[
\text{u}_{ABn} = \text{subs} (\text{u}_{AB}, \text{slist}, \text{nlist});
\]

The statement \( \text{subs(expr, lhs, rhs)} \) replaces \( \text{lhs} \) with \( \text{rhs} \) in the symbolic expression \( \text{expr} \). The numerical results are printed with the following command:

\[
\text{fprintf} (\text{‘u}_{AB} = [\%6.3f \%6.3f \%6.3f] \n’, \text{u}_{ABn})
\]

The escape character \( \backslash n \) specifies new line.

Next the vectors \( \mathbf{r}_A \), \( \mathbf{r}_B \), and \( \mathbf{r}_{AB} \) will be plotted using MATLAB. The numerical values of the vectors \( \mathbf{r}_A \) and \( \mathbf{r}_B \) are obtained with:

\[
\text{rA} = \text{eval (subs (r}_A, \text{slist, nlist))};
\]
\[
\text{rB} = \text{eval (subs (r}_B, \text{slist, nlist))};
\]

The command \( \text{eval(x)} \), where \( x \) is a string, executes the string as an expression. The command is \( \text{axis([xMIN xMAX yMIN yMAX zMIN zMAX])} \) put the scaling for the \( x, y \) and \( z \) axes on the 3-D plot. The statement \( \text{axis ij} \) positions MATLAB into its “matrix” axes mode, the coordinate system origin is at \( y=z=0 \), the \( y \)-axis is numbered from top to bottom, the \( x \)-axis is numbered from left to right, and the \( z \)-axis is vertical with values increasing from bottom to top. For this example the axes are defined by:

\[
\text{a} = 3.5;
\]
\[
\text{axis ([0 a 0 a 0 a])}
\]
\[
\text{axis ij, grid on, hold on}
\]

The MATLAB command \( \text{grid on} \) adds major grid lines to the current axes and \( \text{hold on} \) locks up the current plot and all axis properties so that following graphing commands add to the existing graph. The vectors \( \mathbf{r}_A \) and \( \mathbf{r}_B \) are defined in MATLAB as:

\[
\text{quiver3(0,0,0, r}_A(1),\text{r}_A(2),\text{r}_A(3),1,...
\]
\[
\backslash \text{Color’,’k’,’LineWidth’,1)}
\]
\[
\text{quiver3(0,0,0, r}_B(1),\text{r}_B(2),\text{r}_B(3),1,...
\]
\[
\backslash \text{Color’,’k’,’LineWidth’,1)}
\]
where the statement `quiver3(x, y, z, u, v, w)` represent a vector as arrows at the point \((x, y, z)\) with the components \((u, v, w)\). The dashed line (--) between the points \(A\) and \(B\) is plotted with the command:

```matlab
dashline([rA(1) rB(1)], [rA(2) rB(2)], [rA(3) rB(3)],...
    'LineStyle', '--')
```

and the unit vector \(u\) is represented with:

```matlab
quiver3(...
    rA(1), rA(2), rA(3), u_ABn(1), u_ABn(2), u_ABn(3),...
    1, 'Color', 'r', 'LineWidth', 2)
```

The labels for the vectors and the axes are printed in MATLAB with:

```matlab
text(rA(1)/2, rA(2)/2, rA(3)/2+.3,...
    'r_A', 'fontsize', 14, 'fontweight', 'b')
text(rB(1)/2, rB(2)/2, rB(3)/2+.3,...
    'r_B', 'fontsize', 14, 'fontweight', 'b')
text(...
    (rA(1)+rB(1))/2-.4,...
    (rA(2)+rB(2))/2,...
    (rA(3)+ rB(3))/2+.3,...
    'u', 'fontsize', 14, 'fontweight', 'b')
xlabel('x'), ylabel('y'), zlabel('z')
```

The MATLAB representation of the vectors is shown in Fig. E1.2.
Example 1.3

The vectors \( \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \) and \( \mathbf{V}_4 \) with the magnitude \( |\mathbf{V}_1| = V_1, |\mathbf{V}_2| = V_2, |\mathbf{V}_3| = V_3, \) and \( |\mathbf{V}_4| = V_4 \) are concurrent at the origin \( O(0,0,0) \) and are directed through the points of coordinates \( A_1(x_1, y_1, z_1), A_2(x_2, y_2, z_2), A_3(x_3, y_3, z_3), \) and \( A_4(x_4, y_4, z_4), \) respectively. Determine the resultant vector of the system. Numerical application: \( V_1 = 10, V_2 = 25, V_3 = 15, V_4 = 40, A_1(3, 1, 7), A_2(5, -3, 4), A_3(-4, -3, 1), \) and \( A_4(4, 2, -3) \).

Solution

The magnitudes, \( V_i \), of the vectors \( \mathbf{V}_i \) and the coordinates, \( x_i, y_i, z_i \), of the points \( A_i, i = 1, 2, 3, 4 \) are introduced with MATLAB as:

```matlab
V(1)=10; V(2)=25; V(3)=15; V(4)=40; % magnitudes V_i
x(1)= 3; y(1)= 1; z(1)= 7; % A_1
x(2)= 5; y(2)= -3; z(2)= 4; % A_2
x(3)=-4; y(3)=-3; z(3)= 1; % A_3
x(4)= 4; y(4)= 2; z(4)= -3; % A_4
```

The direction cosines of the vectors \( \mathbf{V}_i \) are

\[
\cos \theta_{ix} = \frac{x_i}{\sqrt{x_i^2+y_i^2+z_i^2}}, \quad \cos \theta_{iy} = \frac{y_i}{\sqrt{x_i^2+y_i^2+z_i^2}}, \quad \cos \theta_{iz} = \frac{z_i}{\sqrt{x_i^2+y_i^2+z_i^2}},
\]

and the \( x, y, z \) components of the vectors \( \mathbf{V}_i \) are

\[
V_{ix} = V_i \cos \theta_{ix}, \quad V_{iy} = V_i \cos \theta_{iy}, \quad V_{iz} = V_i \cos \theta_{iz}.
\]

To calculate the direction cosines and components of the vectors for \( i = 1, 2, 3, 4 \) the MATLAB statement: \texttt{for \( \text{var=starval:step:endval} \), \text{statement} \ end \text{is used. It repeatedly evaluates} \ \text{statement} \ \text{in a loop. The counter variable of the loop is} \ \text{var. At the start the variable is initialized to value} \ \text{starval and is incremented (or decremented when step is negative) by the value step for each iteration. The statement is repeated until var has incremented to the value endval. For the vectors the following applies for i=1:4, Program block, end or:}

\begin{verbatim}
for i = 1:4
% direction cosines of the vector v(i)
c_x(i) = x(i)/sqrt(x(i)^2+y(i)^2+z(i)^2);
c_y(i) = y(i)/sqrt(x(i)^2+y(i)^2+z(i)^2);
c_z(i) = z(i)/sqrt(x(i)^2+y(i)^2+z(i)^2);
% x, y, z components of the vector v(i)
v_x(i) = V(i)*c_x(i);
v_y(i) = V(i)*c_y(i);
v_z(i) = V(i)*c_z(i);
fprintf(‘vector %g: \n’,i)
fprintf(‘direction cosines=’)
fprintf(‘[%6.3f,%6.3f,%6.3f]\n’,c_x(i),c_y(i),c_z(i))
\end{verbatim}
1.9 Examples

```matlab
fprintf('vector V=
')
fprintf('[%6.3f,%6.3f,%6.3f]\n',v_x(i),v_y(i),v_z(i))
fprintf('\n')
end

The results in MATLAB are:

vector 1:
direction cosines=[ 0.391, 0.130, 0.911]
vector V=[ 3.906, 1.302, 9.113]

vector 2:
direction cosines=[ 0.707,-0.424, 0.566]
vector V=[17.678,-10.607,14.142]

vector 3:
direction cosines=[-0.784,-0.588, 0.196]
vector V=[-11.767,-8.825, 2.942]

vector 4:
direction cosines=[ 0.743, 0.371,-0.557]
vector V=[29.711,14.856,-22.283]

or using a table form

<table>
<thead>
<tr>
<th>i</th>
<th>V_i</th>
<th>A_i</th>
<th>cos θ_x</th>
<th>cos θ_y</th>
<th>cos θ_z</th>
<th>V_x</th>
<th>V_y</th>
<th>V_z</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>(3.1,7)</td>
<td>0.391</td>
<td>0.130</td>
<td>0.911</td>
<td>3.906</td>
<td>1.302</td>
<td>9.113</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td>(5,-3,4)</td>
<td>0.700</td>
<td>-0.424</td>
<td>0.566</td>
<td>17.678</td>
<td>-10.607</td>
<td>14.142</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>(-4,-3,1)</td>
<td>-0.784</td>
<td>-0.588</td>
<td>0.196</td>
<td>-11.767</td>
<td>-8.825</td>
<td>2.942</td>
</tr>
<tr>
<td>4</td>
<td>40</td>
<td>(4,2,-3)</td>
<td>0.743</td>
<td>0.371</td>
<td>-0.557</td>
<td>29.711</td>
<td>14.856</td>
<td>-22.283</td>
</tr>
</tbody>
</table>

The vector \( V_i \) can be written as \( V_i = V_{ix} \hat{i} + V_{iy} \hat{j} + V_{iz} \hat{k} \), \( i = 1,2,3,4 \). The resultant of the system is

\[
R = \sqrt{\left(R_x\right)^2 + \left(R_y\right)^2 + \left(R_z\right)^2} = \sqrt{\left(\sum V_{ix}\right)^2 + \left(\sum V_{iy}\right)^2 + \left(\sum V_{iz}\right)^2}.
\]

The direction cosines of the resultant are

\[
\cos \theta_x = \frac{\sum V_{ix}}{R}, \quad \cos \theta_y = \frac{\sum V_{iy}}{R}, \quad \cos \theta_z = \frac{\sum V_{iz}}{R}.
\]

The resultant and the direction cosines in MATLAB are:

```matlab
Rx = sum(v_x); Ry = sum(v_y); Rz = sum(v_z); R = [Rx Ry Rz]; modR = norm(R);```
fprintf('R = V1 + V2 + V3 + V4 = [%6.3f, %6.3f, %6.3f]\n', R)
fprintf('|R| = %6.3g\n', modR)
fprintf('direction cosines = \n', RuR)

The MATLAB results are:
R = V1 + V2 + V3 + V4 = [39.528, -3.274, 3.914]
|R| = 39.9
direction cosines = uR = R/|R| = [0.992, -0.082, 0.098]

or in table form

<table>
<thead>
<tr>
<th>R</th>
<th>R_x</th>
<th>R_y</th>
<th>R_z</th>
<th>cos θ_x</th>
<th>cos θ_y</th>
<th>cos θ_z</th>
</tr>
</thead>
<tbody>
<tr>
<td>39.9</td>
<td>39.528</td>
<td>-3.274</td>
<td>3.914</td>
<td>0.992</td>
<td>-0.082</td>
<td>0.098</td>
</tr>
</tbody>
</table>

The negative value of cos θ_y signifies that the resultant has a negative component in the y direction.

Next the vectors will be plotted using MATLAB. The axes are defined in MATLAB with:

a = 26;
axis([-a a -a a -a a])
axis ij, grid on, hold on
xlabel('x'), ylabel('y'), zlabel('z')
text(0,0,0-1.5,'O','HorizontalAlignment','right')

The vectors \( \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4 \), and \( \mathbf{R} \) are plotted and labeled with the statements:
quiver3(0,0,0,v_x(1),v_y(1),v_z(1),1,...
'Color','k','LineWidth',1.5)
text(v_x(1),v_y(1),v_z(1),'V_1',...
'fontsize',12,'fontWeight','b')
quiver3(0,0,0,v_x(2),v_y(2),v_z(2),1,...
'Color','k','LineWidth',1.5)
text(v_x(2),v_y(2),v_z(2),'V_2',...
'fontsize',12,'fontWeight','b')
quiver3(0,0,0,v_x(3),v_y(3),v_z(3),1,...
'Color','k','LineWidth',1.5)
text(v_x(3),v_y(3),v_z(3)+1,'V_3',...
'fontsize',12,'fontWeight','b')
quiver3(0,0,0,v_x(4),v_y(4),v_z(4),1,...
'Color','k','LineWidth',1.5)
text(v_x(4),v_y(4),v_z(4),'V_4',...
'fontsize',12,'fontWeight','b')
1.9 Examples

The MATLAB representation of the vectors is shown in Fig. E1.3.

\begin{verbatim}
quiver3(0,0,0,Rx,Ry,Rz,1,...
   'Color','r','LineWidth',2.5)
text(Rx,Ry,Rz,' R','fontsize',14,'fontweight','b')
\end{verbatim}

Fig. E1.3 MATLAB figure for Example 1.3

Example 1.4

Two vector system \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \), is shown in Fig. E1.4(a). a) Find the resultant of the system. b) Determine the cross product \( \mathbf{V}_1 \times \mathbf{V}_1 \). c) Find the angle between the vectors \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \). Numerical application: \( |\mathbf{V}_1| = V_1 = 3 \), \( |\mathbf{V}_2| = V_2 = 3 \), \( a = 4 \), \( b = 5 \), and \( c = 3 \).

Solution

a) The vectors \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \) are given by

\[
\mathbf{V}_1 = V_{1x}\mathbf{i} + V_{1y}\mathbf{j} + V_{1z}\mathbf{k} = |\mathbf{V}_1| \frac{\mathbf{r}_{BG}}{|\mathbf{r}_{BG}|} = V_1 \frac{\mathbf{r}_{BG}}{|\mathbf{r}_{BG}|},
\]
\[
\mathbf{V}_2 = V_{2x}\mathbf{i} + V_{2y}\mathbf{j} + V_{2z}\mathbf{k} = |\mathbf{V}_2| \frac{\mathbf{r}_{BP}}{|\mathbf{r}_{BP}|} = V_2 \frac{\mathbf{r}_{BP}}{|\mathbf{r}_{BP}|}.
\]

Next the vectors \( \mathbf{r}_{BG} \) and \( \mathbf{r}_{BP} \) will be calculated. From Fig. E1.4 the coordinates of the points \( B, D, P, \) and \( Q \) are \( B = B(x_B,y_B,z_B) = B(0,b,0) = B(0,5,0) \), \( G = G(x_G,y_G,z_G) = G(a,0,c) = G(4,0,3) \), and \( P = P(x_P,y_P,z_P) = P(a,b/2,0) = P(4,5/2,0) \). The position vectors vectors of the points \( B, G, \) and \( P \) are

\[
\mathbf{r}_B = x_B\mathbf{i} + y_B\mathbf{j} + z_B\mathbf{k} = b\mathbf{j} = 5\mathbf{j},
\]
\[
\mathbf{r}_G = x_G\mathbf{i} + y_G\mathbf{j} + z_G\mathbf{k} = a\mathbf{i} + c\mathbf{k} = 4\mathbf{i} + 3\mathbf{k},
\]
\[
\mathbf{r}_P = x_P\mathbf{i} + y_P\mathbf{j} + z_P\mathbf{k} = a\mathbf{i} + b/2\mathbf{j} = 4\mathbf{i} + 5/2\mathbf{j}.
\]
The MATLAB commands for input data and for $\mathbf{r}_B$, $\mathbf{r}_G$, and $\mathbf{r}_P$ are:

\[
\begin{align*}
V_1 &= 3; \quad V_2 = 3; \\
a &= 4; \quad b = 5; \quad c = 3; \\
x_B &= 0; \quad y_B = b; \quad z_B = 0; \quad \mathbf{r}_B = [x_B, y_B, z_B]; \\
x_G &= a; \quad y_G = 0; \quad z_G = c; \quad \mathbf{r}_G = [x_G, y_G, z_G]; \\
x_P &= a; \quad y_P = b/2; \quad z_P = 0; \quad \mathbf{r}_P = [x_P, y_P, z_P];
\end{align*}
\]

The vectors $\mathbf{r}_{BG}$ and $\mathbf{r}_{BP}$ are

\[
\begin{align*}
\mathbf{r}_{BG} &= \mathbf{r}_G - \mathbf{r}_B = (x_G - x_B) \mathbf{i} + (y_G - y_B) \mathbf{j} + (z_G - z_B) \mathbf{k} \\
&= (a - 0) \mathbf{i} + (0 - b) \mathbf{j} + (c - 0) \mathbf{k} \\
&= a \mathbf{i} - b \mathbf{j} + c \mathbf{k} = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}, \\
\mathbf{r}_{BP} &= \mathbf{r}_P - \mathbf{r}_B = (x_P - x_B) \mathbf{i} + (y_P - y_B) \mathbf{j} + (z_P - z_B) \mathbf{k} \\
&= (a - 0) \mathbf{i} + \left(\frac{b}{2} - b\right) \mathbf{j} + (0 - 0) \mathbf{k} \\
&= a \mathbf{i} - \frac{5}{2} \mathbf{j}.
\end{align*}
\]

The magnitudes of the vectors $\mathbf{r}_{BG}$ and $\mathbf{r}_{BP}$ are

\[
\begin{align*}
|\mathbf{r}_{BG}| &= r_{BG} = \sqrt{(x_G - x_B)^2 + (y_G - y_B)^2 + (z_G - z_B)^2} \\
&= \sqrt{(a - 0)^2 + (0 - b)^2 + (c - 0)^2} = \sqrt{a^2 + b^2 + c^2} \\
&= \sqrt{4^2 + 5^2 + 3^2} = 7.071,
\end{align*}
\]
1.9 Examples

\[ |\mathbf{r}_{BP}| = \mathbf{r}_{BP} = \sqrt{(x_P - x_B)^2 + (y_P - y_B)^2 + (z_P - z_B)^2} \]

\[ = \sqrt{(a - 0)^2 + \left(\frac{b}{2} - b\right)^2 + (0 - 0)^2} = \sqrt{a^2 + \frac{b^2}{4}} \]

\[ = \sqrt{4^2 + \frac{5^2}{4}} = 4.717. \]

The vectors \( \mathbf{r}_{BG} \) and \( \mathbf{r}_{BP} \) and their magnitudes in MATLAB are:

\[ \mathbf{r}_{BG} = \mathbf{r}_G - \mathbf{r}_B; \]
\[ \mathbf{r}_{BP} = \mathbf{r}_P - \mathbf{r}_B; \]
\[ \text{fprintf('r}_{BG} = \begin{bmatrix} %6.3f & %6.3f & %6.3f \end{bmatrix}' \ 
, \mathbf{r}_{BG}) \]
\[ \text{fprintf('r}_{BP} = \begin{bmatrix} %6.3f & %6.3f & %6.3f \end{bmatrix}' \ 
, \mathbf{r}_{BP}) \]
\[ \text{mr}_{BG} = \text{sqrt} \left( \text{dot} \left( \mathbf{r}_{BG}, \mathbf{r}_{BG} \right) \right); \]
\[ \text{mr}_{BP} = \text{sqrt} \left( \text{dot} \left( \mathbf{r}_{BP}, \mathbf{r}_{BP} \right) \right); \]
\[ \text{fprintf('|r}_{BG| = %6.3f}' \ 
, \text{mr}_{BG}) \]
\[ \text{fprintf('|r}_{BP| = %6.3f}' \ 
, \text{mr}_{BP}) \]

The vectors \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \) are

\[ \mathbf{V}_1 = \mathbf{V}_1 \frac{\mathbf{r}_{BG}}{\mathbf{r}_{BG}} = \mathbf{V}_1 \frac{a\mathbf{i} - b\mathbf{j} + c\mathbf{k}}{\sqrt{a^2 + b^2 + c^2}} = \frac{3 \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}}{7.071} \]

\[ = 1.697\mathbf{i} - 2.121\mathbf{j} + 1.273\mathbf{k}, \]

\[ \mathbf{V}_2 = \mathbf{V}_2 \frac{\mathbf{r}_{BP}}{\mathbf{r}_{BP}} = \mathbf{V}_2 \frac{a\mathbf{i} - (b/2)\mathbf{j}}{\sqrt{a^2 + b^2/4}} = \frac{3 \mathbf{i} - (5/2)\mathbf{j}}{4.717} \]

\[ = 2.544\mathbf{i} - 1.590\mathbf{j}, \]

or with MATLAB:

\[ \text{u}_{BD} = \mathbf{r}_{BD}/\text{mr}_{BD}; \]
\[ \text{u}_{PQ} = \mathbf{r}_{PQ}/\text{mr}_{PQ}; \]
\[ \text{V1} = \text{V}_1 \ast \text{u}_{BD}; \]
\[ \text{V2} = \text{V}_2 \ast \text{u}_{PQ}; \]
\[ \text{V1n} = \text{eval} \left( \text{subs} \left( \text{V1}, \text{slist}, \text{nlist} \right) \right); \]
\[ \text{V2n} = \text{eval} \left( \text{subs} \left( \text{V2}, \text{slist}, \text{nlist} \right) \right); \]
\[ \text{fprintf('V1 = \begin{bmatrix} %6.3f & %6.3f & %6.3f \end{bmatrix}' \ 
, \text{V1n}) \]
\[ \text{fprintf('V2 = \begin{bmatrix} %6.3f & %6.3f & %6.3f \end{bmatrix}' \ 
, \text{V2n}) \]

The cartesian components of the vectors \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \) are

\[ V_{1x} = 1.697, \quad V_{1y} = -2.121, \quad V_{1z} = 1.273, \quad V_{2x} = 2.544, \quad V_{2y} = -1.590, \quad V_{2z} = 0. \]

The resultant vector has the components

\[ R_x = \sum V_x = V_{1x} + V_{2x} = 1.697 + 2.544 = 4.241, \]
\[ R_y = \sum V_y = V_{1y} + V_{2y} = -2.121 - 1.590 = -3.711, \]
\[ R_z = \sum V_z = V_{1z} + V_{2z} = 1.273 + 0 = 1.273, \]

and can be written in a vector form as

\[ \mathbf{R} = R_x \mathbf{i} + R_y \mathbf{j} + R_z \mathbf{k} = 4.241 \mathbf{i} - 3.711 \mathbf{j} + 1.273 \mathbf{k}. \]

The magnitude of \( \mathbf{R} \) is

\[ |\mathbf{R}| = R = \sqrt{R_x^2 + R_y^2 + R_z^2} = \sqrt{(4.241)^2 + (-3.711)^2 + (1.273)^2} = 5.778. \]

The angles of the vector \( \mathbf{R} \) with the cartesian axes are calculated from the direction cosines

\[
\begin{align*}
\cos \alpha &= \frac{R_x}{|\mathbf{R}|} = \frac{4.241}{5.778} = 0.734, \\
\cos \beta &= \frac{R_y}{|\mathbf{R}|} = \frac{-3.711}{5.778} = -0.642, \text{ and} \\
\cos \gamma &= \frac{R_z}{|\mathbf{R}|} = \frac{1.273}{5.778} = 0.220.
\end{align*}
\]

The MATLAB commands for the resultant and direction cosines are

```matlab
R_x = V1n(1) + V2n(1); 
R_y = V1n(2) + V2n(2); 
R_z = V1n(3) + V2n(3); 
R = [R_x, R_y, R_z]; 
nR = norm(R); 
U_R = R/nR; % direction cosines 
fprintf('R = [%6.3f %6.3f %6.3f]\n', R) 
fprintf('|R| = %6.3f\n', nR) 
fprintf('U_R = [%6.3f %6.3f %6.3f]\n', U_R)
```

b) The cross product between the vectors \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \) is

\[
\mathbf{V}_1 \times \mathbf{V}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ V_{1x} & V_{1y} & V_{1z} \\ V_{2x} & V_{2y} & V_{2z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & k \\ 1.697 & -2.121 & 1.273 \\ 2.544 & -1.590 & 0 \end{vmatrix} = 2.024 \mathbf{i} + 3.238 \mathbf{j} + 2.698 \mathbf{k},
\]

or with MATLAB:

```matlab
VC = cross(V1, V2); 
fprintf('V1 x V2 = [%6.3f %6.3f %6.3f]\n', VC)
```

c) The angle \( \theta \) between the vectors \( \mathbf{V}_1 \) and \( \mathbf{V}_2 \) is calculated with

\[
\cos \theta = \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{||\mathbf{V}_1|| ||\mathbf{V}_2||} = \frac{V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1z}V_{2z}}{V_1 V_2} \\
= \frac{2.024(2.544) + (-2.121)(-1.590) + 1.273(0)}{3(3)} = 0.8545.
\]
The angle is $\theta = 31.299^\circ$. The MATLAB commands for calculating the angle between the vectors are:

```matlab
costheta = dot(V1, V2)/(V_1*V_2);
fprintf('theta = %6.3f (deg)\n', acosd(costheta))
```

The MATLAB function \texttt{acos(phi)} is the arccosine of the element \texttt{phi} and \texttt{acosd(phi)} is the inverse cosine, expressed in degrees, of the element of \texttt{phi}. Next the vectors \texttt{V1}, \texttt{V2}, \texttt{R}, \texttt{u_R}, and \texttt{V_1 \times V_2} will be plotted using MATLAB. The axes are defined in MATLAB with:

```matlab
axis(1.5*[0 a 0 b 0 c])
grid on, hold on
xlabel('x'), ylabel('y'), zlabel('z')
```

For the default “Cartesian” axes mode the coordinate system origin is at \texttt{x=y=0}. The \texttt{x}-axis is numbered from left to right, the \texttt{y}-axis is numbered from bottom to top, and the \texttt{z}-axis is vertical with values increasing from bottom to top. The coordinates of the points \texttt{A}, \texttt{C}, \texttt{D}, \texttt{E}, and \texttt{F} are:

\begin{align*}
\texttt{x_A} &= a; \quad \texttt{y_A} = 0; \quad \texttt{z_A} = 0; \\
\texttt{x_C} &= 0; \quad \texttt{y_C} = 0; \quad \texttt{z_C} = c; \\
\texttt{x_D} &= a; \quad \texttt{y_D} = b; \quad \texttt{z_D} = c; \\
\texttt{x_E} &= 0; \quad \texttt{y_E} = b; \quad \texttt{z_E} = c; \\
\texttt{x_F} &= a; \quad \texttt{y_F} = b; \quad \texttt{z_F} = 0;
\end{align*}

The labels of the points \texttt{O}, \texttt{A}, \texttt{B}, \texttt{C}, \texttt{D}, \texttt{E}, \texttt{F}, \texttt{G}, and \texttt{P} are:

```matlab
text(0, 0, 0+.1,' O','fontsize',12)
text(x_A, y_A, z_A+.2,' A','fontsize',12)
text(x_B-.2, y_B, z_B-.1,' B','fontsize',12)
text(x_C, y_C, z_C+.2,' C','fontsize',12)
text(x_D, y_D, z_D+.2,' D','fontsize',12)
text(x_E, y_E, z_E+.2,' E','fontsize',12)
text(x_F, y_F, z_F+.2,' F','fontsize',12)
text(x_G, y_G, z_G+.2,' G','fontsize',12)
text(x_P, y_P, z_P+.2,' P','fontsize',12)
```

The parallelepiped \textit{OABCDEFG} is plotted using the MATLAB commands:

```matlab
line([0 x_A],[0 y_A],[0 z_A])
line([0 x_B],[0 y_B],[0 z_B])
line([0 x_C],[0 y_C],[0 z_C])
line([x_B x_E],[y_B y_E],[z_B z_E])
line([x_B x_F],[y_B y_F],[z_B z_F])
line([x_A x_F],[y_A y_F],[z_A z_F])
line([x_A x_G],[y_A y_G],[z_A z_G])
line([x_C x_G],[y_C y_G],[z_C z_G])
line([x_C x_E],[y_C y_E],[z_C z_E])
line([x_D x_G],[y_D y_G],[z_D z_G])
```
Another way of drawing the parallelepiped $OABCDEFG$ is:

```matlab
plot3(...
[x_G x_A x_F x_D x_G x_C x_E x_B 0 x_C],...
[y_G y_A y_F y_D y_G y_C y_E y_B 0 y_C],...
[z_G z_A z_F z_D z_G z_C z_E z_B 0 z_C])
line([0 x_A],[0 y_A],[0 z_A])
line([x_B x_F],[y_B y_F],[z_B z_F])
line([x_D x_E],[y_D y_E],[z_D z_E])
```

where the MATLAB statement `plot3(x,y,z)` plots a line in 3D through the points whose coordinates are the elements of the vectors $x$, $y$, and $z$.

The lines $BG$ and $BP$ are plotted with:

```matlab
line([x_B x_G],[y_B y_G],[z_B z_G],...
 'Color','k','LineStyle','--')
line([x_B x_P],[y_B y_P],[z_B z_P],...
 'Color','k','LineStyle','--')
```

The vectors $V_1$, $V_2$, $R$, $u_R$, and $V_1 \times V_2$ and their labels are described by the following MATLAB commands:

```matlab
quiver3(x_B,y_B,z_B, V1(1),V1(2),V1(3),1,...
 'Color','k','LineWidth',2)
quiver3(x_B,y_B,z_B, V2(1),V2(2),V2(3),1,...
 'Color','k','LineWidth',2)
quiver3(x_B,y_B,z_B, R(1),R(2),R(3),1,...
 'Color','r','LineWidth',3)
quiver3(x_B,y_B,z_B, u_R(1),u_R(2),u_R(3),1,...
 'Color','b','LineWidth',4)
quiver3(x_B,y_B,z_B, VC(1),VC(2),VC(3),1,...
 'Color','m','LineWidth',3)
```

```matlab
text(x_B+V1(1), y_B+V1(2), z_B+V1(3)+.1,...
 'V_1','fontsize',14,'fontweight','b')
text(x_B+V2(1), y_B+V2(2), z_B+V2(3)+.1,...
 'V_2','fontsize',14,'fontweight','b')
text(x_B+R(1), y_B+R(2), z_B+R(3)+.1,...
 'R','fontsize',14,'fontweight','b')
text(x_B+u_R(1), y_B+u_R(2), z_B+u_R(3)+.1,...
 'u_R','fontsize',14,'fontweight','b')
text(x_B+VC(1), y_B+VC(2), z_B+VC(3)+.1,...
 'V_1 x V_2','fontsize',14,'fontweight','b')
```

A rotated MATLAB drawing of the vectors is shown in Fig. E1.4(b).
**Example 1.5**

The vector \( \mathbf{p} \) of magnitude \(|\mathbf{p}| = p\) is located in the \( x - z \) plane and makes an angle \( \theta \) with \( x \)-axis as shown in Fig. E1.5(a). The vector \( \mathbf{q} \) of magnitude \(|\mathbf{q}| = q\) is situated along the \( x \)-axis. Compute the vector (cross) product \( \mathbf{v} = \mathbf{p} \times \mathbf{q} \). Numerical application: \(|\mathbf{p}| = p = 5\), \(|\mathbf{q}| = q = 4\), and \( \theta = 30^\circ \).

**Solution**

The vector product \( \mathbf{v} \) is perpendicular to the vectors \( \mathbf{p} \) and \( \mathbf{q} \) and that is why the vector \( \mathbf{v} \) is along the \( y \)-axis and has the magnitude

\[
|\mathbf{v}| = |\mathbf{p}| |\mathbf{q}| \sin \theta = pq \sin \theta = 5 \cdot 4 \cdot \sin 30^\circ = 10.
\]

From Fig. E1.5(a) the direction of the vector \( \mathbf{v} \) is upward.

The solution could also be obtained by expressing the vector product \( \mathbf{v} = \mathbf{p} \times \mathbf{q} \) of the given vectors \( \mathbf{p} \) and \( \mathbf{q} \) in terms of their rectangular components. Resolving \( \mathbf{p} \) and \( \mathbf{q} \) into components, one can write

\[
\mathbf{v} = \mathbf{p} \times \mathbf{q} = (p_x \mathbf{i} + p_y \mathbf{j} + p_z \mathbf{k}) \times (q_x \mathbf{i} + q_y \mathbf{j} + q_z \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{vmatrix}
\]

\[
= (p_y q_z - p_z q_y) \mathbf{i} + (p_z q_x - p_x q_z) \mathbf{j} + (p_x q_y - p_y q_x) \mathbf{k}.
\]

The components \( p_x \), \( p_y \), and \( p_z \) of the vector \( \mathbf{p} \) are

\[
p_x = |\mathbf{p}| \cos \theta = p \cos \theta = 5 \cos 30^\circ = \frac{5\sqrt{3}}{2},
\]

\[
p_y = 0,
\]

and

\[
p_z = |\mathbf{p}| \sin \theta = p \sin \theta = 5 \sin 30^\circ = \frac{5}{2}.
\]
\[ p_z = |p| \sin \theta = p \sin \theta = 5 \left( \frac{1}{2} \right) = \frac{5}{2}. \]
The components \( q_x, q_y, \) and \( q_z \) of the vector \( q \) are \( q_x = q = 4, q_y = 0 \) and \( q_z = 0. \)

It results

\[
\mathbf{v} = \mathbf{p} \times \mathbf{q} = (p_y q_z - p_z q_y) \mathbf{i} + (p_z q_x - p_x q_z) \mathbf{j} + (p_x q_y - p_y q_x) \mathbf{k}
\]
\[= \left( 0(0) - \frac{5}{2}(0) \right) \mathbf{i} + \left( \frac{5}{2}(4) - \frac{5\sqrt{3}}{2}(0) \right) \mathbf{j} + \left( \frac{5\sqrt{3}}{2}(0) - 0(4) \right) \mathbf{k}
\]
\[= \frac{5}{2}(4) \mathbf{j} = 10 \mathbf{j}. \]

The MATLAB program for the cross product \( \mathbf{v} = \mathbf{p} \times \mathbf{q} \) is:

```
syms p q theta real
p_x = p*cos(theta); p_y = 0; p_z = p*sin(theta);
q_x = q; q_y = 0; q_z = 0;
v = cross([p_x p_y p_z],[q_x q_y q_z]);
slist = [p, q, theta]; nlist = [5, 4, pi/6];
vn = subs(v, slist, nlist);
fprintf('p x q = ')
fprintf('[%s %s %s]',char(v(1)),char(v(2)),char(v(3)))
fprintf(' = [%g %g %g] \n', vn)
```

and the output is:

```
p x q = [0 p*sin(theta)*q 0] = [0 10 0]
```

The function \( \text{char}(\mathbf{x}) \) converts the array \( \mathbf{x} \) into MATLAB character array.

Next the vectors \( \mathbf{p}, \mathbf{q}, \) and \( \mathbf{p} \times \mathbf{q} \) will be plotted using MATLAB. The numerical values of the components of the vectors \( \mathbf{p} \) and \( \mathbf{q} \) are calculated with:

```
p_xn=double(subs(p_x,slist,nlist));
p_yn=double(subs(p_y,slist,nlist));
p_zn=double(subs(p_z,slist,nlist));
q_xn=double(subs(q_x,slist,nlist));
q_yn=double(subs(q_y,slist,nlist));
q_zn=double(subs(q_z,slist,nlist));
```

The statement \( \text{double}(\mathbf{x}) \) converts the symbolic matrix \( \mathbf{x} \) to a matrix of double precision floating point numbers. The Cartesian axes \( x, y, z \) are plotted with:

```
axis ([0 6 0 8 0 5])
axis auto, grid on, hold on
xlabel(\'it x\'), ylabel(\'it y\'), zlabel(\'it z\')
quiver3(0,0,0,6,0,0,1,'Color','k','LineWidth',1)
text(\'Interpreter\','latex\','String\',' $x$\', \'Position\',[6,0,0],'FontSize',14)
```
The statement `axis auto` returns the axis scaling to its default automatic mode. The vectors \( \mathbf{p} \), \( \mathbf{q} \), and \( \mathbf{v} = \mathbf{p} \times \mathbf{q} \) are plotted with the MATLAB commands:

```matlab
quiver3(0,0,0,0,p_xn,p_yn,p_zn,1,...
    'Color','b','LineWidth',1.5)
quiver3(0,0,0,q_xn,q_yn,q_zn,1,...
    'Color','b','LineWidth',1.5)
quiver3(0,0,0,vn(1),vn(2),vn(3),1,...
    'Color','r','LineWidth',2.5)
text('Interpreter','latex','String',' $\bf{q}$',...
    'Position',[q_xn,q_yn,q_zn],...
    'FontSize',14)
text('Interpreter','latex','String',' $\bf{p}$',...
    'Position',[p_xn,p_yn,p_zn],...
    'FontSize',14)
text('Interpreter','latex','String',' $\bf{p} \times \bf{q}$',...
    'Position',[vn(1)+.5,vn(2),vn(3)],...
    'FontSize',14)
text('Interpreter','latex','String',' $O$',...
    'Position',[0,0,0-.5],'
    'FontSize',1)
```

The MATLAB drawing of the vectors is shown in Fig. E1.5(b).

**Example 1.6**
Compute \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \), \( (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \) and \( (\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a} \) where \( \mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \), \( \mathbf{b} = b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k} \), and \( \mathbf{c} = c_x \mathbf{i} + c_y \mathbf{j} + c_z \mathbf{k} \). Numerical application: \( a_x = 2, a_y = 1, a_z = 3, b_x = 2, b_y = 1, b_z = 0, c_x = 2, c_y = 0, \) and \( c_z = 0 \).

**Solution**
The scalar \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \) is
\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \cdot \begin{vmatrix} 1 & j & k \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \]

\[ = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \]

\[ = a_x (b_y c_z - b_z c_y) + a_y (b_z c_x - b_x c_z) + a_z (b_x c_y - b_y c_x) \]

\[ = 2 (1(0) - 0(0)) + 1 (0(2) - 2(0)) + 3 (2(0) - 1(2)) = -6 \]

The scalar \((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\) is

\[ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} 1 & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \cdot (c_x \mathbf{i} + c_y \mathbf{j} + c_z \mathbf{k}) \]

\[ = (c_x a_x + c_y a_y + c_z a_z) \begin{vmatrix} 1 & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \]

\[ = a_x (b_y c_z - b_z c_y) + a_y (b_z c_x - b_x c_z) + a_z (b_x c_y - b_y c_x) \]

\[ = 2 (1(0) - 0(0)) + 1 (0(2) - 2(0)) + 3 (2(0) - 1(2)) = -6 \]

The scalar \((\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a}\) is

\[ (\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a} = \begin{vmatrix} 1 & j & k \\ c_x & c_y & c_z \\ b_x & b_y & b_z \end{vmatrix} \cdot (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \]

\[ = (a_x c_x + a_y c_y + a_z c_z) \begin{vmatrix} 1 & j & k \\ c_x & c_y & c_z \\ b_x & b_y & b_z \end{vmatrix} \]

\[ = - [a_x (b_y c_z - b_z c_y) + a_y (b_z c_x - b_x c_z) + a_z (b_x c_y - b_y c_x)] \]

\[ = -2 (1(0) - 0(0)) + 1 (0(2) - 2(0)) + 3 (2(0) - 1(2)) = 6 \]

Note that: \(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = - (\mathbf{c} \times \mathbf{b}) \cdot \mathbf{a}\).

The MATLAB program for the example is

```matlab
syms a_x a_y a_z b_x b_y b_z c_x c_y c_z real
a=[a_x a_y a_z]; b=[b_x b_y b_z]; c=[c_x c_y c_z];
d = dot(a, cross(b, c)); % a . (b x c)
```
1.9 Examples

\[ e = \text{dot}(\text{cross}(a, b), c); \% (a \times b) \cdot c \]
\[ f = \text{dot}(\text{cross}(c, b), a); \% (c \times b) \cdot a \]
\[ \text{fprintf}(''a.(b \times c)-(a \times b).c=%s\n'', \text{char} (\text{simplify} (d-e))) \]
\[ \text{fprintf}(''a.(b \times c)+(c \times b).a=%s\n'', \text{char} (\text{simplify} (d+f))) \]
\[ \text{slist} = \{a_x, a_y, a_z, b_x, b_y, b_z, c_x, c_y, c_z\}; \]
\[ \text{nlist} = \{2, 1, 3, 2, 1, 0, 2, 0, 0\}; \]
\[ \text{fprintf}(''a.(b \times c)=%g\n'', \text{subs} (d, \text{slist}, \text{nlist})) \]
\[ \text{fprintf}(''(a \times b).c=%g\n'', \text{subs} (e, \text{slist}, \text{nlist})) \]
\[ \text{fprintf}(''(c \times b).a=%g\n'', \text{subs} (f, \text{slist}, \text{nlist})) \]

Example 1.7

Find the \(c_z\) component of the vector \(c\) such as the vectors \(a = a_1 \mathbf{i} + a_j + a_z \mathbf{k}, \ b = b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k},\) and \(c = c_x \mathbf{i} + c_j + c_z \mathbf{k}\) are coplanar. Numerical application: \(a_x = 2, a_y = 3, a_z = 0, b_x = 3, b_y = 2, b_z = -2, c_x = 2,\) and \(c_y = 3.\)

**Solution**

The three vectors are coplanar if \(a \cdot (b \times c) = 0.\) The scalar \(a \cdot (b \times c)\) is

\[
\begin{vmatrix}
1 & j & k \\
b_x & b_y & b_z \\
c_x & c_y & c_z
\end{vmatrix}
= a_x (b_y c_z - b_z c_y) + a_y (b_z c_x - b_x c_z) + a_z (b_x c_y - b_y c_x)
= a_x b_y c_z - a_x b_z c_y + a_y b_z c_x - a_y b_x c_z + a_z b_x c_y - a_z b_y c_x
= a_x b_x c_z - a_y b_y c_z - a_z b_z c_z + a_y b_z c_y + a_z b_y c_x + a_z b_x c_y - a_x b_y c_x
= c_z (a_x b_y - a_y b_x) - a_x b_y c_z + a_y b_z c_x + a_z b_z c_y - a_x b_z c_y.
\]

The scalar triple product of the three vectors in MATLAB is given by

\[ \text{syms} \ a_x \ a_y \ a_z \ b_x \ b_y \ b_z \ c_x \ c_y \ c_z \ \text{real} \]
\[ \text{a} = \{a_x \ a_y \ a_z\}; \quad \text{b} = \{b_x \ b_y \ b_z\}; \quad \text{c} = \{c_x \ c_y \ c_z\}; \]
\[ \text{d} = \text{det}(\text{a}; \text{b}; \text{c}); \quad \% \ (a \times b \times c) \]

The vectors \(a, \ b,\) and \(c\) are coplanar if

\[ a \cdot (b \times c) = 0 \iff c_z (a_x b_y - a_y b_x) - a_x b_y c_z + a_y b_z c_x + a_z b_z c_y - a_x b_z c_y = 0, \]

or

\[ c_z = \frac{a_x b_z c_y - a_y b_z c_x + a_z b_x c_y}{a_x b_x - a_y b_y}. \]

Substituting with the numerical values it results

\[ c_z = \frac{2(-2)(3) - 3(-2)(2) - 0(3)(3) + 0(2)(2)}{2(2) - 3(3)} = \frac{-12 + 12 - 0 + 0}{4 - 9} = 0. \]

The given numerical vectors \(a, \ b,\) and \(c\) are coplanar if \(c_z = 0.\)
To solve the equation \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0 \), a specific MATLAB command will be used. The command
\[
\text{solve('eqn1', 'eqn2', ..., 'eqnN', 'var1', 'var2', ..., 'varN')}
\]
attempts to solve an equation or set of equations \( 'eqn1', 'eqn2', ..., 'eqnN' \) for the variables \( 'eqnN', 'var1', 'var2', ..., 'varN' \). The set of equations are symbolic expressions or strings specifying equations. The MATLAB command to find the solution \( c_z \) of the equation \( \det([\mathbf{a}; \mathbf{b}; \mathbf{c}])=0 \) is
\[
x = \text{solve(d, c_z)};
\]
and the numerical solution for \( c_z \) is displayed with
\[
\text{slist=\{a_x,a_y,a_z,b_x,b_y,b_z,c_x,c_y\};
\text{nlist=\{2,3,0,3,2,-2,2,3\};
\text{fprintf('c_z= \%g\n', \text{subs(x, slist, nlist)})}}
\]

### 1.10 Problems

1.1 a) Find the angle \( \theta \) made by the vector \( \mathbf{v} = -10\mathbf{i} + 5\mathbf{j} \) with the positive \( x \)-axis and determine the unit vector in the direction of \( \mathbf{v} \). The angle \( \theta \) is measured counterclockwise (ccw) and has the values \( 0 \leq \theta \leq 2\pi \) or \( -\pi \leq \theta \leq \pi \).

b) Determine the magnitude of the resultant \( \mathbf{p} = \mathbf{v}_1 + \mathbf{v}_2 \) and the angle that \( \mathbf{p} \) makes with the positive \( x \)-axis, where the vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are shown in Fig. P1.1. The magnitudes of the vectors are \( |\mathbf{v}_1| = 10 \), \( |\mathbf{v}_2| = 5 \), and the angles of the vectors with the positive \( x \)-axis are \( \theta_1 = 30^\circ \) and \( \theta_2 = 60^\circ \).

![Fig. P1.1 Problem 1.1](image)

1.2 The planar vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) are given in \( xOy \) plane as shown in Fig. P1.2. The magnitude of the vectors are \( a = P, b = 2P, \) and \( c = P\sqrt{2} \). The angles in the figure are \( \alpha = 45^\circ, \beta = 120^\circ, \) and \( \gamma = 30^\circ \). Determine the resultant \( \mathbf{v} = \mathbf{a} + \mathbf{b} + \mathbf{c} \) and the angle that \( \mathbf{v} \) makes with the positive \( x \)-axis.
1.3 The cube in Fig. P1.3 has the sides equal to \( l = 1 \). a) Find the direction cosines of the resultant \( \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 \). b) Determine the angle between the vectors \( \mathbf{v}_2 \) and \( \mathbf{v}_3 \). c) Find the projection of the vector \( \mathbf{v}_2 \) on the vector \( \mathbf{v}_4 \). d) Calculate \( \mathbf{v}_2 \cdot \mathbf{v}_4 \), \( \mathbf{v}_2 \times \mathbf{v}_4 \), \( \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) \), \( (\mathbf{v}_2 \times \mathbf{v}_3) \times \mathbf{v}_4 \), and \( \mathbf{v}_2 \times (\mathbf{v}_3 \times \mathbf{v}_4) \).

1.4 The vectors \( \mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \) and \( \mathbf{F}_4 \), shown in Fig. P1.4, act on the sides of a cube (the side of the cube is \( l = 2 \)). The magnitudes of the vectors are \( \mathbf{F}_1 = \mathbf{F}_2 = \mathbf{F}_3 = \mathbf{F}_4 = F = 1 \), and \( \mathbf{F}_3 = \mathbf{F}_4 = F \sqrt{2} \). a) Find the resultant \( \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 \). b) Find the direction cosines of the vector \( \mathbf{F}_4 \). c) Determine the angle between the vectors \( \mathbf{F}_1 \) and \( \mathbf{F}_3 \). d) Find the projection of the vector \( \mathbf{F}_2 \) on the vector \( \mathbf{F}_4 \). e) Calculate \( \mathbf{F}_1 \cdot \mathbf{F}_3 \), \( \mathbf{F}_2 \times \mathbf{F}_4 \), and \( \mathbf{F}_1 \cdot (\mathbf{F}_2 \times \mathbf{F}_3) \).
1.5 Figure P1.5 represents the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ acting on a cube with the side $l = 2$. The magnitude of the forces are $\mathbf{v}_1 = V = 2$ and $\mathbf{v}_2 = \mathbf{v}_3 = \mathbf{v}_4 = 2V$. a) Find the resultant and the direction cosines of the resultant $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$. b) Determine the angle between the vectors $\mathbf{v}_1$ and $\mathbf{v}_3$. c) Find the projection of the vector $\mathbf{v}_4$ on the resultant vector $\mathbf{v}$. d) Calculate $\mathbf{v}_2 \cdot \mathbf{v}$, $\mathbf{v}_1 \times \mathbf{v}_2$, and $\mathbf{v}_2 \times \mathbf{v}_4$.

1.6 Repeat the previous problem for Fig. P1.6.
1.7 The parallelepiped shown in Fig. P1.7 has the sides $l = 1$ m, $w = 2$ m, and $h = 3$ m. The magnitude of the vectors are $\mathbf{F}_1 = \mathbf{F}_2 = 10$ N, and $\mathbf{F}_3 = \mathbf{F}_4 = 20$ N. 

a) Find the resultant $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$. b) Find the unit vectors of the vectors $\mathbf{F}_1$ and $\mathbf{F}_4$. c) Determine the angle between the vectors $\mathbf{F}_1$ and $\mathbf{F}_4$. d) Find the projection of the vector $\mathbf{F}_2$ on the vector $\mathbf{F}_4$. e) Calculate $\mathbf{F}_1 \cdot \mathbf{F}_4$, $\mathbf{F}_2 \times \mathbf{F}_3$, and $\mathbf{F}_1 \cdot (\mathbf{F}_2 \times \mathbf{F}_3)$.

![Fig. P1.7 Problem 1.7](image)

1.8 A uniform rectangular plate of length $l$ and width $w$ is held open by a cable (Fig. P1.8). The plate is hinged about an axis parallel to the plate edge of length $l$. Points $A$ and $B$ are at the extreme ends of this hinged edge. Points $D$ and $C$ are at the ends of the other edge of length $l$ and are respectively adjacent to points $A$ and $B$. Points $D$ and $C$ move as the plate opens. In the closed position, the plate is in a horizontal plane. When held open by a cable, the plate has rotated through an angle $\theta$ relative to the closed position. The supporting cable runs from point $D$ to point $E$ where point $E$ is located a height $h$ directly above the point $B$ on the hinged edge of the plate. The cable tension required to hold the plate open is $T$. Find the projection of the tension force onto the diagonal axis $AC$ of the plate. Numerical application: $l = 1.0$ m, $w = 0.5$ m, $\theta = 45^\circ$, $h = 1.0$ m, and $T = 100$ N.

![Fig. P1.8 Problem 1.8](image)
1.9 The following spatial vectors are given: \( \mathbf{v}_1 = -3 \mathbf{i} + 4 \mathbf{j} - 3 \mathbf{k} \), \( \mathbf{v}_2 = 3 \mathbf{i} + 3 \mathbf{k} \), and \( \mathbf{v}_3 = 1 \mathbf{i} + 2 \mathbf{j} + 3 \mathbf{k} \). Find the expressions \( E_1 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \), \( E_2 = \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 \), \( E_3 = (\mathbf{v}_1 \times \mathbf{v}_2) \times \mathbf{v}_3 \), and \( E_4 = (\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3 \).

1.10 Find the angle between the vectors \( \mathbf{v}_1 = 2 \mathbf{i} - 4 \mathbf{j} + 4 \mathbf{k} \) and \( \mathbf{v}_2 = 4 \mathbf{i} + 2 \mathbf{j} + 4 \mathbf{k} \). Find the expressions \( \mathbf{v}_1 \times \mathbf{v}_2 \) and \( \mathbf{v}_1 \cdot \mathbf{v}_2 \).

1.11 The following vectors are given \( \mathbf{v}_1 = 2 \mathbf{i} + 4 \mathbf{j} + 6 \mathbf{k} \), \( \mathbf{v}_2 = 1 \mathbf{i} + 3 \mathbf{j} + 5 \mathbf{k} \), and \( \mathbf{v}_3 = -2 \mathbf{i} + 2 \mathbf{k} \). Find the vector triple product of \( \mathbf{v}_1 \), \( \mathbf{v}_2 \), and \( \mathbf{v}_3 \), and explain the result.

1.12 Solve the vectorial equation \( \mathbf{x} \times \mathbf{a} = \mathbf{x} \times \mathbf{b} \), where \( \mathbf{a} \) and \( \mathbf{b} \) are two known given vectors.

1.13 Solve the vectorial equation \( \mathbf{v} = \mathbf{a} \times \mathbf{x} \), where \( \mathbf{v} \) and \( \mathbf{a} \) are two known given vectors.

1.14 Solve the vectorial equation \( \mathbf{a} \cdot \mathbf{x} = m \), where \( \mathbf{a} \) is a known given vector and \( m \) is a known given scalar.