Sample Problem 2/5

The curvilinear motion of a particle is defined by \( v_x = 50 - 16t \) and \( y = 100 - 4t^2 \), where \( v_x \) is in meters per second, \( y \) is in meters, and \( t \) is in seconds. It is also known that \( x = 0 \) when \( t = 0 \). Plot the path of the particle and determine its velocity and acceleration when the position \( y = 0 \) is reached.

Solution. The \( x \)-coordinate is obtained by integrating the expression for \( v_x \), and the \( x \)-component of the acceleration is obtained by differentiating \( v_x \). Thus,

\[
\int dx = \int v_x \, dt \\
\int_0^x dx = \int_0^t (50 - 16t) \, dt \\
x = 50t - 8t^2 \text{ m}
\]

\( [a_x = \dot{v}_x] \)

\( a_x = \frac{d}{dt} (50 - 16t) \quad a_x = -16 \text{ m/s}^2 \)

The \( y \)-components of velocity and acceleration are

\( [v_y = \dot{y}] \)

\( v_y = \frac{d}{dt} (100 - 4t^2) \quad v_y = -8t \text{ m/s} \)

\( [a_y = \ddot{y}] \)

\( a_y = \frac{d}{dt} (-8t) \quad a_y = -8 \text{ m/s}^2 \)

We now calculate corresponding values of \( x \) and \( y \) for various values of \( t \) and plot \( x \) against \( y \) to obtain the path as shown. When \( y = 0 \), \( 0 = 100 - 4t^2 \), so \( t = 5 \text{ s} \). For this value of the time, we have

\( v_x = 50 - 16(5) = -30 \text{ m/s} \)

\( v_y = -8(5) = -40 \text{ m/s} \)

\( v = \sqrt{(-30)^2 + (-40)^2} = 50 \text{ m/s} \)

\( a = \sqrt{(-16)^2 + (-8)^2} = 17.89 \text{ m/s}^2 \)

The velocity and acceleration components and their resultants are shown on the separate diagrams for point \( A \), where \( y = 0 \). Thus, for this condition we may write

\( \mathbf{v} = -30\mathbf{i} - 40\mathbf{j} \text{ m/s} \quad \text{Ans.} \)

\( \mathbf{a} = -16\mathbf{i} - 8\mathbf{j} \text{ m/s}^2 \quad \text{Ans.} \)

Helpful Hint

We observe that the velocity vector lies along the tangent to the path as it should, but that the acceleration vector is not tangent to the path. Note especially that the acceleration vector has a component that points toward the inside of the curved path. We concluded from our diagram in Fig. 2/5 that it is impossible for the acceleration to have a component that points toward the outside of the curve.
Sample Problem 2/7

To anticipate the dip and hump in the road, the driver of a car applies her brakes to produce a uniform deceleration. Her speed is 100 km/h at the bottom A of the dip and 50 km/h at the top C of the hump, which is 120 m along the road from A. If the passengers experience a total acceleration of 3 m/s² at A and if the radius of curvature of the hump at C is 150 m, calculate (a) the radius of curvature at A, (b) the acceleration at the inflection point B, and (c) the total acceleration at C.

Solution. The dimensions of the car are small compared with those of the path, so we will treat the car as a particle. The velocities are

\[ v_A = \frac{100 \text{ km/h}}{3600 \text{ s}} \times 1000 \text{ m/km} = 27.8 \text{ m/s} \]
\[ v_C = \frac{50 \times 1000}{3600} = 13.89 \text{ m/s} \]

We find the constant deceleration along the path from

\[ \int v \, dv = \int a_n \, ds \]
\[ \int_{v_A}^{v_C} v \, dv = a_n \int_0^s \, ds \]
\[ a_t = \frac{1}{2} (v_C^2 - v_A^2) = \frac{(13.89)^2 - (27.8)^2}{2(120)} = -2.41 \text{ m/s}^2 \]

(a) Condition at A. With the total acceleration given and \( a_t \) determined, we can easily compute \( a_n \) and hence \( \rho \) from

\[ a^2 = a_n^2 + a_t^2 \]
\[ a_n^2 = 3^2 - (2.41)^2 = 3.19 \]
\[ a_n = 1.785 \text{ m/s}^2 \]
\[ a_n = v^2/\rho \]
\[ \rho = v^2/a_n = (27.8)^2/1.785 = 432 \text{ m} \]

Ans.

(b) Condition at B. Since the radius of curvature is infinite at the inflection point, \( a_n = 0 \) and

\[ a = a_t = -2.41 \text{ m/s}^2 \]

Ans.

(c) Condition at C. The normal acceleration becomes

\[ a_n = v^2/\rho \]
\[ a_n = (13.89)^2/150 = 1.286 \text{ m/s}^2 \]

With unit vectors \( \mathbf{e}_n \) and \( \mathbf{e}_t \) in the n- and t-directions, the acceleration may be written

\[ \mathbf{a} = 1.286 \mathbf{e}_n - 2.41 \mathbf{e}_t \text{ m/s}^2 \]

where the magnitude of \( \mathbf{a} \) is

\[ a = \sqrt{a_n^2 + a_t^2} = \sqrt{(1.286)^2 + (-2.41)^2} = 2.73 \text{ m/s}^2 \]

Ans.

The acceleration vectors representing the conditions at each of the three points are shown for clarification.

Helpful Hint

Actually, the radius of curvature to the road differs by about 1 m from that to the path followed by the center of mass of the passengers, but we have neglected this relatively small difference.
Sample Problem 2/8

A certain rocket maintains a horizontal attitude of its axis during the powered phase of its flight at high altitude. The thrust imparts a horizontal component of acceleration of 20 ft/sec², and the downward acceleration component is the acceleration due to gravity at that altitude, which is \( g = 30 \text{ ft/sec}^2 \). At the instant represented, the velocity of the mass center \( G \) of the rocket along the 15° direction of its trajectory is 12,000 mi/hr. For this position determine (a) the radius of curvature of the flight trajectory, (b) the rate at which the speed \( v \) is increasing, (c) the angular rate of line \( GC \) depends on \( v \) and \( \rho \) and is given by (d) the vector expression for the total acceleration \( a \) of the rocket.

Solution. We observe that the radius of curvature appears in the expression for the normal component of acceleration, so we use \( n \)- and \( t \)-coordinates to describe the motion of \( G \). The \( n \)- and \( t \)-components of the total acceleration are obtained by resolving the given horizontal and vertical accelerations into their \( n \)- and \( t \)-components and then combining. From the figure we get

\[
\begin{align*}
(a) & \quad a_n = 30 \cos 15° - 20 \sin 15° = 23.8 \text{ ft/sec}^2 \\
& \quad a_t = 30 \sin 15° + 20 \cos 15° = 27.1 \text{ ft/sec}^2
\end{align*}
\]

\( \hat{n} \) We may now compute the radius of curvature from

\[
\begin{align*}
(b) & \quad \rho = \frac{v^2}{a_n} = \frac{[12,000/(44/30)]^2}{23.8} = 13.01(10^6) \text{ ft} \\
& \quad \hat{v} = a_t \\
& \quad \dot{v} = 27.1 \text{ ft/sec}^2
\end{align*}
\]

\( \hat{b} \) The angular rate \( \dot{\beta} \) of line \( GC \) depends on \( v \) and \( \rho \) and is given by

\[
\begin{align*}
(c) & \quad \dot{\beta} = \frac{\dot{v}}{\rho} = \frac{12,000(44/30)}{13.01(10^6)} = 13.53(10^{-4}) \text{ rad/sec} \\
& \quad \hat{\beta} = \dot{d} \\
\end{align*}
\]

\( \hat{d} \) With unit vectors \( \vec{e}_n \) and \( \vec{e}_t \) for the \( n \)- and \( t \)-directions, respectively, the total acceleration becomes

\[
\begin{align*}
(d) & \quad \vec{a} = 23.8\vec{e}_n + 27.1\vec{e}_t \text{ ft/sec}^2
\end{align*}
\]
Sample Problem 2/9

Rotation of the radially slotted arm is governed by \( \theta = 0.2t + 0.02t^3 \), where \( \theta \) is in radians and \( t \) is in seconds. Simultaneously, the power screw in the arm engages the slider \( B \) and controls its distance from \( O \) according to \( r = 0.2 + 0.04t^2 \), where \( r \) is in meters and \( t \) is in seconds. Calculate the magnitudes of the velocity and acceleration of the slider for the instant when \( t = 3 \) s.

**Solution.** The coordinates and their time derivatives which appear in the expressions for velocity and acceleration in polar coordinates are obtained first and evaluated for \( t = 3 \) s.

\[
\begin{align*}
    & r = 0.2 + 0.04t^2 \\
    & \dot{r} = 0.08t \\
    & \ddot{r} = 0.08 \text{ m/s}^2 \\
    & \theta = 0.2t + 0.02t^3 \\
    & \dot{\theta} = 0.2 + 0.06t^2 \\
    & \ddot{\theta} = 0.12t \\
\end{align*}
\]

The velocity components are obtained from Eq. 2/13 and for \( t = 3 \) s are

\[
\begin{align*}
    [v_r = \dot{r}] & \quad v_r = 0.24 \text{ m/s} \\
    [v_\theta = r\dot{\theta}] & \quad v_\theta = 0.56(0.74) = 0.414 \text{ m/s} \\
    [v = \sqrt{v_r^2 + v_\theta^2}] & \quad v = \sqrt{(0.24)^2 + (0.414)^2} = 0.479 \text{ m/s} \quad \text{Ans.}
\end{align*}
\]

The acceleration components are obtained from Eq. 2/14 and for \( t = 3 \) s are

\[
\begin{align*}
    [a_r = \ddot{r} - r\ddot{\theta}] & \quad a_r = 0.08 - 0.56(0.74)^2 = -0.227 \text{ m/s}^2 \\
    [a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}] & \quad a_\theta = 0.56(0.36) + 2(0.24)(0.74) = 0.557 \text{ m/s}^2 \\
    [a = \sqrt{a_r^2 + a_\theta^2}] & \quad a = \sqrt{(-0.227)^2 + (0.557)^2} = 0.601 \text{ m/s}^2 \quad \text{Ans.}
\end{align*}
\]

The acceleration and its components are also shown for the 65.3° position of the arm.

Plotted in the final figure is the path of the slider \( B \) over the time interval \( 0 \leq t \leq 5 \) s. This plot is generated by varying \( t \) in the given expressions for \( r \) and \( \theta \). Conversion from polar to rectangular coordinates is given by

\[
x = r \cos \theta \quad y = r \sin \theta
\]

**Helpful Hint**

We see that this problem is an example of constrained motion where the center \( B \) of the slider is mechanically constrained by the rotation of the slotted arm and by engagement with the turning screw.
Sample Problem 2/10

A tracking radar lies in the vertical plane of the path of a rocket which is coasting in unpowered flight above the atmosphere. For the instant when \( \theta = 30^\circ \), the tracking data give \( r = 25(10^4) \) ft, \( \dot{r} = 4000 \) ft/sec, and \( \ddot{\theta} = 0.80 \) deg/sec. The acceleration of the rocket is due only to gravitational attraction and for its particular altitude is 31.4 ft/sec\(^2\) vertically down. For these conditions determine the velocity \( v \) of the rocket and the values of \( r \) and \( \theta \).

**Solution.** The components of velocity from Eq. 2/13 are

\[
\begin{align*}
\dot{v}_r &= r \\
\dot{v}_\theta &= r \dot{\theta} \\
v &= \sqrt{\dot{v}_r^2 + \dot{v}_\theta^2}
\end{align*}
\]

1. \( \dot{v}_r = r \dot{\theta} \)
   \( \dot{v}_r = 25(10^4)(0.80) \left( \frac{\pi}{180} \right) = 3490 \) ft/sec  

2. \( v = \sqrt{(4000)^2 + (3490)^2} = 5310 \) ft/sec  
   Ans.

Since the total acceleration of the rocket is \( g = 31.4 \) ft/sec\(^2\) down, we can easily find its \( r \)- and \( \theta \)-components for the given position. As shown in the figure, they are

\[
\begin{align*}
a_r &= -31.4 \cos 30^\circ = -27.2 \text{ ft/sec}^2 \\
a_\theta &= 31.4 \sin 30^\circ = 15.70 \text{ ft/sec}^2
\end{align*}
\]

We now equate these values to the polar-coordinate expressions for \( a_r \) and \( a_\theta \) which contain the unknowns \( \ddot{r} \) and \( \ddot{\theta} \). Thus, from Eq. 2/14

\[
\begin{align*}
\ddot{r} &= \frac{-27.2}{r} - \dot{r} - 25(10^4)(0.80) \left( \frac{\pi}{180} \right)^2 \\
\ddot{\theta} &= 21.5 \text{ ft/sec}^2  \\
\dot{a}_\theta &= 2r \ddot{\theta} + r \dddot{\theta} = 15.70 = 2(25)(0.80) \left( \frac{\pi}{180} \right) \\
\dddot{\theta} &= -3.84 \times 10^{-4} \text{ rad/sec}^2  \\
\end{align*}
\]

Ans.

**Helpful Hints**

1. We observe that the angle \( \theta \) in polar coordinates need not always be taken positive in a counterclockwise sense.
2. Note that the \( r \)-component of acceleration is in the negative \( r \)-direction, so it carries a minus sign.
3. We must be careful to convert \( \dot{\theta} \) from deg/sec to rad/sec.
Sample Problem 2/13

Passengers in the jet transport $A$ flying east at a speed of 800 km/h observe a second jet plane $B$ that passes under the transport in horizontal flight. Although the nose of $B$ is pointed in the 45° northeast direction, plane $B$ appears to the passengers in $A$ to be moving away from the transport at the 60° angle as shown. Determine the true velocity of $B$.

Solution. The moving reference axes $x$-$y$ are attached to $A$, from which the relative observations are made. We write, therefore,

1. $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA}$

Next we identify the knowns and unknowns. The velocity $\mathbf{v}_A$ is given in both magnitude and direction. The 60° direction of $\mathbf{v}_{BA}$, the velocity which $B$ appears to have to the moving observers in $A$, is known, and the true velocity of $B$ is in the 45° direction in which it is heading. The two remaining unknowns are the magnitudes of $\mathbf{v}_B$ and $\mathbf{v}_{BA}$. We may solve the vector equation in any one of three ways.

(I) Graphical. We start the vector sum at some point $P$ by drawing $\mathbf{v}_A$ to a convenient scale and then construct a line through the tip of $\mathbf{v}_A$ with the known direction of $\mathbf{v}_{BA}$. The known direction of $\mathbf{v}_B$ is then drawn through $P$, and the intersection $C$ yields the unique solution enabling us to complete the vector triangle and scale off the unknown magnitudes, which are found to be

$$v_{BA} = 586 \text{ km/h} \quad \text{and} \quad v_B = 717 \text{ km/h} \quad \text{Ans.}$$

(II) Trigonometric. A sketch of the vector triangle is made to reveal the trigonometry, which gives

$$\frac{v_B}{\sin 60°} = \frac{v_A}{\sin 75°} \quad v_B = 800 \frac{\sin 60°}{\sin 75°} = 717 \text{ km/h} \quad \text{Ans.}$$

(III) Vector Algebra. Using unit vectors $\mathbf{i}$ and $\mathbf{j}$, we express the velocities in vector form as

$$\mathbf{v}_A = 800\mathbf{i} \text{ km/h} \quad \mathbf{v}_B = (v_B \cos 45°)\mathbf{i} + (v_B \sin 45°)\mathbf{j}$$

$$\mathbf{v}_{BA} = (v_{BA} \cos 60°)(-\mathbf{i}) + (v_{BA} \sin 60°)\mathbf{j}$$

Substituting these relations into the relative-velocity equation and solving separately for the $\mathbf{i}$ and $\mathbf{j}$ terms give

(i-terms) \quad $v_B \cos 45° = 800 - v_{BA} \cos 60°$

(j-terms) \quad $v_B \sin 45° = v_{BA} \sin 60°$

Solving simultaneously yields the unknown velocity magnitudes

$$v_{BA} = 586 \text{ km/h} \quad \text{and} \quad v_B = 717 \text{ km/h} \quad \text{Ans.}$$

It is worth noting the solution of this problem from the viewpoint of an observer in $B$. With reference axes attached to $B$, we would write $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB}$. The apparent velocity of $A$ as observed by $B$ is then $\mathbf{v}_{AB}$, which is the negative of $\mathbf{v}_{BA}$.

Helpful Hints

1. We treat each airplane as a particle.
2. We assume no side slip due to cross wind.
3. Students should become familiar with all three solutions.
4. We must be prepared to recognize the appropriate trigonometric relation, which here is the law of sines.
5. We can see that the graphical or trigonometric solution is shorter than the vector algebra solution in this particular problem.
Sample Problem 2/14

Car A is accelerating in the direction of its motion at the rate of 3 ft/sec$^2$. Car B is rounding a curve of 440-ft radius at a constant speed of 30 mi/hr. Determine the velocity and acceleration which car B appears to have to an observer in car A if car A has reached a speed of 45 mi/hr for the positions represented.

Solution. We choose nonrotating reference axes attached to car A since the motion of B with respect to A is desired.

**Velocity.** The relative-velocity equation is

\[ \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} \]

and the velocities of A and B for the position considered have the magnitudes

\[ v_A = 45 \frac{5280}{60^2} = 45 \frac{44}{30} = 66 \text{ ft/sec} \quad v_B = 30 \frac{44}{30} = 44 \text{ ft/sec} \]

The triangle of velocity vectors is drawn in the sequence required by the equation, and application of the law of cosines and the law of sines gives

\[ v_{BA} = 58.2 \text{ ft/sec} \quad \theta = 40.9^\circ \]

**Acceleration.** The relative-acceleration equation is

\[ \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} \]

The acceleration of A is given, and the acceleration of B is normal to the curve in the n-direction and has the magnitude

\[ a_n = \frac{v^2}{\rho} \quad a_B = \frac{(44)^2}{440} = 4.4 \text{ ft/sec}^2 \]

The triangle of acceleration vectors is drawn in the sequence required by the equation as illustrated. Solving for the x- and y-components of \( \mathbf{a}_{BA} \) gives us

\[ (a_{BA})_x = 4.4 \cos 30^\circ - 3 = 0.810 \text{ ft/sec}^2 \]
\[ (a_{BA})_y = 4.4 \sin 30^\circ = 2.2 \text{ ft/sec}^2 \]

from which \( a_{BA} = \sqrt{(0.810)^2 + (2.2)^2} = 2.34 \text{ ft/sec}^2 \)

The direction of \( \mathbf{a}_{BA} \) may be specified by the angle \( \beta \) which, by the law of sines, becomes

\[ \frac{4.4}{\sin \beta} = \frac{2.34}{\sin 30^\circ} \quad \beta = \sin^{-1} \left( \frac{4.4}{2.34} \cdot 0.5 \right) = 110.2^\circ \]

Helpful Hints

1. Alternatively, we could use either a graphical or a vector algebraic solution.
2. Be careful to choose between the two values 69.8° and 180° – 69.8° = 110.2°.

Suggestion: To gain familiarity with the manipulation of vector equations, it is suggested that the student rewrite the relative-motion equations in the form \( \mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A \) and \( \mathbf{a}_{BA} = \mathbf{a}_B - \mathbf{a}_A \) and redraw the vector polygons to conform with these alternative relations.

Caution: So far we are only prepared to handle motion relative to nonrotating axes. If we had attached the reference axes rigidly to car B, they would rotate with the car, and we would find that the velocity and acceleration terms relative to the rotating axes are not the negative of those measured from the nonrotating axes moving with A. Rotating axes are treated in Art. 5/7.