# Chapter 6
# Friction

If a body rests on an incline plane, the friction force exerted on it by the surface prevents it from sliding down the incline. The question is, what is the steepest incline on which the body can rest?

A body is placed on a horizontal surface. The body is pushed with a small horizontal force $F$. If the force $F$ is sufficiently small, the body does not move.

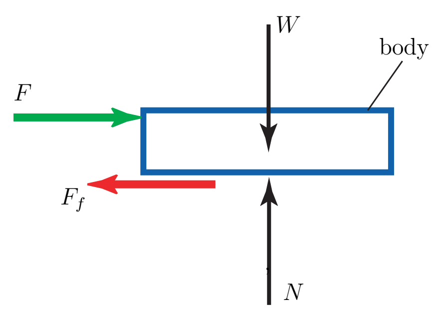


**Fig. 6.1** Free-body diagram of the body

Figure 6.1 shows the free-body diagram of the body, where the force $W$ is the weight force of the body, and $N$ is the normal force exerted by the surface on the body. The force $F$ is the horizontal force, and $F_f$ is the friction force exerted by the surface. Friction force arises in part from the interactions of the roughness, or asperities, of the contacting surfaces. The body is in equilibrium and $F_f = F$.

The force $F$ is slowly increased. As long as the body remains in equilibrium, the friction force $F_f$ must increase correspondingly, since it equals the force $F$. The body slips on the surface. The friction force, after reaching the maximum value, cannot maintain the body in equilibrium. The force applied to keep the body moving on the surface is smaller than the force required to cause it to slip. Why more force is required to start the body sliding on a surface than to keep it sliding is explained in part by the necessity to break the asperities of the contacting surfaces before sliding can begin.

The theory of dry friction, or *Coulomb friction*, predicts:

• the maximum friction forces that can be exerted by dry, contacting surfaces that are stationary relative to each other;

• the friction forces exerted by the surfaces when they are in relative motion, or sliding.

**Static Coefficient of Friction**

The magnitude of the maximum friction force, $F_f$, that can be exerted between two plane dry surfaces in contact is

$$F_f = \mu_s N, \tag{6.1}$$

where $\mu_s$ is a constant, the *static coefficient of friction*, and $N$ is the normal component of the contact force between the surfaces. The value of the static coefficient of friction, $\mu_s$, depends on:

• the materials of the contacting surfaces;

• the conditions of the contacting surfaces namely smoothness and degree of contamination.

Typical values of $\mu_s$ for various materials are shown in Table 6.1.

Table 6.1. Typical values of the static coefficient of friction.

| Materials | $\mu_s$ |
|---|---|
| metal on metal | 0.15 - 0.20 |
| metal on wood | 0.20 - 0.60 |
| metal on masonry | 0.30 - 0.70 |
| wood on wood | 0.25 - 0.50 |
| masonry on masonry | 0.60 - 0.70 |
| rubber on concrete | 0.50 - 0.90 |

Equation (6.1) gives the maximum friction force that the two surfaces can exert without causing it to slip. If the static coefficient of friction $\mu_s$ between the body and the surface is known, the largest value of $F$ one can apply to the body without causing it to slip is $F = F_f = \mu_s N$. Equation (6.1) determines the magnitude of the maximum friction force but not its direction. The friction force resists the impending motion.

**Kinetic coefficient of friction**

The magnitude of the friction force between two plane dry contacting surfaces that are in motion relative to each other is

$$F_f = \mu_k N, \tag{6.2}$$

where $\mu_k$ is the *kinetic coefficient of friction* and $N$ is the normal force between the surfaces. The value of the kinetic coefficient of friction is generally smaller than the value of the static coefficient of friction, $\mu_s$.

To keep the body in Fig. 6.1 in uniform motion (sliding on the surface) the force exerted must be $F = F_f = \mu_k N$. The friction force resists the relative motion, when two surfaces are sliding relative to each other.

The body $RB$ shown in Fig. 6.2(a) is moving on the fixed surface 0.

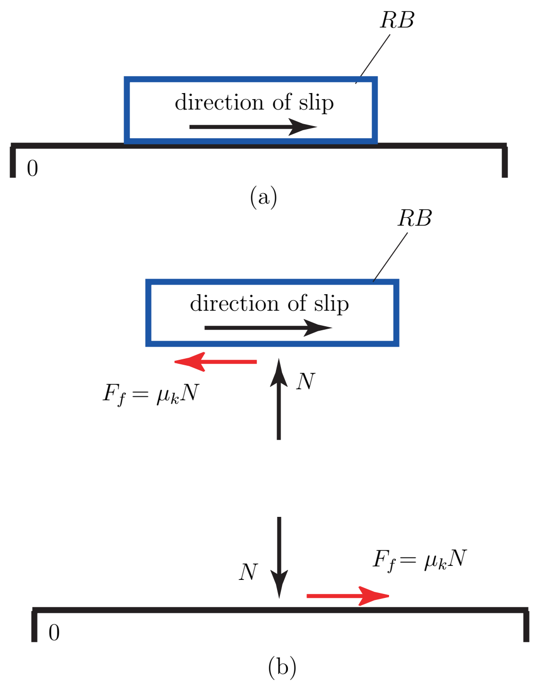


**Fig. 6.2** Directions of the friction forces

The direction of motion of $RB$ is the positive axis $x$. The friction force on the body $RB$ acts in the direction opposite to its motion, and the friction force on the fixed surface is in the opposite direction as shown in Fig. 6.2(b).

**Angles of Friction**

The *angle of friction*, $\theta$, is the angle between the friction force, $F_f = |\mathbf{F}_f|$, and the normal force to the surface $N = |\mathbf{N}|$, as shown in Fig. 6.3.

The magnitudes of the normal force and friction force, and $\theta$ are related by

$$F_f = R \sin\theta,$$
$$N = R \cos\theta,$$

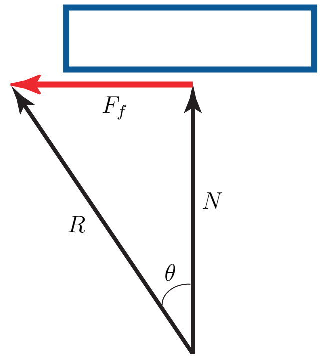


**Fig. 6.3** Angle of friction, $\theta$

where $R = |\mathbf{R}| = |\mathbf{N} + \mathbf{F}_f|$.

The value of the angle of friction when slip is impending is called the *static angle of friction*, $\theta_s$,

$$\tan\theta_s = \mu_s.$$

The value of the angle of friction when the surfaces are sliding relative to each other is called the the *kinetic angle of friction*, $\theta_k$,

$$\tan\theta_k = \mu_k.$$

## 6.1 Screws

Threaded fasteners include bolts, studs, and various forms of screws. Fixed fasteners include welds, solders, brazing, adhesives, and rivets. Threaded fasteners such as screws, nuts, and bolts are important components of mechanical structures and machines. Screws may be used as removable fasteners or as devices for moving loads.

### *6.1.1 Screw Thread*

A screw thread is a uniform wedge-shaped section in the form of a helix on the external or internal surface of a cylinder (straight thread) or a cone (taper thread). The basic arrangement of a helical thread wound around a cylinder is illustrated in Fig 6.4

The terminology of an external screw threads is:

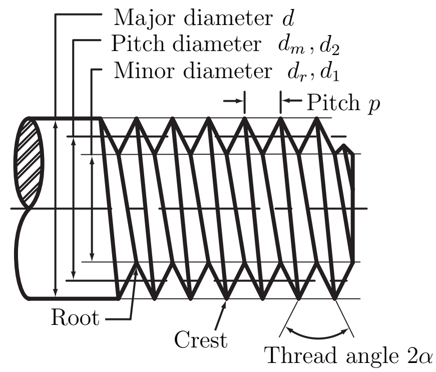


**Fig. 6.4** Screw thread

- *pitch* denoted by $p$ is the distance, parallel to the screw axis, between corresponding points on adjacent thread forms having uniform spacing.
- *major diameter* denoted by $d$ is the largest (outside) diameter of a screw thread.
- *minor diameter* denoted by $d_r$ or $d_1$, is the smallest diameter of a screw thread.
- *pitch diameter* denoted by $d_m$ or $d_2$ is the imaginary diameter for which the width of the threads and the grooves are equal.

The *lead* denoted by $l$ is the distance the nut moves parallel to the screw axis when the nut is given one turn (distance a threaded section moves axially in one revolution). A screw with two or more threads cut beside each other is called a *multiple-threaded* screw. The lead is equal to twice the pitch for a double-threaded screw, and up to 3 times the pitch for a triple-threaded screw. The pitch $p$, lead $l$, and lead angle $\lambda$ are represented in Fig. 6.5.

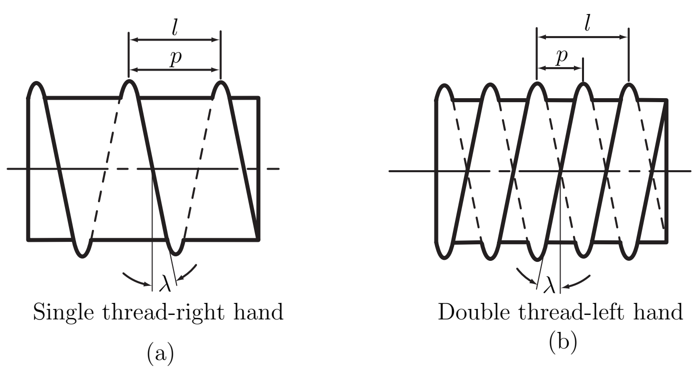


**Fig. 6.5** Pitch $p$, lead $l$, and lead angle $\lambda$

Figure 6.5(a) shows a single thread right-hand screw and Fig. 6.5(b) shows a double-threaded left-hand screw. If a thread traverses a path in a clockwise and receding direction when viewed axially, it is a *right-hand thread*. All threads are assumed to be right-hand, unless otherwise specified.

Metric threads are specified by the letter M preceding the nominal major diameter in millimeters and the pitch in millimeters per thread. For example: M 14 × 2, M is the SI thread designation, 14 mm is the outside (major) diameter, and the pitch is 2 mm per thread. Screw size in the Unified system is designated by the size number for major diameter (in.), the number of treads per in., and the thread form and series, like this: $\frac{5''}{8} - 18$, UNF $\frac{5''}{8}$ is the the outside (major) diameter where the double tick marks mean inches, and 18 threads per in. Some Unified thread series are: UNC Unified National Coarse, UNEF Unified National Extra Fine, UNF Unified National Fine, UNS Unified National Special, and UNR Unified National Round (round root).

### *6.1.2 Power Screws*

Power screws are used to convert rotary motion to linear motion of the meeting member along the screw axis. These screws are used to lift weights (screw-type jacks) or exert large forces (presses, tensile testing machines). The power screws can also be used to obtain precise positioning of the axial movement.

A square-threaded power screw with a single thread having the pitch diameter $d_m$, the pitch $p$, and the helix angle $\lambda$ is considered in Fig. 6.6. A square thread profile is shown in Fig. 6.7.

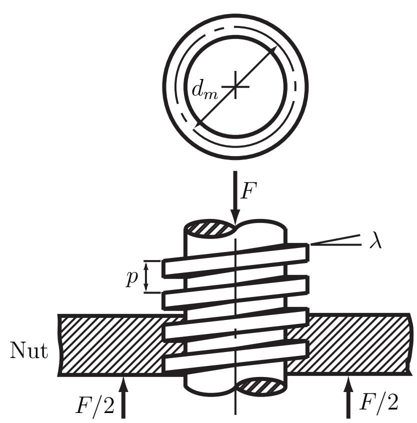


**Fig. 6.6** Power screw with a single thread

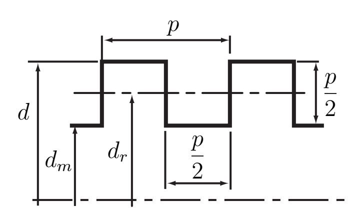


**Fig. 6.7** Square thread

Consider that a single thread of the screw is unrolled for exactly one turn. The edge of the thread is the hypotenuse of a right triangle and the height is the lead. The base of the right triangle is the circumference of the pitch diameter circle (Fig. 6.8).

The lead angle $\lambda$ is the helix angle of the thread. The screw is loaded by an axial compressive force $F$ (Figs. 6.6 and 6.8). The force diagram for lifting the load is

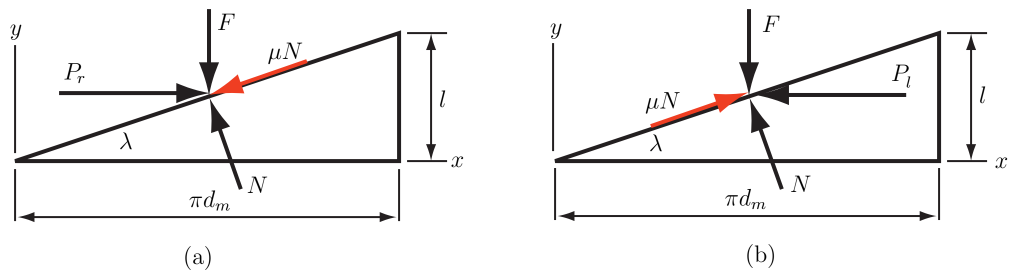


**Fig. 6.8** (a) Force diagram for lifting the load and (b)force diagram for lowering the load

shown in Fig. 6.8(a), (the force $P_r$ is positive). The force diagram for lowering the load is shown in Fig. 6.8(b), (the force $P_l$ is negative). The friction force is

$$F_f = \mu N,$$

where $\mu$ is the coefficient of dry friction and $N$ is the normal force. The friction force is acting opposite to the motion. The equilibrium of forces for raising the load gives

$$\sum F_x = P_r - N\ \sin\lambda - \mu\ N\ \cos\lambda = 0, \tag{6.3}$$
$$\sum F_y = F + \mu\ N \sin\lambda - N\ \cos\lambda = 0. \tag{6.4}$$

Similarly, for lowering the load one may write the equations

$$\sum F_x = -P_l - N\ \sin\lambda + \mu\ N\ \cos\lambda = 0, \tag{6.5}$$
$$\sum F_y = F - \mu\ N\ \sin\lambda - N\ \cos\lambda = 0. \tag{6.6}$$

Eliminating $N$ and solving for $P_r$

$$P_r = \frac{F\ (\sin\lambda + \mu\ \cos\lambda)}{\cos\lambda - \mu\ \sin\lambda}, \tag{6.7}$$

and for lowering the load

$$P_l = \frac{F\ (\mu\ \cos\lambda - \sin\lambda)}{\cos\lambda + \mu\ \sin\lambda}. \tag{6.8}$$

Using the relation

$$\tan\lambda = l/(\pi d_m),$$

and dividing the equations by $\cos\lambda$ one may obtain

$$P_r = \frac{F\ [(l\ \pi\ d_m) + \mu]}{1 - (\mu\ l\ \pi\ d_m)}, \quad (6.9)$$

$$P_l = \frac{F\ [\mu - (l\ \pi\ d_m)]}{1 + (\mu\ l\ \pi\ d_m)}. \quad (6.10)$$

The moment required to overcome the thread friction and to raise the load is

$$M_r = P_r \frac{d_m}{2} = \frac{F\ d_m}{2} \left( \frac{l + \pi\ \mu\ d_m}{\pi\ d_m - \mu\ l} \right). \quad (6.11)$$

The moment required to lower the load (and to overcome a part of the friction) is

$$M_l = \frac{F\ d_m}{2} \left( \frac{\pi\ \mu\ d_m - l}{\pi\ d_m + \mu\ l} \right). \quad (6.12)$$

When the lead, $l$, is large or the friction, $\mu$, is low the load will lower itself. In this case the screw will spin without any external effort, and the moment $M_l$ in Eq. (6.12) will be negative or zero. When the moment is positive, $M_l > 0$, the screw is said to be *self-locking*. The condition for self-locking is

$$\pi\ \mu\ d_m > l.$$

Dividing both sides of this inequality by $\pi\ d_m$, and using $l/(\pi d_m) = \tan\lambda$, yields

$$\mu > \tan\lambda. \quad (6.13)$$

The self-locking is obtained whenever the coefficient of friction is equal to or greater than the tangent of the thread lead angle.

The moment, $M_0$, required only to raise the load when the friction is zero, $\mu = 0$, is obtained from Eq. (6.11):

$$M_0 = \frac{F\ l}{2\ \pi}. \quad (6.14)$$

The screw efficiency $e$ can be defined as

$$e = \frac{M_0}{M_r} = \frac{F\ l}{2\ \pi\ M_r}. \quad (6.15)$$

For square threads the normal thread load, $F$, is parallel to the axis of the screw. The preceding equations can be applied for square threads.

### 6.1.3 Force Analysis for a Square-Threaded Screw

Consider a square-threaded jack under the action of a axial load $F$ and a moment $M$ about the axis of the screw, Fig. 6.9. The screw has the mean radius $r_m$ and the lead

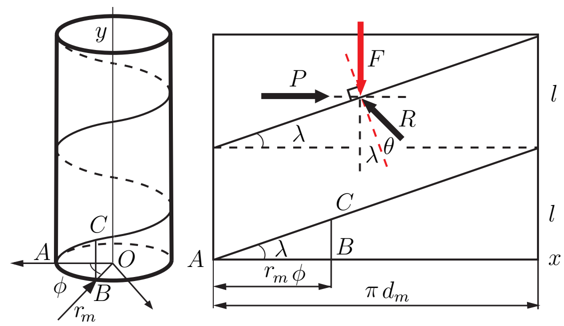


**Fig. 6.9** Force diagram for a square-fhreaded screw

$l$. The force exerted by the frame thread on the screw thread is $R$. The angle $\theta$ made by $R$ with the normal to the thread is the angle of friction (see Fig. 6.9)

$$\tan\theta = \mu = \frac{F_f}{N}.$$

The unwrapped thread of the screw shown in Fig. 6.9 is for lifting the load. The force equilibrium equation in the axial direction is

$$F = R\ \cos(\lambda+\theta),$$

where $\lambda$ is the helix angle, $\tan\lambda = l/(2\ \pi\ r_m)$. The moment of $R$ about the vertical axis of the screw is $R\,r_m\ \sin(\lambda+\theta)$. The moment equilibrium equation for the screw becomes

$$M = R\ r_m\ \sin(\lambda+\theta).$$

Combining the expression for $F$ and $M$ gives

$$M = M_r = F\ r_m\ \tan(\lambda+\theta). \tag{6.16}$$

The force required to push the thread up is $P = M/r_m$. The moment required to lower the load by unwinding the screw is obtained in a similar manner:

$$M = M_l = F\ r_m\ \tan(\theta-\lambda). \tag{6.17}$$

If $\theta < \lambda$ the screw will unwind by itself.

In general, when the screw is loaded axially, a thrust bearing or thrust collar may be used between the rotating and stationary links to carry the axial component

(Fig. 6.10). The load is concentrated at the mean collar diameter $d_c$. The moment

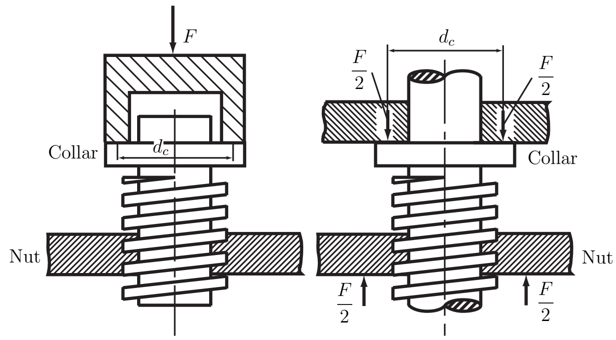


**Fig. 6.10** Thrust collar

required is

$$M_c = \frac{F\ \mu_c\ d_c}{2}, \tag{6.18}$$

where $\mu_c$ is the coefficient of collar friction.

## *6.1.4 Examples*

*Example 6.1.1*: Double square-thread power screw.
A double square-thread power screw (Fig. 6.11) has the major diameter $d = 40$ mm and the pitch $p = 6$ mm. The coefficient of friction of the thread is $\mu = 0.08$ and the coefficient of collar friction is $\mu_c = 0.1$. The mean collar diameter is $d_c = 45$ mm. The external load on the screw is $F = 8$ kN. Find: a) the lead, the pitch (mean) diameter and the minor diameter; b) the moment required to raise the load; c) the moment required to lower the load; d) the efficiency of the device.

*Solution*
a) From Fig. 6.7:
the minor diameter is $d_r = d - p = 40 - 6 = 34$ mm,
the pitch (mean) diameter is $d_m = d - p/2 = 40 - 3 = 37$ mm,
the lead is $l = 2\,p = 2\,(6) = 12$ mm.
b) The moment required to raise the load is [Eqs. (6.11) and (6.18)]

$$M_r = \frac{F d_m}{2}\left(\frac{l + \pi\mu d_m}{\pi d_m - \mu l}\right) + \frac{F \mu_c d_c}{2}$$

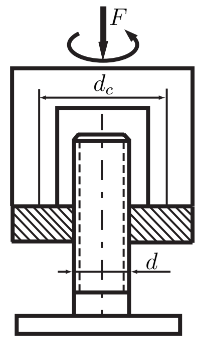


**Fig. 6.11** Example 6.1.1

$$= \frac{8\,(10^3)\,(37)\,(10^{-3})}{2}\left[\frac{12+0.08(37)\pi}{37\pi-0.08\,(12)}\right]+\frac{8\,(10^3)\,(0.1)\,(45)\,(10^{-3})}{2}$$
$$= 45.344 \text{ N m}.$$

c) The moment required to lower the load is [Eqs. (6.12) and (6.18)]:

$$M_l = \frac{F d_m}{2}\left(\frac{\pi\mu d_m - l}{\pi d_m + \mu l}\right) + \frac{F\mu_c d_c}{2}$$
$$= \frac{8\,(10^3)\,(37)\,(10^{-3})}{2}\left[\frac{0.08(37)\pi-12}{37\pi+0.08\,(12)}\right]+\frac{8\,(10^3)\,(0.1)\,(45)\,(10^{-3})}{2}$$
$$= 14.589 \text{ N m}.$$

The screw is not self-locking: $\pi\mu d_m - l = 0.08(37)\pi - 12 = -2.700 < 0$.
d) The overall efficiency is [Eq. (6.15)]:

$$e = \frac{F l}{2\pi M_r} = \frac{8\,(10^3)\,(12)(10^{-3})}{2(45.344)\pi} = 0.336.$$

### *6.1.5 Problems*

6.1.1 The double square-threaded screw has the major diameter $d = 1$ in. and the pitch $p = 0.2$ in. The coefficient of friction in the threads is 0.15. A moment $M = 60$ lb-in. is applied about the axis of the screw, Fig. 6.12. Find the axial force required to advance the screw: a) to the right, and b) to the left.

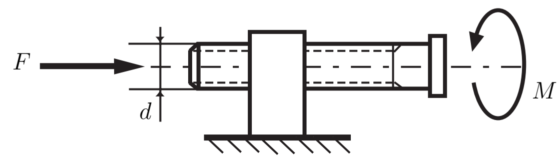


**Fig. 6.12** Problem 6.1.1

6.1.2 A double square-thread power screw has a pitch (mean) diameter of 30 mm and a pitch of 4 mm. The coefficient of friction of the thread is 0.08 and the coefficient of collar friction is also 0.08. The mean collar diameter is 40 mm. The external load on the screw is 6.4 kN. Determine the moment required to lower the load and the overall efficiency.

6.1.3 A power screw has a double square thread with a mean diameter of 40 mm and a pitch of 12 mm. The coefficient of friction in the thread is 0.15. Determine if the the screw is self-locking.

6.1.4 The single-threaded screw of a vise has a mean diameter of 1 in. and has 5 square threads per in. The coefficient of static friction in the threads is 0.20. Determine the helix angle and the fiction angle for the thread.

6.1.5 A triple-thread screw is used in a jack to raise a load of 4000 lb. The major diameter of the screw is 3 in. A plain thrust collar is used. The mean diameter of the collar is 4 in. The coefficient of friction of the thread is 0.08 and the coefficient of collar friction is 0.1. Determine: a) the screw pitch, lead, thread depth, mean pitch diameter, and helix angle; b) the starting moment for raising and for lowering the load; c) the efficiency of the jack.

6.1.6 A C-clamp develops a 250 lb clamping force (Fig. 6.13). The clamp uses a 1/2 in. Acme single thread. The collar of the clamp has a mean diameter of 5/8 in. The coefficients of running friction are estimated as 0.1 for both the collar and the screw. Estimate the force required at the end of a 6 in. handle.

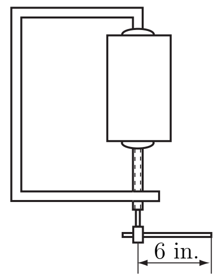


**Fig. 6.13** Problem 6.1.6

## 6.2 Disk Friction and Flexible Belts

### *6.2.1 Disk Friction*

The sliding surfaces are present in most machine components (bearings, gears, cams, etc.) and it is desirable to minimize the friction in order to reduce energy loss and wear. In contrast, clutches and brakes depend on friction in order to function. The function of a clutch is to permit smooth, gradual connection and disconnection of two elements having a common axis of rotation. A brake acts similarly except that one of the elements is fixed.

In pivot bearings, clutch plates, and disk brakes there is friction between circular surfaces under distributed normal pressure. Two flat circular disks are considered in Fig. 6.14. The figure shows a simple disk clutch with one driving and one driven

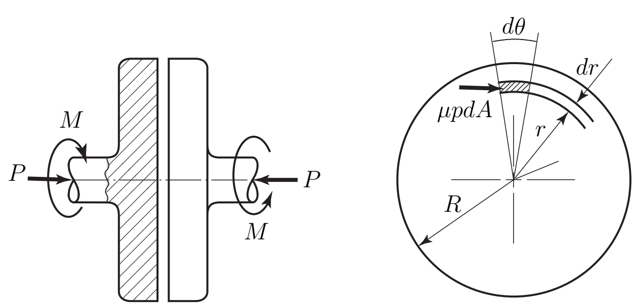


**Fig. 6.14** Simple disk clutch

surface. Driving friction between the two develops when they are forced together.

The disks can be brought into contact under an axial force $P$. The maximum moment that this clutch can transmit is equal to the moment $M$ required to slip one disk against the other. The elemental frictional force acting on an elemental area is

$$dF_f = \mu\, p\, dA,$$

where $p$ is the normal pressure at any location between the plates, $\mu$ is the coefficient of friction, and $dA = r\, dr\, d\theta$ is the area of the element.

The moment of this elemental friction force about the shaft axis is

$$dM = \mu\, p\, r\, dA,$$

and the total moment is

$$M = \int\int \mu\, p\, r\, dA,$$

where the integral is evaluated over the area of the disk.

The coefficient of friction, $\mu$, is assumed to be constant. If the disk surfaces are new, flat, and well supported it is assumed that the pressure $p$ is uniform over the entire surface so that

$$P = \pi R^2 p.$$

The total frictional moment becomes

$$M = \int\int \mu \frac{\mu P}{\pi R^2} r\, dA = \frac{\mu P}{\pi R^2} \int_0^{2\pi} \int_0^R r^2\, dr\, d\theta = \frac{2}{3} \mu P R. \qquad (6.19)$$

The total moment is equal to a friction force $\mu P$ acting at a distance $2R/3$ from the shaft center. If the friction disks are rings, as shown in Fig. 6.15, the frictional moment is

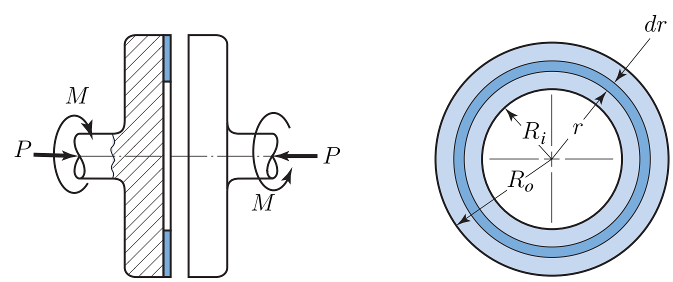


**Fig. 6.15** Disk clutch with ring friction disks

$$M = \frac{\mu P}{\pi R^2} \int_0^{2\pi} \int_{R_i}^{R_o} r^2 \, dr \, d\theta = \frac{2}{3} \mu P \frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}, \tag{6.20}$$

where $R_o$ and $R_i$ are the inside and outside radii.

It is reasonable to assume that after the initial wearing-in period is over, the surfaces retain their new relative shape and further wear is therefore constant over the surface. This wear depends on both the pressure $p$ and the circumferential distance traveled. The distance traveled is proportional to $r$. Therefore the following expression may be written:

$$r\,p = K,$$

where $K$ is a constant that is determined from the equilibrium condition for the axial forces

$$P = \int p \, dA = K \int_0^{2\pi} \int_0^R dr \, d\theta = 2\pi K R.$$

The constant $K$ is

$$K = \frac{P}{2\pi R}.$$

With $p\,r = P/(2\pi R)$, the frictional moment is

$$M = \int \int \mu \, p \, r \, dA = \frac{\mu P}{2\pi R} \int_0^{2\pi} \int_0^R r \, dr \, d\theta = \frac{1}{2} \mu P R. \tag{6.21}$$

The frictional moment for worn-in plates is, therefore, only (1/2)/(2/3)=3/4, as much as for new surfaces. If the friction disks are rings of inside radius $R_i$ and outside radius $R_o$, the frictional moment for worn-in surfaces is

$$M = \frac{1}{2} \mu P (R_o + R_i). \tag{6.22}$$

Actual clutches employ $N$ friction interfaces transmitting torque in parallel. The number of friction interfaces $N$ is an even number. For a clutch with $N$ friction interfaces, Eq. (6.22) is modified to give

$$M = \frac{1}{2} \mu P (R_o + R_i) N. \tag{6.23}$$

The ratio of inside to outside radius is a parameter in the design of clutches. The maximum moment for a given outside radius is obtained when [**?**]

$$R_i = R_0 \sqrt{\frac{1}{3}} = 0.58 R_0, \tag{6.24}$$

and the proportions commonly used range from $R_i = 0.45 R_0$ to $R_i = 0.80 R_0$.

Disk clutches can be designed to operate either “dry” or “wet” with oil. Most multiple-disk clutches, including those used in automotive automatic transmissions, operate wet.

### 6.2.2 Flexible Belts

In the design of belt drives and band brakes the impending slippage of flexible cables, belts, and ropes over sheaves and drums is important. Figure 6.16(a) shows a drum subjected to the two belt tensions $T_1$ and $T_2$, the moment $M$ necessary to prevent rotation, and a bearing reaction $R$. Figure 6.16(b) shows the free-body diagram of an element of the belt of length $r\,d\theta$. The forces acting on the differential element are calculated using the equilibrium of the element. The tension increases from $T$ at the angle $\theta$ to $T + dT$ at the angle $\theta + d\theta$. The normal force which acts on the differential element of area is a differential $dN$. The friction force, $\mu\,dN$ is impending motion and acts on the belt in a direction to oppose slipping.

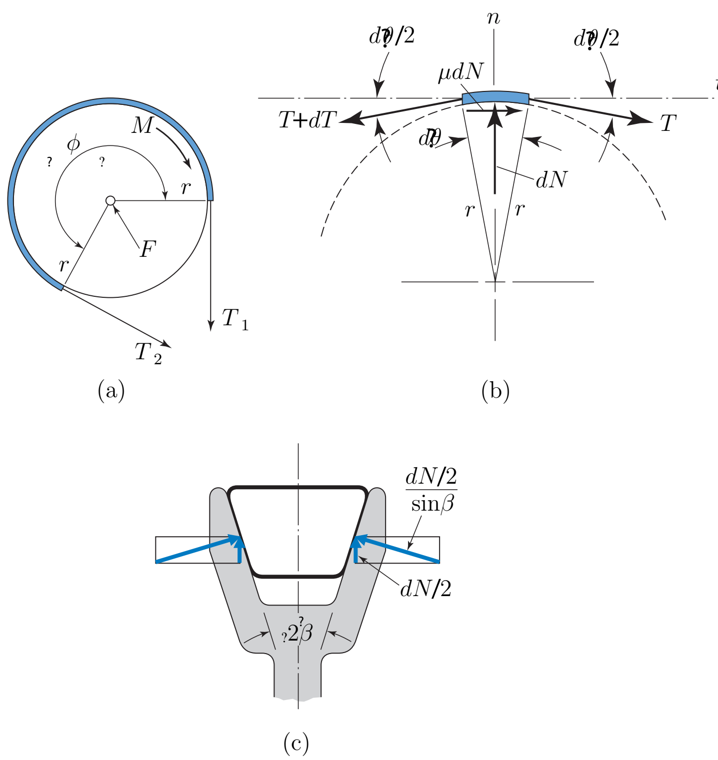


**Fig. 6.16** a) Drum subjected to belt tensions; b) free-body diagram of an element of the belt; c) V-belt of angle $\beta$

The equation for the equilibrium of forces in the $t$-direction gives

$$T\cos\frac{d\theta}{2} + \mu\,dN = (T + dT)\cos\frac{d\theta}{2} \quad \text{or} \quad \mu\,dN = dT, \tag{6.25}$$

where the cosine of the differential quantity is unity in the limit ($\cos d\theta/2 \approx 1$).

Equilibrium of forces in the $n$-direction gives

$$dN = (T + dT)\,\sin\frac{d\theta}{2} + T\,\sin\frac{d\theta}{2} \quad \text{or} \quad dN = T\,d\theta, \tag{6.26}$$

where the sine of the differential angle is the angle in the limit ($\sin d\theta/2 \approx d\theta/2$) and the product of two differentials is neglected in the limit compared with the first-order differentials ($dT\,d\theta \approx 0$).

The two equilibrium relations Eqs. (6.25), (6.26) give

$$\frac{dT}{T} = \mu\,d\theta,$$

and integrating between corresponding limits $T_1$ and $T_2$ [with $M$ in the direction shown in Fig. 6.16(a) $\Longrightarrow T_2 > T_1$]:

$$\int_{T_1}^{T_2} \frac{dT}{T} = \int_0^{\phi} \mu\,d\theta \quad \text{or} \quad \ln\frac{T_2}{T_1} = \mu\,\phi,$$

where $\phi$ is the total angle of belt contact expressed in radians.

The tension $T_2$ is

$$T_2 = T_1\,e^{\mu\,\phi}. \tag{6.27}$$

If a rope were wrapped around a drum $n$ times, the total angle of belt contact is

$$\phi = 2\,\pi\,n.$$

Equation (6.27) also applies to belt drives where both the belt and the pulley are rotating at constant speed and describes the ratio of belt tensions for impending slippage (or slippage).

The centrifugal force acting on a flat belt creates a tension of

$$T_c = m'\,V^2 = m'\,\omega^2\,r^2, \tag{6.28}$$

where $m'$ is the mass per unit length of belt, $V$ is the belt speed, and $r$ is the pulley radius. Equation (6.27) becomes

$$\frac{T_2 - T_c}{T_1 - T_c} = e^{\mu\,\phi}. \tag{6.29}$$

The centrifugal force tends to reduce the angles of wrap $\phi$.

For a V-belt of angle $\beta$, see Fig. 6.16(c), Eq. (6.29) becomes

$$\frac{T_2 - T_c}{T_1 - T_c} = e^{\mu\,\phi/\sin\beta}. \tag{6.30}$$

### 6.2.3 Examples

**Example 6.2.1.** The automobile disk brake, shown in Fig. 6.2.4, consists of a flat-faced rotor and caliper which contains a disk pad on each side of the rotor. The inside radius is $R_i$ and the outside radius is $R_o$. The forces behind the two pads are eqaul to $P$ and $\mu$ is the coefficient of friction. The normal pressure $p$ is uniform distributed over the pad. Show that the moment applied to the hub is independent of the angular span $\alpha$ of the pads.

Solution.

The force acting on the pads is

$$P = pA = p\int_0^{\alpha}\int_{R_i}^{R_o} r\,dr\,d\theta = \frac{p}{2}\int_0^{\alpha}(R_o^2 - R_i^2)\,d\theta = \frac{p}{2}\,(R_o^2 - R_i^2)\,\alpha.$$

The moment applied to the hub is

$$M = 2\int \mu\, p\, r\, dA = 2\mu\, p\int_0^{\alpha}\int_{R_i}^{R_o} r^2\,dr\,d\theta = \frac{2\mu\, p}{3}\,(R_o^3 - R_i^3)\,\alpha$$
$$= \frac{2\mu}{3}\,\frac{2P}{(R_o^2 - R_i^2)\,\alpha}\,(R_o^3 - R_i^3)\,\alpha = \frac{4\mu\, P}{3}\,\frac{R_o^3 - R_i^3}{R_o^2 - R_i^2}.$$

The expression of the moment $M$ shows no dependence with the angular span $\alpha$ of the pads. The pressure variation with the angle $\theta$ would not change the moment $M$.

**Example 6.2.2.** The basic disk clutch, shown in Fig. 6.2.2, has the outside disk diameter of 6 in. The kinetic coefficient of friction is 0.3 and the maximum disk allowable pressure is 100 psi. The disk clutch is designed to transmit a moment of 400 lb·in. Determine the appropriate value of the inside diameter and the clamping force.

Solution.

The maximum moment for a given outside radius is obtained from Eq. (6.24)

$$R_i = 0.58\,R_0 = 0.58\,(3) = 1.74 \text{ in.}$$

The greatest pressure occurs at the inside radius. The design of a clutch of inside radius $R_i$ and allowable pressure $p_{max}$ is based on

$$p\,r = K = p_{max}\,R_i. \tag{6.31}$$

The total moment that can be developed over the entire interface is

$$M = \int \mu\, p\, r\, dA = \int_{R_i}^{R_o} \mu\,(p\,r)\,(2\pi\, r\, dr) = \int_{R_i}^{R_o} 2\pi\,\mu\, p_{max}\,R_i\, r\, dr$$
$$= \pi\,\mu\, p_{max}\,R_i\,(R_o^2 - R_i^2), \tag{6.32}$$

or

$$M = \pi\,(0.3)\,(100)\,(1.74)\,(3^2 - 1.74^2) = 979.421 \text{ lb}\cdot\text{in.}$$

For $R_i = 1.74$ in. and $p_{max} = 100$ psi, the clutch is overdesigned based on the output moment by a factor of 979.421/400=2.448.

Accepting the overdesign, the clamping force is calculated from Eq. (6.22) for $M = 400$ lb·in. as

$$P = \frac{2M}{\mu\,(R_i + R_o)} = \frac{2\,(400)}{0.3\,(1.74+3)} = 562.588 \text{ lb.}$$

**Example 6.2.3.** Determine the force $F$ on the handle 1 of the differential band brake [Fig. 6.2.5(a)] that will prevent the wheel 2 from turning on its shaft. The external moment $M = 200$ N·m is applied to the shaft. The coefficient of friction between the band and the wheel of radius $r = 100$ mm is 0.45. The following dimensions are given: $l = 500$ mm, $h = 80$ mm, and $\theta = 30°$.

Solution.

The free-body diagrams for the handle 1 and the wheel 2 are given in Fig. 6.2.5(b). For the band the tension $T_2$ is given by Eq. (6.27):

$$T_2 = T_1\,e^{\mu\,\phi} = T_1\,e^{7\pi\mu/6} = 5.203\,T_1, \qquad (6.33)$$

where $\phi = \theta + \pi$.

For the wheel 2 the sum of the moments with respect to its center $C$ gives

$$\sum M_C^{(2)} = M - r\,(T_2 - T_1) = 200 - 0.1\,(T_2 - T_1) = 0. \qquad (6.34)$$

The tensions $T_1$ and $T_2$ are obtained from Eqs. (6.33) and (6.34):

$$T_1 = 475.791 \text{ N} \quad \text{and} \quad T_2 = 2475.79 \text{ N.}$$

For the link 1 the sum of the moments with respect to point $O$ gives

$$\sum M_O^{(1)} = r\,T_2 - l\,F - h\,T_1\,\sin\theta = 0,$$

and the force $F$ is

$$F = \frac{r\,T_2 - h\,T_1\,\sin\theta}{l} = \frac{0.1(2475.79) - 0.08(475.791)\sin 30°}{0.5} = 457.095 \text{ N.}$$

**Example 6.2.4.** A 3000 rpm motor drives a machine through a V-belt with an angle $\beta = 18°$ and a unit weight of 1.75 N/m (Fig. 6.2.6). The pulley on the motor shaft has a 0.1 m pitch radius and the angle of wrap is 170°. The maximum belt tension should be limited to 1000 N and the coefficient of friction is at least 0.3. Find the maximum power that can be transmitted by the smaller pulley of the V-belt drive.

Solution.
The speed of the belt in m/s is

$$V = \frac{\pi\, d\, n}{60} = \frac{\pi\,(0.2)\,(3000)}{60} = 31.415 \text{ m/s}, \tag{6.35}$$

where $d = 2\,r = 2\,(0.1) = 0.2$ m and $n = 3000$ rpm.
Equation (6.28) gives the tension created by the centrifugal force

$$T_c = m'\,V^2 = \left(0.178\frac{\text{kg}}{\text{m}}\right)\left(31.415\frac{\text{m}}{\text{s}}\right)^2 = 176.063 \text{ N}, \tag{6.36}$$

where $m'$ is the mass unit length of belt:

$$m' = \frac{1.75 \text{ N/m}}{9.81 \text{ m/s}^2} = 0.178 \text{ kg/m}. \tag{6.37}$$

From Eq. (6.29), with $T_1 = T_{max} = 1000$ N, the tension $T_2$ is

$$T_2 = T_c + \frac{T_1 - T_c}{e^{\mu\,\phi/\sin\beta}} = 176.063 + \frac{1000 - 176.063}{e^{0.3\,(170)\left(\frac{\pi}{180}\right)/\sin\left[18\left(\frac{\pi}{180}\right)\right]}}$$
$$= 222.292 \text{ N}.$$

The moment on the pulley is

$$M = (T_1 - T_2)\,r = (1000 - 222.292)\,(0.1) = 77.770 \text{ N}\cdot\text{m}. \tag{6.38}$$

The power transmitted by the pulley is

$$H = \frac{M\,n}{9549} = \frac{77.770\ (3000)}{9549} = 24.433 \text{ kW}. \tag{6.39}$$

**Example 6.2.5.** A 30 hp, 2000 rpm electric motor drives a machine through a multiple V-belt as shown in Fig. 6.2.7. The belts have an angle $\beta = 18^\circ$ and a unit weight of 0.012 lb/in. The pulley on the motor shaft has a diameter of 6 in. and the angle of wrap is 165°. The maximum belt tension should be limited to 110 lb and the coefficient of friction is at least 0.2. Determine how many belts are required.
Solution.
The speed of the belt in m/s is

$$V = \frac{\pi\, d\, n}{60} = \frac{\pi\,(6)\,(2000)}{60} = 628.319 \text{ in./s}.$$

Equation (6.28) gives the tension created by the centrifugal force:

$$T_c = m' V^2 = \left(0.000031 \frac{\text{lb} \cdot \text{s}^2}{\text{in.}^2}\right) \left(628.319 \frac{\text{in.}}{\text{s}}\right)^2 = 12.260 \text{ lb},$$

where $m'$ is the mass unit length of belt:

$$m' = \frac{0.012 \text{ lb/in.}}{(32.2 \text{ ft/s}^2)(12 \text{ in./ft})} = 0.000031 \text{ lb} \cdot \text{s}^2/\text{in}^2. \tag{6.40}$$

From Eq. (6.29), with $T_1 = T_{max} = 110$ lb, the tension $T_2$ is

$$T_2 = T_c + \frac{T_1 - T_c}{e^{\mu \phi / \sin \beta}} = 12.260 + \frac{110 - 12.260}{e^{0.2(165)\left(\frac{\pi}{180}\right)/\sin\left[18\left(\frac{\pi}{180}\right)\right]}}$$
$$= 27.417 \text{ lb}.$$

The moment on the pulley is

$$M = (T_1 - T_2)\, d/2 = (110 - 27.417)(6/2) = 247.748 \text{ lb} \cdot \text{in}.$$

The power per belt transmitted by the pulley is

$$H = \frac{M n}{5252} = \frac{247.748\ (2000)}{5252\ (12)} = 7.862 \text{ hp/belt}.$$

The number of belts is

$$N = \frac{30}{7.862} = 3.815,$$

and 4 belts are needed.

### 6.2.4 Problems

6.2.1 The circular disk 1 is placed on top of disk 2 as shown in Fig. 6.2.8. The disk 2 is on a supporting surface 3. The diameters of 1 and 2 are 10 in. and 14 in., respectively. A compressive force of 100 lb acts on disk 1. The coefficient of friction between 1 and 2 is 0.30. Determine: a) the couple that will cause 1 to slip on 2; b) the minimum coefficient of friction between the disk 2 and the supporting surface 3 that will prevent 3 from rotating.

6.2.2 A shaft and a hoisting drum are used to raise the 600 kg load at constant speed as shown in Fig. 6.2.9. The diameter of the shaft is 40 mm and the diameter of the drum is 300 mm. The drum and shaft together have a mass of 100 kg and the coefficient of friction for the bearing is 0.3. Find the torque that must be applied to the shaft to raise the load.

6.2.3 The disks shown in Fig. 6.2.10 can be brought into contact under an axial force $P$. The pressure $p$ between the disks follows the relation $p = k/r$, where $k$ is a constant. The coefficient of friction *mu* is constant over the entire surface. Derive the expression for the torque $M$ required to turn the upper disk on the fixed lower in therm of $P$, $\mu$, and the inside and outside radii $R_o$ and $R_i$.

6.2.4 The cable reel in Fig. 6.2.11 has a mass of 300 kg and a diameter of 600 mm and is mounted on a shaft with the diameter $d = 2r = 100$ mm. The coefficient of friction between the shaft and its bearing is 0.20. Find the horizontal tension $T$ required to turn the reel.

6.2.5 For the V-belt in Fig. 6.2.3(c) derive the expression among the belt tension, the angle of contact $\beta$, and the coefficient of friction when slipping impends.

6.2.6 A cable supports a load of 200 kg and is subjected to a force $F = 600$ N which makes with the horizontal axis the angle $\theta$, as shown in Fig. 6.2.12. The coefficient of friction between the cable and the fixed drum is 0.2. Find the minimum value of $\theta$ before the load begins to slip.

6.2.7 A band brake is shown in Fig. 6.2.13. The band itself is usually made of steel, lined with a woven friction material for flexibility. The drum has a clockwise rotation. The width of the band is $b$, the coefficient of friction is $\mu$, and the angle of band contact is $\phi$. Find the brake torque and the corresponding actuating force $F$ if the maximum lining pressure is $p_{max}$. Use the following numerical application: $b = 80$ mm, $r = 300$ mm, $h = 150$ mm, $l = 800$ mm, $\phi = 270°$, $p_{max} = 0.6$ MPa, and $\mu = 0.3$.

6.2.8 Figure 6.2.14 shows a simple band brake operated by an applied force $F$ of 250 N. The band is 30 mm wide and is lined with a woven material with a coefficient of friction of 0.4. The drum radius is $r = 550$ mm. Find the angle of wrap $\phi$ necessary to obtain a brake torque of 900 N·m and determine the corresponding maximum lining pressure.

6.2.9 A 25 hp, 1800 rpm electric motor drives a machine through a multiple V-belt as shown in Fig. 6.2.6. The belts have an angle $\beta = 18°$ and a unit weight of 0.012 lb/in. The pulley on the motor shaft has a diameter of 3.7 in. and the angle of wrap is 165°. The maximum belt tension should be limited to 200 lb and the coefficient of friction is at least 0.3. Determine how many belts are required.

6.2.10 A 3500 rpm motor drives a machine through a V-belt with an angle $\beta = 18^\circ$ and a unit weight of 2.2 N/m (see Fig. 6.2.6). The pulley on the motor shaft has a 180 mm diameter and the angle of wrap is $160^\circ$. The maximum belt tension should be limited to 1300 N and the coefficient of friction is at least 0.33. Find the maximum power that can be transmitted by the smaller pulley of the V-belt drive.