2.1 Moment of a Vector About a Point

Definition. The moment of a bound vector \( \mathbf{v} \) about a point \( A \) is the vector

\[
\mathbf{M}_A^v = \mathbf{r}_{AB} \times \mathbf{v},
\]

(2.1)

where \( \mathbf{r}_{AB} \) is the position vector of \( B \) relative to \( A \), and \( B \) is any point of line of action, \( \Delta \), of the vector \( \mathbf{v} \) (Fig. 2.1). The vector \( \mathbf{M}_A^v = \mathbf{0} \) if and only the line of action of \( \mathbf{v} \) passes through \( A \) or \( \mathbf{v} = \mathbf{0} \). The magnitude of \( \mathbf{M}_A^v \) is

\[
|M_A^v| = |M_A^v| = |\mathbf{r}_{AB}| |\mathbf{v}| \sin \theta = |\mathbf{r}_{AB}| \mathbf{v} \sin \theta,
\]
where $\theta$ is the angle between $\mathbf{r}_{AB}$ and $\mathbf{v}$ when they are placed tail to tail. The perpendicular distance from $A$ to the line of action of $\mathbf{v}$ is

$$d = |\mathbf{r}_{AB}| \sin \theta = r_{AB} \sin \theta,$$

and the magnitude of $\mathbf{M}_A^v$ is

$$|\mathbf{M}_A^v| = M_A^v = |\mathbf{v}|d = vd.$$

The vector $\mathbf{M}_A^v$ is perpendicular to both $\mathbf{r}_{AB}$ and $\mathbf{v}$: $\mathbf{M}_A^v \perp \mathbf{r}_{AB}$ and $\mathbf{M}_A^v \perp \mathbf{v}$. The vector $\mathbf{M}_A^v$ being perpendicular to $\mathbf{r}_{AB}$ and $\mathbf{v}$ is perpendicular to the plane containing $\mathbf{r}_{AB}$ and $\mathbf{v}$.

The moment given by Eq. (2.1) does not depend on the point $B$ of the line of action of $\mathbf{v}$, $\triangle$, where $\mathbf{r}_{AB}$ intersects $\triangle$. Instead of using the point $B$ the point $B'$ (Fig. 2.1) can be used. The position vector of $B'$ relative to $A$ is $\mathbf{r}_{AB} = \mathbf{r}_{AB'} + \mathbf{r}_{B'B}$ where the vector $\mathbf{r}_{B'B}$ is parallel to $\mathbf{v}$, $\mathbf{r}_{B'B} \parallel \mathbf{v}$. Therefore,

$$\mathbf{M}_A^v = \mathbf{r}_{AB} \times \mathbf{v} = (\mathbf{r}_{AB'} + \mathbf{r}_{B'B}) \times \mathbf{v} = \mathbf{r}_{AB'} \times \mathbf{v} + \mathbf{r}_{B'B} \times \mathbf{v} = \mathbf{r}_{AB'} \times \mathbf{v},$$

(2.2)

because $\mathbf{r}_{B'B} \times \mathbf{v} = 0$. The moment of a vector about a point (which is also the moment about a defined axis through the point) is a sliding vector whose direction is along the axis through the point.

Next, using MATLAB®, it will be shown the validity of Eq. (2.2). Three points $A$, $B$, and $C$ are defined by three symbolic position vectors $\mathbf{r}_A$, $\mathbf{r}_B$, and $\mathbf{r}_C$:

```matlab
syms x_A y_A z_A x_B y_B z_B x_C y_C z_C real
r_A = [x_A y_A z_A];
r_B = [x_B y_B z_B];
r_C = [x_C y_C z_C];
```

The vector $\mathbf{v}$ is $\mathbf{v} = \mathbf{r}_C - \mathbf{r}_B$, or in MATLAB:

```matlab
v = r_C - r_B;
```

The line of action of the vector $\mathbf{v}$ is defined as the line segment $BC$. A generic point $B'$ (in MATLAB $Bp$) divides the line segment joining two given points $B$ and $C$ in a given ratio. The position vector of the point $B'$ is $\mathbf{r}_{B'}$:

```matlab
syms k real % k is a given real number
r_Bp = r_B + k*(r_C - r_B);
```

The moment of the vector $\mathbf{v}$ with respect to $A$ is calculated as $\mathbf{r}_{AB} \times \mathbf{v}$, $\mathbf{r}_{AB'} \times \mathbf{v}$, and $\mathbf{r}_{AC} \times \mathbf{v}$, or with MATLAB:

```matlab
Mv_AB = cross(r_B-r_A, v); % r_AB x v
Mv_ABp = cross(r_Bp-r_A, v); % r_ABp x v
Mv_AC = cross(r_C-r_A, v); % r_AC x v
```
2.1 Moment of a Vector About a Point

To prove that $M_A^v = \mathbf{r}_{AB} \times \mathbf{v} = \mathbf{r}_{AB} = \mathbf{r}_{AC} \times \mathbf{v}$ the following MATLAB commands are used:

```matlab
simplify(Mv_AB) == simplify(Mv_ABp)
simplify(Mv_AB) == simplify(Mv_AC)
```

To represent the vectors $\mathbf{r}_{AB}, \mathbf{r}_{AC}, \mathbf{v}$ and $M_A^v$ the following numerical data are used: $x_A = y_A = z_A = 0, x_B = 1, y_B = 2, z_B = 0, x_C = 3, y_C = 3, z_C = 0, \text{ and } k = 0.75$.

The numerical values for the vectors $\mathbf{r}_A, \mathbf{r}_B, \mathbf{r}_C, \mathbf{r}_{BP}, \mathbf{v}, Mv_{AB}, Mv_{ABp}, \text{ and } Mv_{AC}$ are calculated in MATLAB with:

```matlab
slist={x_A,y_A,z_A, x_B,y_B,z_B, x_C,y_C,z_C, k};
nlist={0,0,0, 1,2,0, 3,3,0, .75};
rA = double(subs(r_A,slist,nlist))
rB = double(subs(r_B,slist,nlist))
rC = double(subs(r_C,slist,nlist))
rBp = double(subs(r_Bp,slist,nlist))
V = double(subs(v,slist,nlist))
MvA = double(subs(Mv_AB,slist,nlist))
MvBp = double(subs(Mv_ABp,slist,nlist))
MvC = double(subs(Mv_AC,slist,nlist))
```

The MATLAB commands for the current axes and for the Cartesian reference with the origin at $A$ are:

```matlab
a=3; axis([0 a 0 a -a a]), grid on, hold on

% Cartesian axes
quiver3(rA(1),rA(2),rA(3),a-.5,0,0,1,...
'Color','k','LineWidth',1)
text('Interpreter','latex','String',' $x$',...
'Position',[a-.5,0,0],FontSize',12)
quiver3(rA(1),rA(2),rA(3),0,a-.5,0,1,...
'Color','k','LineWidth',1)
text('Interpreter','latex','String',' $y$',...
'Position',[0,a-.5,0],FontSize',12)
quiver3(rA(1),rA(2),rA(3),0,0,a-.5,1,...
'Color','k','LineWidth',1)
text('Interpreter','latex','String',' $z$',...
'Position',[0,0,a-.5],FontSize',12)

The fonts for the labels $x, y, \text{ and } z$ are LaTeX fonts. The vectors $\mathbf{r}_B, \mathbf{r}_C, \mathbf{r}_{BP}, \mathbf{V}$, and the line $BC$ are plotted with:

```matlab
quiver3(rA(1),rA(2),rA(3), rB(1),rB(2),rB(3),1,...
'Color','k','LineWidth',1)
quiver3(rA(1),rA(2),rA(3), rC(1),rC(2),rC(3),1,...
'Color','k','LineWidth',1)
quiver3(rA(1),rA(2),rA(3), rBp(1),rBp(2),rBp(3),1,...
'Color','k','LineWidth',1)
```
The vectors $\mathbf{M}_A$, $\mathbf{M}_{ABp}$, and $\mathbf{M}_{AC}$ are plotted with:

```matlab
quiver3(0,0,0, MvA(1),MvA(2),MvA(3),1,...
    'Color','r','LineWidth',2)
quiver3(0,0,0, MvBp(1),MvBp(2),MvBp(3),1,...
    'Color','g','LineWidth',2)
quiver3(0,0,0, MvC(1),MvC(2),MvC(3),1,...
    'Color','r','LineWidth',2)
```

The labels for the vectors are printed with

```matlab
text('Interpreter','latex','String',' $A=O$',...
    'Position',[0,0,0],'FontSize',12)
text('Interpreter','latex','String',' $B$',...
    'Position',[rB(1),rB(2),rB(3)],'FontSize',12)
text('Interpreter','latex','String','$B^\prime$','Position',[rBp(1),rBp(2),rBp(3)],...
    'FontSize',12)
text('Interpreter','latex','String',' $C$','Position',[rC(1),rC(2),rC(3)],'FontSize',12)
text('Interpreter','latex','String','$\mathbf{M}_A^\mathbf{v}$','Position',...
    [MvA(1),MvA(2),MvA(3)+.5],'FontSize',12)
```

The MATLAB representation of the vectors is shown in Fig. 2.2.

**Fig. 2.2** Moment of $\mathbf{v} = \mathbf{r}_{BC}$ about $A$: $\mathbf{M}_A^\mathbf{v} = \mathbf{r}_{AB} \times \mathbf{v} = \mathbf{r}_{AB} \times \mathbf{v} = \mathbf{r}_{AC} \times \mathbf{v}$
2.1 Moment of a Vector About a Point

**Moment of a Vector About a Line**

**Definition.** The moment $M_{\Omega}^v$ of a vector $v$ about a line $\Omega$ is the $\Omega$ resolute ($\Omega$ component) of the moment $v$ about any point on $\Omega$, see Fig. 2.3(a). The $M_{\Omega}^v$ is the $\Omega$ resolute of $M_A^v$.

$$M_{\Omega}^v = n \cdot M_A^v n = n \cdot (r \times v) n = [n, r, v] n,$$

where $n$ is a unit vector parallel to $\Omega$, and $r$ is the position vector of a point on the line of action of $v$ relative to a point on $\Omega$. The magnitude of $M_{\Omega}^v$ is given by

$$|M_{\Omega}^v| = M_{\Omega}^v = |[n, r, v]|.$$

The moment of a vector about a line is a free vector. If a line $\Omega$ is parallel to the line of action $\Delta$ of a vector $v$, then $[n, r, v] n = 0$ and $M_{\Omega}^v = 0$. If a line $\Omega$ intersects the line of action $\Delta$ of $v$, then $r$ can be chosen in such a way that $r = 0$ and $M_{\Omega}^v = 0$. If a line $\Omega$ is perpendicular to the line of action $\Delta$ of a vector $v$, and $d$ is the shortest distance between these two lines, Fig. 2.3(b), then

$$|M_{\Omega}^v| = |[n, r, v]| = |n \cdot (r \times v)| = |n \cdot |r||v| \sin(r, v) n| = |r||v| = d|v|.$$

**Moment of a System of Vectors**

**Definition.** The moment of a system $\{S\}$ of vectors $v_i$, $\{S\} = \{v_1, v_2, \ldots, v_n\} = \{v_i\}_{i=1,2,\ldots,n}$ about a point $A$ is

$$M_{A}^{\{S\}} = \sum_{i=1}^{n} M_A^{v_i}.$$

**Definition.** The moment of a system $\{S\}$ of vectors $v_i$, $\{S\} = \{v_1, v_2, \ldots, v_n\} = \{v_i\}_{i=1,2,\ldots,n}$ about a line $\Omega$ is

$$M_{\Omega}^{\{S\}} = \sum_{i=1}^{n} M_{\Omega}^{v_i}.$$
The moments $M_A^{(S)}$ and $M_P^{(S)}$ of a system $\{S\}$, $\{S\} = \{v_i\}_{i=1,2,...,n}$, of vectors, $v_i$, about two points $A$ and $P$, are related to each other as follows,

$$M_A^{(S)} = M_P^{(S)} + r_{AP} \times R,$$

(2.3)

where $r_{AP}$ is the position vector of $P$ relative to $A$, and $R$ is the resultant of $\{S\}$.

**Proof.** Let $B_i$ a point on the line of action of the vector $v_i$, $r_{ABi}$ and $r_{PBi}$ the position vectors of $B_i$ relative to $A$ and $P$, Fig. 2.4. Thus,

$$M_A^{(S)} = \sum_{i=1}^{n} M_A^{v_i} = \sum_{i=1}^{n} r_{ABi} \times v_i$$

$$= \sum_{i=1}^{n} (r_{AP} + r_{PBi}) \times v_i = \sum_{i=1}^{n} (r_{AP} \times v_i + r_{PBi} \times v_i) = \sum_{i=1}^{n} r_{AP} \times v_i + \sum_{i=1}^{n} r_{PBi} \times v_i$$

$$= r_{AP} \times \sum_{i=1}^{n} v_i + \sum_{i=1}^{n} r_{PBi} \times v_i = r_{AP} \times R + \sum_{i=1}^{n} M_P^{v_i} = r_{AP} \times R + M_P^{(S)}.$$

The proof of Eq. (2.3) for a system of three vectors $v_1$, $v_2$, and $v_3$ is given by the following MATLAB commands:

```matlab
% vectors vi i=1,2,3
v1 = sym('[v1x v1y v1z]');
v2 = sym('[v2x v2y v2z]');
v3 = sym('[v3x v3y v3z]');

% application points Bi of vi
r_B1 = sym('[xB1 yB1 zB1]');
r_B2 = sym('[xB2 yB2 zB2]');
r_B3 = sym('[xB3 yB3 zB3]');

% any two points A and P
r_A = sym('[xA yA zA]');
```

Fig. 2.4 Moments of a system of vectors, $v_i$, about two points $A$ and $P$.
2.1 Moment of a Vector About a Point

\[
\begin{align*}
r_P &= \text{sym}('[x_P \ y_P \ z_P]'); \\
r_{AP} &= r_P - r_A; \\
r_{PB1} &= r_{B1} - r_P; \\
r_{PB2} &= r_{B2} - r_P; \\
r_{PB3} &= r_{B3} - r_P; \\
r_{AB1} &= r_{AP} + r_{PB1}; \\
r_{AB2} &= r_{AP} + r_{PB2}; \\
r_{AB3} &= r_{AP} + r_{PB3}; \\
R &= v1 + v2 + v3;  \\
\% \ M_A = \sum(AB_i \times v_i) \ i=1,2,3 \\
M_A &= \text{cross}(r_{AB1}, v1) + \ldots \\
&\phantom{=} \text{cross}(r_{AB2}, v2) + \ldots \\
&\phantom{=} \text{cross}(r_{AB3}, v3); \\
\% \ M_P = \sum(PB_i \times v_i) \ i=1,2,3 \\
M_P &= \text{cross}(r_{B1} - r_P, v1) + \ldots \\
&\phantom{=} \text{cross}(r_{B2} - r_P, v2) + \ldots \\
&\phantom{=} \text{cross}(r_{B3} - r_P, v3); \\
\% \ M_A = AP \times R + M_P \\
simplify(M_A) &= \ldots \\
simplify(cross(r_P - r_A, R) + M_P)
\end{align*}
\]

If the resultant \( R \) of a system \( \{S\} \) of vectors is not equal to zero, \( R \neq 0 \), the points about which \( \{S\} \) has a minimum moment \( M_{min} \) lie on a line called the central axis, \((CA)\), of \( \{S\} \), which is parallel to \( R \) and passes through a point \( P \) whose position vector \( r \) relative to an arbitrarily selected reference point \( O \) is given by

\[
r = \frac{R \times M_O^{\{S\}}}{R^2}.
\]

The minimum moment \( M_{min} \) is given by

\[
M_{min} = \frac{R \cdot M_O^{\{S\}}}{R^2} R.
\]
2.2 Couples

Definition. A couple is a system of bound vectors whose resultant is equal to zero and whose moment about some point is not equal to zero. A system of vectors is not a vector, therefore couples are not vectors. A couple consisting of only two vectors is called a simple couple. The vectors of a simple couple have equal magnitudes, parallel lines of action, and opposite senses. Writers use the word “couple” to denote the simple couple. The moment of a couple about a point is called the torque of the couple, \( M \) or \( T \). The moment of a couple about any other point is equal to the moment of the couple about any other point, i.e., it is unnecessary to refer to a specific point. The moment of a couple is a free vector.

The torques are vectors and the magnitude of a torque of a simple couple is given by

\[
|M| = d|v| = dv,
\]

where \( d \) is the distance between the lines of action of the two vectors comprising the couple, and \( v \) is one of these vectors.

\[\text{Fig. 2.5 Couple of the vectors } v \text{ and } -v, \text{ simple couple}\]

Proof. In Fig. 2.5, the torque \( M \) is the sum of the moments of \( v \) and \( -v \) about any point. The moments about point \( A \) are

\[
M = M_A^v + M_A^{-v} = r \times v + 0.
\]

Hence,

\[
|M| = |r \times v| = |r||v|\sin(r, v) = d|v|.
\]

The direction of the torque of a simple couple can be determined by inspection: \( M \) is perpendicular to the plane determined by the lines of action of the two vectors comprising the couple, and the sense of \( M \) is the same as that of \( r \times v \).
2.3 Equivalence of Systems

The moment of a couple about a line \( \Omega \) is equal to the \( \Omega \) resolute of the torque of the couple. The moments of a couple about two parallel lines are equal to each other.

2.3 Equivalence of Systems

Definition. Two systems \( \{S\} \) and \( \{S'\} \) of vectors are said to be equivalent if and only if

1. the resultant of \( \{S\} \), \( R \), is equal to the resultant of \( \{S'\} \), \( R' \)

\[ R = R' \]

2. there exists at least one point about which \( \{S\} \) and \( \{S'\} \) have equal moments

\[ \exists \text{ } P: \text{ } M^{\{S\}}_P = M^{\{S'\}}_P. \]

Figures 2.6(a) and 2.6(b) each show a rod subjected to the action of a pair of forces. The two pairs of forces are equivalent, but their effects on the rod are different from each other. The word “equivalence” is not to be regarded as implying physical equivalence. For given a line \( \Omega \) and two equivalent systems \( \{S\} \) and \( \{S'\} \) of vectors,

\[ F \]

\[ F \]

\[ F \]

\[ F \]

\[ (a) \]

\[ (b) \]

Fig. 2.6 Rod subjected to the action of a pair of forces

the sum of the \( \Omega \) resolutes of the vectors in \( \{S\} \) is equal to the sum of the \( \Omega \) resolutes of the vectors in \( \{S'\} \). The moments of two equivalent systems of vectors, about any point, are equal to each other. The moments of two equivalent systems \( \{S\} \) and \( \{S'\} \) of vectors, about any line \( \Omega \), are equal to each other.

Transitivity of the equivalence relation. If \( \{S\} \) is equivalent to \( \{S'\} \), and \( \{S'\} \) is equivalent to \( \{S''\} \), then \( \{S\} \) is equivalent to \( \{S''\} \).

Every system \( \{S\} \) of bound vectors with the resultant \( R \) can be replaced with a system consisting of a couple \( C \) and a single vector \( v \) whose line of action passes through an arbitrarily selected base point \( O \). The torque \( M \) of \( C \) depends on the choice of base point \( M = M^{\{S\}}_O \). The vector \( v \) is independent of the choice of base point, \( v = R \).

A couple \( C \) can be replaced with any system of couples, the sum of whose torque is equal to the torque of \( C \).

When a system of vectors consists of a couple of torque \( M \) and a single resultant vector parallel to \( M \), it is called a wrench.
2.4 Force Vector and Moment of a Force

Force is a vector quantity, having both magnitude and direction. Force is commonly explained in terms of Newton’s three laws of motion set forth in his *Principia Mathematica* (1687). Newton’s first principle: a body that is at rest or moving at a uniform rate in a straight line will remain in that state until some force is applied to it. Newton’s second law of motion states that a particle acted on by forces whose resultant is not zero will move in such a way that the time rate of change of its momentum will at any instant be proportional to the resultant force. Newton’s third law states that when one body exerts a force on another body, the second body exerts an equal force on the first body. This is the principle of action and reaction.

Because force is a vector quantity it can be represented graphically as a directed line segment. The representation of forces by vectors implies that they are concentrated either at a single point or along a single line. The force of gravity is invariably distributed throughout the volume of a body. Nonetheless, when the equilibrium of a body is the primary consideration, it is generally valid as well as convenient to assume that the forces are concentrated at a single point. In the case of gravitational force, the total weight of a body may be assumed to be concentrated at its center of gravity.

Force is measured in newtons (N); a force of 1 N will accelerate a mass of one kilogram at a rate of one meter per second. The newton is a unit of the International System (SI) used for measuring force.

Using the English system, the force is measured in pounds. One pound of force imparts to a one-pound object an acceleration of 32.17 feet per second squared.

![Fig. 2.7](image-url) (a) Moment of a force about (with respect to) a point and (b) couple of two forces
The moment of the force $F$ can be expressed in terms of a cartesian reference frame, with the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, Fig. 2.7(a)

$$
\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}.
$$

(2.4)

The components of the force in the $x$, $y$, and $z$ directions are $F_x$, $F_y$, and $F_z$. The resultant of two forces: $\mathbf{F}_1 = F_{1x} \mathbf{i} + F_{1y} \mathbf{j} + F_{1z} \mathbf{k}$ and $\mathbf{F}_2 = F_{2x} \mathbf{i} + F_{2y} \mathbf{j} + F_{2z} \mathbf{k}$ is the vector sum of those forces

$$
\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = (F_{1x} + F_{2x}) \mathbf{i} + (F_{1y} + F_{2y}) \mathbf{j} + (F_{1z} + F_{2z}) \mathbf{k}.
$$

(2.5)

A moment is defined as the moment of a force about (with respect to) a point. The moment of the force $\mathbf{F}$ about the point $O$ is the cross product vector

$$
\mathbf{M}_O^F = \mathbf{r} \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & 0 & 0 \\
F_x & F_y & F_z \\
\end{vmatrix}
= (r_y F_z - r_z F_y) \mathbf{i} + (r_z F_x - r_x F_z) \mathbf{j} + (r_x F_y - r_y F_x) \mathbf{k}.
$$

(2.6)

where $\mathbf{r} = r_x \mathbf{i} + r_y \mathbf{j} + r_z \mathbf{k}$ is a position vector directed from the point about which the moment is taken ($O$ in this case) to any point $A$ on the line of action of the force, see Fig. 2.7(a). If the coordinates of $O$ are $x_O, y_O, z_O$ and the coordinates of $A$ are $x_A, y_A, z_A$, then $\mathbf{r} = \mathbf{r}_{OA} = (x_A - x_O) \mathbf{i} + (y_A - y_O) \mathbf{j} + (z_A - z_O) \mathbf{k}$ and the moment of the force $\mathbf{F}$ about the point $O$ is

$$
\mathbf{M}_O^F = \mathbf{r}_{OA} \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
x_A - x_O & y_A - y_O & z_A - z_O \\
F_x & F_y & F_z \\
\end{vmatrix}.
$$

The magnitude of $\mathbf{M}_O^F$ is

$$
|\mathbf{M}_O^F| = M_O^F = r F \sin \theta,
$$

where $\theta = \angle(\mathbf{r}, \mathbf{F})$ is the angle between vectors $\mathbf{r}$ and $\mathbf{F}$, and $r = |\mathbf{r}|$ and $F = |\mathbf{F}|$ are the magnitudes of the vectors. The line of action of $\mathbf{M}_O^F$ is perpendicular to the plane containing $\mathbf{r}$ and $\mathbf{F}$ ($\mathbf{M}_O^F \perp \mathbf{r} \& \mathbf{M}_O^F \perp \mathbf{F}$) and the sense is given by the right-hand rule.

The moment of the force $\mathbf{F}$ about another point $P$ is

$$
\mathbf{M}_P^F = \mathbf{r}_{PA} \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
x_A - x_P & y_A - y_P & z_A - z_P \\
F_x & F_y & F_z \\
\end{vmatrix},
$$

where $x_P, y_P, z_P$ are the coordinates of the point $P$.

The system of two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, which have equal magnitudes $|\mathbf{F}_1| = |\mathbf{F}_2|$, opposite senses $\mathbf{F}_1 = -\mathbf{F}_2$, and parallel directions ($\mathbf{F}_1 \parallel \mathbf{F}_2$) is a couple. The resul-
tant force of a couple is zero \( \mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = 0 \). The resultant moment \( \mathbf{M} \neq \mathbf{0} \) about an arbitrary point is

\[
\mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2,
\]

or

\[
\mathbf{M} = \mathbf{r}_1 \times (-\mathbf{F}_2) + \mathbf{r}_2 \times \mathbf{F}_2 = (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}_2 = \mathbf{r} \times \mathbf{F}_2,
\]

(2.7)

where \( \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \) is a vector from any point on the line of action of \( \mathbf{F}_1 \) to any point of the line of action of \( \mathbf{F}_2 \). The direction of the torque of the couple is perpendicular to the plane of the couple and the magnitude is given by, Fig. 2.7(b)

\[
|M| = M = r F_2 |\sin \theta| = h F_2,
\]

(2.8)

where \( h = r |\sin \theta| \) is the perpendicular distance between the lines of action. The resultant moment of a couple is independent of the point with respect to which moments are taken.

### 2.5 Representing Systems by Equivalent Systems

To simplify the analysis of the forces and moments acting on a given system one can represent the system by an equivalent a less complicated one. The actual forces and moments can be replaced with a total force and a total moment.

![Fig. 2.8 Equivalent systems](image)

Figure 2.8 shows an arbitrary system of forces and moments, \{system I\}, and a point \( P \). This system can be represented by a system, \{system II\}, consisting of a single force \( \mathbf{F} \) acting at \( P \) and a single couple of torque \( \mathbf{M} \). The conditions for equivalence are

\[
\sum \mathbf{F} \{\text{system II}\} = \sum \mathbf{F} \{\text{system I}\} \implies \mathbf{F} = \sum \mathbf{F} \{\text{system I}\},
\]

and
2.5 Representing Systems by Equivalent Systems

\[ \sum M_p^{\text{system II}} = \sum M_p^{\text{system I}} \implies M = \sum M_p^{\text{system I}}. \]

These conditions are satisfied if \( F \) equals the sum of the forces in \{system I\}, and \( M \) equals the sum of the moments about \( P \) in \{system I\}. Thus, no matter how complicated a system of forces and moments may be, it can be represented by a single force acting at a given point and a single couple. Three particular cases occur frequently in practice.

**Force Represented by a Force and a Couple**

A force \( F_I \) acting at a point \( I \) \{system I\} in Fig. 2.9 can be represented by a force \( F \) acting at a different point \( P \) and a couple of torque \( M \), \{system II\}. The moment about \( P \) is \( r_{PI} \times F_I \), where \( r_{PI} \) is the vector from \( P \) to \( I \). The conditions for equivalence are

\[ \sum F^{\text{system II}} = \sum F^{\text{system I}} \implies F = F_I, \]

and

\[ \sum M_p^{\text{system II}} = \sum M_p^{\text{system I}} \implies M = M_p^{F_I} = r_{PI} \times F_I. \]

The systems are equivalent if the force \( F \) equals the force \( F_I \) and the couple of torque \( M_p^{F_I} \) equals the moment of \( F_I \) about \( P \).

**Concurrent Forces Represented by a Force**

A system of concurrent forces whose lines of action intersect at a point \( P \) \{system I\} in Fig. 2.10, can be represented by a single force whose line of action intersects \( P \), \{system II\}. The sums of the forces in the two systems are equal if

\[ F = F_1 + F_2 + \ldots + F_n. \]

The sum of the moments about \( P \) equals zero for each system, so the systems are equivalent if the force \( F \) equals the sum of the forces in \{system I\}.
Parallel Forces Represented by a Force
A system of parallel forces whose sum is not zero can be represented by a single force $F$ shown in Fig. 2.11.

System Represented by a Wrench
In general any system of forces and moments can be represented by a single force acting at a given point and a single couple. Figure 2.12 shows an arbitrary force $F$ acting at a point $I$ and an arbitrary couple of torque $M$, \{system I\}. This system can be represented by a simpler one, i.e., one may represent the force $F$ acting at a different point $P$ and the component of $M$ that is parallel to $F$. A coordinate system is chosen so that $F$ is along the $y$ axis

$$F = F_j,$$

and $M$ is contained in the $xy$ plane

$$M = M_xi + M_yj.$$

The equivalent system, \{system II\}, consists of the force $F$ acting at a point $P$ on the $z$ axis

$$F = F_j,$$

and the component of $M$ parallel to $F$
2.5 Representing Systems by Equivalent Systems

\[ \{ \text{system I} \} \]
\[ \{ \text{system II} \} \]

\[ F = F_j \]
\[ M = M_{xI} + M_{yJ} \]

\[ |r_{IP}| = IP = M_x / F \]

Fig. 2.12 System represented by a wrench

\[ M_p = M_{yJ} \]

The distance \( IP \) is chosen so that \( |r_{IP}| = IP = M_x / F \). The \{system I\} is equivalent to \{system II\}. The sum of the forces in each system is the same \( F \). The sum of the moments about \( I \) in \{system I\} is \( M \), and the sum of the moments about \( I \) in \{system II\} is

\[ \sum M_{j}^{(\text{system II})} = r_{IP} \times F + M_{yJ} = [-|IP| \mathbf{k}] \times (F_j) + M_{yJ} = M_{xI} + M_{yJ} = M. \]

The system of the force \( F = F_j \) and the couple \( M_p = M_{yJ} \) that is parallel to \( F \) is a wrench. A wrench is the simplest system that can be equivalent to an arbitrary system of forces and moments.

Fig. 2.13 Steps required to represent a system of forces and moments by a wrench
The representation of a given system of forces and moments by a wrench requires the following steps:

1. Choose a convenient point \( I \) and represent the system by a force \( \mathbf{F} \) acting at \( P \) and a couple \( \mathbf{M} \), see Fig. 2.13(a).

2. Determine the components of \( \mathbf{M} \) parallel and normal to \( \mathbf{F} \), see Fig. 2.13(b):

\[
\mathbf{M} = \mathbf{M}_p + \mathbf{M}_n, \text{ where } \mathbf{M}_p \parallel \mathbf{F}.
\]

3. The wrench consists of the force \( \mathbf{F} \) acting at a point \( P \) and the parallel component \( \mathbf{M}_p \), see Fig. 2.13(c). For equivalence, the following condition must be satisfied:

\[
\mathbf{r}_{IP} \times \mathbf{F} = \mathbf{M}_n,
\]

where \( \mathbf{M}_n \) is the normal component of \( \mathbf{M} \).

In general, the \( \{ \text{system I} \} \) cannot be represented by a force \( \mathbf{F} \) alone.
2.6 Examples

Example 2.1
Calculate the moment about the base point $O$ of the force $F$, as shown in Fig. E2.1(a). Numerical application: $F = 500\, \text{N}$, $\theta = 45^\circ$, $a = 1\, \text{m}$, and $b = 5\, \text{m}$.

![Diagram](image)

**Solution**
A cartesian reference frame with the origin at $O$, as shown in Fig. E2.1(a), is selected. The moment of the force $F$ with respect to the point $O$ is
\[
\mathbf{M}_O = \mathbf{r}_{OA} \times \mathbf{F} = \begin{vmatrix}
1 & j & k \\
x_A-x_O & y_A-y_O & 0 \\
F_x & F_y & 0
\end{vmatrix} = \begin{vmatrix}
1 & j & k \\
a & b & 0 \\
F \cos \theta & F \sin \theta & 0
\end{vmatrix}
\]

\[
= (aF \cos \theta - bF \sin \theta) \mathbf{k} = [(1) 5 \cos 45^\circ - (5) 500 \sin 45^\circ] \mathbf{k}
\]

\[
= -1414.214 \mathbf{k} \text{ N m.}
\]

The minus sign indicates that the moment vector is in the negative \(z\)-direction. The MATLAB program for determining the moment of the force \(\mathbf{F}\) about the point \(O\) is:

```matlab
syms F theta a b real
rA = [a b 0];
FA = [F*cos(theta) F*sin(theta) 0];
MO = cross(rA, FA);
MOz = MO(3);
sl = {F, theta, a, b};
nl = {5, pi/4, 1, 5};
fprintf('MOz = %s =', char(MOz));
fprintf('%6.3f (kN m)
', subs(MOz, sl, nl))
```

and the output of the program is

\[
\text{MOz} = aF \sin \theta - bF \cos \theta = -14.142 \text{ (kN m)}
\]

The MATLAB program for plotting the vectors is:

```matlab
% numerical values
A = double(subs(rA, sl, nl));
F = double(subs(FA, sl, nl));
M = subs(MO, sl, nl);

% vector plotting
axis([0 5 0 10 -18 0])
xlabel('x'), ylabel('y'), zlabel('z')
hold on, grid on

% Cartesian axes
text(0,0,0,'O','fontsize',14,'fontweight','b')
quiver3(0,0,4,0,0,1,'Color','b')
text(4.1,0,0,'x')
quiver3(0,0,0,9,0,1,'Color','b')
text(0,9.4,0,'y')
quiver3(0,0,0,0,5,1,'Color','b')
text(0,0,5.5,'z')
line([0 0],[0 A(2)], [0,0], 'LineStyle', '--', 'Color','k', 'LineWidth', 4)
line([0 A(1)],[A(2) A(2)], [0,0], 'LineStyle', '--', 'Color','k', 'LineWidth', 4)
```
2.6 Examples

Example 2.1

The vector representation with MATLAB is shown in Fig. E2.1(b).

Example 2.2

The pole in Fig. E2.29(a) is subjected to a tension that is directed from A to B. Find the moment created by the force about the support at O. Numerical application: \( T = 10 \text{ kN}, a = 12 \text{ m}, b = 9 \text{ m}, \text{ and } c = 15 \text{ m}. \)

Solution

The vector expression for the tension \( T \) is

\[
T = T \mathbf{u}_{AB} = T \frac{\mathbf{r}_{AB}}{|\mathbf{r}_{AB}|} = T \left( \frac{x_B - x_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}} \mathbf{i} + \frac{y_B - y_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}} \mathbf{j} + \frac{z_B - z_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}} \mathbf{k} \right)
\]

\[
= T \frac{a_1 + b_1 - c_1}{\sqrt{a^2 + b^2 + c^2}} = \frac{12a + 9b - 15c}{\sqrt{12^2 + 9^2 + 15^2}} = 5.6571 + 4.243j - 7.071k \text{ kN},
\]

where \( r_B = x_B \mathbf{i} + y_B \mathbf{j} + z_B \mathbf{k} = a \mathbf{i} + b \mathbf{j} \) and \( r_C = x_C \mathbf{i} + y_C \mathbf{j} + z_C \mathbf{k} = c \mathbf{k}. \) The moment of the tension \( T \) with respect to the point \( O \) is

\[
M_O^T = \mathbf{r}_{OA} \times T = \begin{vmatrix} 1 & j & k \\ x_A & y_A & z_A \\ T_x & T_y & T_z \end{vmatrix} = \frac{T}{\sqrt{a^2 + b^2 + c^2}} \begin{vmatrix} 1 & j & k \\ 0 & 0 & c \\ a & b & -c \end{vmatrix} = \frac{T(-bc \mathbf{i} + ac \mathbf{j})}{\sqrt{a^2 + b^2 + c^2}}
\]
Fig. E2.2 a) Example 2.2 and b) MATLAB figure

\[
\frac{10 \left[ -9(15) \mathbf{i} + 12(9) \mathbf{j} \right]}{\sqrt{12^2 + 9^2 + 15^2}} = -63.640 \mathbf{i} + 84.853 \mathbf{j} \text{ kN m},
\]

and \(|\mathbf{M}_O^T| = 106.066 \text{ kN m. The MATLAB program is given by}\n
```matlab
syms T a b c real
rB = [a b 0];
rA = [0 0 c];
rAB = rB-rA;
uAB = rAB/sqrt(dot(rAB, rAB));
TAB = T*uAB;
MO = cross(rA, TAB);
sl = {T, a, b, c};
nl = {10, 12, 9, 15};
Tn = subs(TAB,sl,nl);
Mn = subs(MO,sl,nl);
```
2.6 Examples

\[
T = \begin{bmatrix} 5.657 & 4.243 & -7.071 \end{bmatrix} \text{(kN)}
\]

\[
\begin{align*}
\text{MOx} & = -cTb/(a^2+b^2+c^2)^{1/2} = -63.640 \text{ (kN m)} \\
\text{MOy} & = cTa/(a^2+b^2+c^2)^{1/2} = 84.853 \text{ (kN m)} \\
\text{MOz} & = 0 = 0.000 \text{ (kN m)} \\
|\text{MO}| & = 106.066 \text{ (kN m)}
\end{align*}
\]

The MATLAB program for plotting the vectors is

```matlab
axis([-70 15 -10 90 0 15])
xlabel('x'), ylabel('y'), zlabel('z')
hold on, grid on
A = double(subs(rA,sl,nl));
B = double(subs(rB,sl,nl));
text(0,0,0,'O','fontsize',14)
text(A(1),A(2),A(3),'A','fontsize',14)
text(B(1),B(2),B(3),'B','fontsize',14)
line([0 A(1)], [0 A(2)], [0 A(3)], 'LineWidth',4)
line([A(1) B(1)], [A(2) B(2)], [A(3) B(3)], ...
    'LineStyle','-','Marker','o','LineWidth',1)
quiver3(A(1),A(2),A(3),Tn(1),Tn(2),Tn(3),1,...
    'Color','k','LineWidth',4)
text(A(1)+Tn(1),A(2)+Tn(2),A(3)+Tn(3),'T',...
    'fontsize',14,'fontweight','b')
quiver3(0,0,0,Mn(1),0,0,1,...
    'Color','r','LineWidth',2)
text(Mn(1),0,0,'M_{Ox}^T','fontsize',14,...
    'fontweight','b')
quiver3(0,0,0,Mn(2),0,1,...
    'Color','r','LineWidth',2)
text(0,Mn(2),0,'M_{Oy}^T','fontsize',14,...
    'fontweight','b')
quiver3(0,0,0,Mn(1),Mn(2),Mn(3),1,...
Example 2.3
Determine the moment of the force $F$ about $A$ as shown in Fig. E2.3(a). Numerical application: $F = 1 \, \text{kN}$, $a = 1 \, \text{m}$, $b = 3 \, \text{m}$, and $c = 2 \, \text{m}$.

![Fig. E2.3 a) Example 2.3 and b) MATLAB figure](image)

Solution

The moment of a force about a point is given by the cross product of a position vector with the force vector. The position vector must run from the point about which the moment is being calculated to a point on the line of action of the force. Figure E2.3(a) shows the location of the point $A$, the force $F$, and the line of action of the force. Point $B$ is on the line of action of the force. Thus the position vector of interest is the vector from point $A$ to point $B$. From the figure this position vector can be seen to be $a$ units in the $-x$ followed by $b$ units in the positive $y$.

$$\mathbf{r}_{AB} = -a \mathbf{i} + b \mathbf{j}$$

The force vector is parallel to the $z$-axis with magnitude $F$. Thus it can be expressed in vector form as: $\mathbf{F} = -F \mathbf{k}$. The desired moment is the cross product of these two vectors.
\[ M_F^A = (-a\mathbf{i} + b\mathbf{j}) \times (-F\mathbf{k}). \]

Recalling that \( \mathbf{i} \times \mathbf{k} = -\mathbf{j} \) and \( \mathbf{j} \times \mathbf{k} = \mathbf{i} \) yields

\[ M_F^A = -bF \mathbf{i} - aF \mathbf{j}. \]

The MATLAB program for the moment of the force \( \mathbf{F} \) about point \( A \) is given by

```matlab
syms a b c F
rA = [a 0 0];
rB = [0 b 0];
rE = [0 b c];
rAE = rE - rA;
rAB = rB - rA;
f = [0 0 -F];
ME = cross(rAE, f); % M = rAE x F
MB = cross(rAB, f); % M = rAB x F
ME == MB; % rAB x F = rAE x F

fprintf('M = rAB x F = rAE x F 
')
fprintf('Mx = %s; ',char(ME(1)))
fprintf('My = %s; ',char(ME(2)))
fprintf('Mz = %s.
',char(ME(3)))

% numerical calculation
sl = {a, b, c, F};
nl = {1, 3, 2, 1};
MEn = double(subs(ME,sl,nl));
MBn = double(subs(MB,sl,nl));

fprintf('ME = [%6.3f %6.3f %6.3f] (kN m) 
',MEn)
fprintf('MB = [%6.3f %6.3f %6.3f] (kN m) 
',MBn)
```

The output of the MATLAB program is

\[ M = rAB \times F = rAE \times F \]
\[ Mx = -F*b; \quad My = -F*a; \quad Mz = 0. \]
\[ ME = [-3.000 -1.000 0] \quad (kN m) \]
\[ MB = [-3.000 -1.000 0] \quad (kN m) \]

The MATLAB program for plotting the vectors and the triangular prism is

```matlab
F=1; % kN
a=1; b=3; c=2; % m
axis([-2 2 -1 4 0 2])
hold on, grid on
```
% Cartesian axes
line([0 4],[0 0],[0,0],’Color’,’b’,’LineWidth’,1.5)
text(3,0,0,’x’,’fontweight’,’b’) 

line([0 0],[0 4],[0,0],’Color’,’b’,’LineWidth’,1.5)
text(0,4.1,0,’y’,’fontweight’,’b’) 

line([0 0],[0 0],[0,2.5],’Color’,’b’,’LineWidth’,1.5)
text(0,0,2.6,’z’,’fontweight’,’b’) 

text(-.45,0,0,’O(1)’,’fontweight’,’b’) 
text(a+.1,0,0,’A(2)’,’fontweight’,’b’) 
text(.1,b-.1,0,’B(3)’,’fontweight’,’b’) 
text(-.45,0,c-.1,’C(4)’,’fontweight’,’b’) 
text(a+.1,0,c,’D(5)’,’fontweight’,’b’) 
text(0,b+.05,c-.1,’E(6)’,’fontweight’,’b’) 

text((a+.1)/3,.3,0,’a’,’fontweight’,’b’) 
text(.05,(b-.1)/2,.17,’b’,’fontweight’,’b’) 
text(-.16,0,(c-.1)/2,’c’,’fontweight’,’b’) 

view(42,34); 
% view(AZ,EL) set the angle of the view from which 
% an observer sees the current 3-D plot 
% AZ is the azimuth or horizontal rotation and 
% EL is the vertical elevation (both in degrees) 

% Generate data 
vert=[0 0 0; a 0 0; 0 b 0; 0 0 c; a 0 c; 0 b c]; 
% define the matrix of the vertices 
% O: 0,0,0 defined as vertex 1 
% A: a,0,0 defined as vertex 2 
% B: 0,b,0 defined as vertex 3 
% C: 0,0,c defined as vertex 4 
% D: a,0,c defined as vertex 5 
% E: 0,b,c defined as vertex 6 

face_up=[1 2 3; 4 5 6];
% define the lower and upper face of the triangular prism 
% lower face is defined by vertices 1, 2, 3 (O, A, B) 
% upper face is defined by vertices 4, 5, 6 (C, D, E)
face_l=[1 2 5 4; 2 3 6 5; 1 3 6 4];
% generate the lateral faces
% lateral face 1 is defined by 1, 2, 5, 4
% lateral face 2 is defined by 2, 3, 6, 5
% lateral face 3 is defined by 1, 3, 6, 4
% when defined a face the order of the vertices
% has to be given clockwise or counterclockwise

% draw the lower and upper triangular patches
patch...
('Vertices',vert,'Faces',face_up,'facecolor','b')
% patch(x,y,C) adds the "patch" or
% filled 2-D polygon defined by
% vectors x and y to the current axes.
% C specifies the color of the face(s)
% X represents the matrix vert
% Y represents the matrix face_up

% draw the lateral rectangular patches
patch...
('Vertices',vert,'Faces',face_l,'facecolor','b')
quiver3(0,b,F+c,0,0,-F,1,'Color','r','LineWidth',1.75)
text(-.3,b,c+.2,' F','fontsize',14,'fontweight','b')
quiver3(a,0,0,MBn(1),MBn(2),MBn(3),1,...
    'Color','k','LineWidth',2)
text((a+MBn(1))/2,MBn(2)/2,MBn(3)/2,...
    ' M','fontsize',14,'fontweight','b')
quiver3(a,0,0,MBn(1),0,0,1,'Color','r','LineWidth',2)
text((a+MBn(1))/1.3,0,0,...
    ' M_x','fontsize',14,'fontweight','b')
quiver3(a,0,0,MBn(2),0,1,'Color','r','LineWidth',2)
text(a+.3,MBn(2),0,...
    ' M_y','fontsize',14,'fontweight','b')
light('Position',[1 2 3]);
% light('PropertyName',propertyvalue,...)
% light creates a light object in the current axes.
% Lights affect only patch and surface objects.
% light the peaks surface plot with a light source
% located at infinity and oriented along the
% direction defined by the vector [1 2 3]
material shiny
% material shiny makes the objects shiny
alpha('color');
% alpha get or set alpha properties for
% objects in the current axis
% alpha('color') set the alphadata to be
% the same as the color data.

The vector representation with MATLAB is shown in Fig. E2.3(b).

Example 2.4
A force \( F \) acts on a link at the point \( A \) as shown in Fig. E2.4(a). Find an equivalent system consisting of a force at \( O \) and a couple. Numerical application: \( F = 100 \) lb, \( OA = l = 1 \) ft, \( \theta = 45^\circ \), and \( \alpha = 100^\circ \).

**Solution**
The original \( F \) force is equivalent to the force at \( O \) as shown in Fig. E2.4(b)

\[
\mathbf{R} = \mathbf{F} = -F \cos(\alpha - \theta) \mathbf{i} + F \sin(\alpha - \theta) \mathbf{j} = \\
-100 \cos(100^\circ - 45^\circ) \mathbf{i} + 100 \sin(100^\circ - 45^\circ) \mathbf{j} = -57.358 \mathbf{i} + 81.915 \mathbf{j} \text{ lb.}
\]

The moment of the force \( \mathbf{F} \) with respect to the point \( O \), as shown in Fig. E2.4(b), is

\[
\mathbf{M} = \mathbf{M}_O^F = \mathbf{r}_{OA} \times \mathbf{F} = \\
\begin{vmatrix}
1 & 1 & k \\
x_A & y_A & 0 \\
F_x & F_y & 0
\end{vmatrix} = \\
\begin{vmatrix}
1 & 1 & k \\
l \cos \theta & l \sin \theta & 0 \\
-F \cos(\alpha - \theta) & F \sin(\alpha - \theta) & 0
\end{vmatrix} = \\
|lF (\cos \theta) \sin(\alpha - \theta) + lF (\sin \theta) \cos(\alpha - \theta)| \mathbf{k} = \\
|1(100)(\cos 45^\circ)(\sin(100^\circ - 45^\circ)) + 1(100)(\sin 45^\circ)(\cos(100^\circ - 45^\circ))| \mathbf{k} = \\
98.481 \mathbf{k} \text{ lb ft.}
\]

The MATLAB program is

\[
syms \ F \ l \ theta \ alfa \ real \\
sl = \{F, l, theta, alfa\};
\]
2.6 Examples

\[ nl = \{100, 1, \pi/4, \pi/1.8\}; \]
\[ FA = [-F \cdot \cos(\alpha - \theta), F \cdot \sin(\alpha - \theta), 0]; \]
\[ rA = [\cos(\theta), \sin(\theta), 0]; \]
\[ FAn = \text{double(subs(FA, sl, nl))}; \]
\[ \text{fprintf('R = [\%6.3f \%6.3f \%g](lb)\n', FAn)} \]
\[ MO = \text{cross(rA, FA)}; \]
\[ MOz = \text{simplify(MO(3))}; \]
\[ MOn = \text{double(subs(MOz, sl, nl))}; \]
\[ \text{fprintf('MOz = \n')}; \]
\[ \text{fprintf('%s\n', char(MOz))} \]
\[ \text{fprintf('MOz = \%6.3f (lb ft)\n', MOn)} \]

and the results are

\[ R = [-57.358 81.915 0](lb) \]
\[ MOz = -l \cdot F \cdot (\cos(\theta) \cdot \sin(-\alpha + \theta) - \sin(\theta) \cdot \cos(-\alpha + \theta)) \]
\[ MOz = 98.481 (lb \text{ ft}) \]

Example 2.5

Three forces \( F_A, F_B, \) and \( F_C, \) as shown in Fig. E2.5, are acting on a rectangular planar plate \((F_A||Oz, F_B||Oy, F_C||Ox)\). The three forces acting on the plate are replaced by a wrench. Find: a) the resultant force for the wrench; b) the magnitude of couple moment, \( M, \) for the wrench and the point \( T(x, z) \) where its line of action intersects the plate. Numerical application: \( F_A = 900 \text{ lb}, \) \( F_B = 500 \text{ lb}, \) \( F_C = 300 \text{ lb}, \) \( a = BC = 4 \text{ ft}, \) and \( b = AB = 6 \text{ ft}. \)

Solution

a) The direction cosines of the resultant force \( R, \) are the same as those of the moment \( M \) of the couple of the wrench, assuming that the wrench is positive. The resultant force is

\[ R = F_A + F_B + F_C = F_C \mathbf{i} + F_B \mathbf{j} - F_A \mathbf{k} = 300 \mathbf{i} + 500 \mathbf{j} - 900 \mathbf{k} \text{ lb} \]
\[ R = |R| = \sqrt{F_A^2 + F_B^2 + F_C^2} = \sqrt{300^2 + 500^2 + 900^2} = 1072.381 \text{ lb} = 1.072 \text{ kip}. \]

The direction cosines of the resultant force are

\[ \cos \theta_x = \frac{F_C}{R} = 0.280, \quad \cos \theta_y = \frac{F_B}{R} = 0.466, \quad \cos \theta_z = \frac{-F_A}{R} = -0.839. \]

The MATLAB program for calculating the direction cosines or the components of the unit vector of the resultant force are

\[ \text{syms a b FA FB FC x z M}; \]
\[ \text{sl = \{a, b, FA, FB, FC\};} \]
\[ \text{nl = \{4, 6, 0.9, 0.5, 0.3\};} \]
\[ \text{F_A = [0 0 -FA]; rA = [a/2 0 0];} \]
\[ \text{F_B = [0 FB 0]; rB = [a 0 b];} \]
Fig. E2.5 a) Example 2.5 and b) MATLAB figure

\[ F_C = \begin{bmatrix} FC \\ 0 \\ 0 \end{bmatrix}; \quad r_C = [0 \ 0 \ b]; \]
\[ R = F_A + F_B + F_C; \]
\[ Rn = \text{double} (\text{subs}(R, \ sl, \ nl)); \]
\[ uR = R/\text{magn}(R); \quad uRn = \text{double} (\text{subs}(uR, \ sl, \ nl)); \]
\[ \text{fprintf}(''R = [\%6.3f \%6.3f \%6.3f] (kip)\n', \text{Rn}); \]
\[ \text{fprintf}(''|R| = \%6.3f (kip)\n', \text{magn}(\text{Rn})); \]
\[ \text{fprintf}(''uR = [\%6.3f \%6.3f \%6.3f] (kip)\n', \text{uR}); \]

The function magn was defined in the previous chapter.

b) The moment of the wrench couple must equal the sum of the moments of the given forces about point \( T \) through which the resultant passes. The moments about \( T(x, 0, z) \) of the three forces are

\[ \mathbf{M}_T = \mathbf{M}_T^{F_A} + \mathbf{M}_T^{F_B} + \mathbf{M}_T^{F_C}, \]

where
The total moment about the point $T$ of the forces is

$$M = (z - b)F_B \mathbf{i} + [(a - x)F_A + (b - z)F_C] \mathbf{j} + (a - x)F_B\mathbf{k}.$$
digits(3)
fprintf('first equation:\n')
pretty(eq1)
fprintf('%s = 0 \n\n',char(vpa(eq1n)))

fprintf('second equation:\n')
pretty(eq2)
fprintf('%s = 0 \n\n',char(vpa(eq2n)))

fprintf('third equation:\n')
pretty(eq3)
pretty(eq3)
pretty('%s = 0 \n\n',char(vpa(eq3n)))

sol = solve(eq1, eq2, eq3,’x, z, M’);
Ms = sol.M;
Mn = subs(Ms, sl, nl);
xs = sol.x;
xn = subs(xs, sl, nl);
zs = sol.z;
zn = subs(zs, sl, nl);

fprintf('M = ')
pretty(Ms)
pretty('M = %6.3f (kip ft)\n’, double(Mn))

fprintf('x = ')
pretty(xs)
pretty('x = %6.3f (ft)\n’, double(xn))

fprintf('z = ')
pretty(zs)
pretty('z = %6.3f (ft)\n’, double(zn))

The function pretty(x) prints the symbolic expression x in a format that looks like type-set mathematics. The results obtained with MATLAB are
first equation:
\[
\begin{align*}
&\frac{FC}{2} \quad \frac{FB (b - z)}{M} \\
&\frac{(FA + FB + FC)}{2} \quad \frac{(0.5*(z - 6.0))/M - 0.28}{0} = 0
\end{align*}
\]

second equation:
\[
\begin{align*}
&\frac{FC (b - z) + FA | - - x |}{2} \quad \frac{- FB}{M} \\
&\frac{- (1.0*(0.9*x + 0.3*z - 3.6))/M - 0.466}{0} = 0
\end{align*}
\]

third equation:
\[
\begin{align*}
&\frac{FA FB (a - x)}{2} \quad \frac{FB (a - x)}{M} \\
&\frac{(FA + FB + FC)}{2} \quad \frac{0.839 - (0.5*(x - 4.0))/M}{0} = 0
\end{align*}
\]

\[
M = \frac{FA FB a (FA + FB + FC)}{2} \quad \frac{2 FA + 2 FB + 2 FC}{2} \\
M = -0.839 \text{ (kip ft)}
\]

\[
x = \frac{a FA + 2 a FB + 2 a FC}{2} \quad \frac{2 FA + 2 FB + 2 FC}{2} \\
x = 2.591 \text{ (ft)}
\]

\[
z = \frac{2 b FA - a FA FC + 2 b FB + 2 b FC}{2} \quad \frac{2 FA + 2 FB + 2 FC}{2} \\
z = 5.530 \text{ (ft)}
\]
The moment $M = -839.254 \text{ lb ft} = -0.839 \text{ kip ft}$ is negative, and that is why the couple vector is pointing in the direction opposite to $R$, which makes the wrench negative. The MATLAB program for plotting the vectors is

```matlab
a=4; b=6;

axis([-2*a 2*a -b b -2*b 2*b])
xlabel('x'), ylabel('y'), zlabel('z')
hold on, grid on

% Cartesian axes
quiver3(0,0,0,2*a,0,0,1,'Color','b'), text(2*a,0,0,' x')
quiver3(0,0,0,0,b,0,1,'Color','b'), text(0,b,0,' y')
quiver3(0,0,0,0,0,2*b,1,'Color','b'), text(0,0,2*b,' z')

xA=a/2; yA=0; zA=0;
xB=a; yB=0; zB=b;
xC=0; yC=0; zC=b;
xD=a; yD=0; zD=0;
XT=double(xn); YT=0; ZT=double(zn);

line([0 xC],[0 yC],[0,zC],'Color','b','LineWidth',2)
line([0 xD],[0 yD],[0,zD],'Color','b','LineWidth',2)
line([xD xB],[yD yB],[zD,zB],'Color','b','LineWidth',2)
line([xC xB],[yC yB],[zC,zB],'Color','b','LineWidth',2)

text(0,0,0,' O')
text(xA,yA,zA,' A')
text(xB,yB,zB,' B')
text(xC,yC,zC,' C')
text(xD,yD,zD,' D')
text(xT,yT,zT-1,' T')

fs=10; % force scale
FA = fs*double(subs(F_A, sl, nl));
FB = fs*double(subs(F_B, sl, nl));
FC = fs*double(subs(F_C, sl, nl));
Rt = fs*Rn;
Mt = fs*double(Mn)*uRn;

quiver3(xA,yA,zA,FA(1),FA(2),FA(3),1,'Color','k','LineWidth',2)
quiver3(xB,yB,zB,FB(1),FB(2),FB(3),1,'Color','k','LineWidth',2)
quiver3(xC,yC,zC,FC(1),FC(2),FC(3),1,'Color','k','LineWidth',2)
quiver3(xT,yT,zT,Rt(1),Rt(2),Rt(3),1,'Color','r','LineWidth',2)
quiver3(xT,yT,zT,Mt(1),Mt(2),Mt(3),1,'Color','G','LineWidth',2)
```
text (xA+FA(1),yA+FA(2),zA+FA(3),...
    ' F_A','fontsize',12,'fontweight','b')
text (xB+FB(1),yB+FB(2),zB+FB(3),...
    ' F_B','fontsize',12,'fontweight','b')
text (xT+Rt(1),yT+Rt(2),zT+Rt(3),...
    ' R','fontsize',14,'fontweight','b')
text (xT+Mt(1),yT+Mt(2),zT+Mt(3),...
    ' M','fontsize',14,'fontweight','b')
view(-68,30);

The vector representation with MATLAB is shown in Fig. E2.5(b).
2.7 Problems

2.1 a) Determine the resultant of the forces \( F_1 = F_{1x} \mathbf{i} + F_{1y} \mathbf{j} + F_{1z} \mathbf{k} \), \( F_2 = F_{2x} \mathbf{i} + F_{2y} \mathbf{j} + F_{2z} \mathbf{k} \), and \( F_3 = F_{3x} \mathbf{i} + F_{3y} \mathbf{j} + F_{3z} \mathbf{k} \), which are concurrent at the point \( P(x_P, y_P, z_P) \), where \( F_{1x} = 2 \), \( F_{1y} = 3.5 \), \( F_{1z} = -3 \), \( F_{2x} = -1.5 \), \( F_{2y} = 4.5 \), \( F_{2z} = -3 \), \( F_{3x} = 7 \), \( F_{3y} = -6 \), \( F_{3z} = 5 \), \( x_P = 1 \), \( y_P = 2 \), and \( z_P = 3 \).

b) Find the total moment of the given forces about the origin \( O(0, 0, 0) \). The units for the forces are in Newtons and for the coordinates are given in meters.

2.2 a) Determine the resultant of the three forces shown in Fig. P2.2. The force \( F_1 \) acts along the \( x \)-axis, the force \( F_2 \) acts along the \( z \)-axis, and the direction of the force \( F_3 \) is given by the line \( O_3P_3 \), where \( O_3 = O(x_{O_3}, y_{O_3}, z_{O_3}) \) and \( P_3 = P(x_{P_3}, y_{P_3}, z_{P_3}) \). The application point of the forces \( F_1 \) and \( F_2 \) is the origin \( O(0, 0, 0) \) of the reference frame as shown in Fig. P2.2. b) Find the total moment of the given forces about the point \( P_3 \). Numerical application: \( |F_1| = F_1 = 250 \text{ N} \), \( |F_2| = F_2 = 300 \text{ N} \), \( |F_3| = F_3 = 300 \text{ N} \), \( O_3 = O_3(1, 2, 3) \) and \( P_3 = P_3(5, 7, 9) \). The coordinates are given in meters.

![Fig. P2.2 Problem 2.2](image)

2.3 Replace the three forces \( F_1, F_2, \) and \( F_3 \), shown in Fig. P2.3, by a resultant force, \( R \), through \( O \) and a couple. The force \( F_2 \) acts along the \( x \)-axis, the force \( F_1 \) is parallel to the \( y \)-axis, and the force \( F_3 \) is parallel to the \( z \)-axis. The application point of the forces \( F_2 \) is \( O \), the application point of the forces \( F_1 \) is \( B \), and the application points of the force \( F_3 \) is \( A \). The distance between \( O \) and \( A \) is \( d_1 \) and the distance between \( A \) and \( B \) is \( d_2 \) as shown in Fig. P2.3. Numerical application: \( |F_1| = F_1 = 250 \text{ N} \), \( |F_2| = F_2 = 300 \text{ N} \), \( |F_3| = F_3 = 400 \text{ N} \), \( d_1 = 1.5 \text{ m} \) and \( d_2 = 2 \text{ m} \).

![Fig. P2.3 Problem 2.3](image)
2.4 Two forces $F_1$ and $F_2$ and a couple of moment $M$ in the $xy$ plane are given. The force $F_1 = F_{1x}\mathbf{i} + F_{1y}\mathbf{j} + F_{1z}\mathbf{k}$ acts at the point $P_1 = P_1(x_1, y_1, z_1)$ and the force $F_2 = F_{2x}\mathbf{i} + F_{2y}\mathbf{j} + F_{2z}\mathbf{k}$ acts at the point $P_2 = P_2(x_2, y_2, z_2)$. Find the resultant force-couple system at the origin $O(0, 0, 0)$. 

Numerical application: $F_{1x} = 10$, $F_{1y} = 5$, $F_{1z} = 40$, $F_{2x} = 30$, $F_{2y} = 10$, $F_{2z} = -30$, $F_{3x} = 7$, $F_{3y} = -6$, $F_{3z} = 5$, $P_1 = P_1(0, 1, -1)$, $P_2 = P_2(1, 1, 1)$ and $M = -30$ N·m. The units for the forces are in Newtons and for the coordinates are given in meters.

2.5 Replace the three forces $F_1$, $F_2$, and $F_3$, shown in Fig. P2.5, by a resultant force at the origin $O$ of the reference frame and a couple. The force $F_1$ acts along the $x$-axis, the force $F_2$ is parallel with the $z$-axis, and the force $F_3$ is parallel with the $y$-axis. The application point of the force $F_1$ is at $O$, the application point of the forces $F_2$ is at $A$, and the application points of the force $F_3$ is at $B$. The distance between the origin $O$ and the point $A$ is $d_1$ and the distance between the point $A$ and the point $B$ is $d_2$. The line $AB$ is parallel with the $z$-axis. Numerical application: $|F_1| = F_1 = 50$ N, $|F_2| = F_2 = 30$ N, $|F_3| = F_3 = 60$ N, $d_1 = 1$ m, and $d_2 = 0.7$ m

![Fig. P2.5 Problem 2.5](image)

2.6 Three forces $F_1$, $F_2$ and $F_3$ act on a beam as shown in Fig. P2.6. The directions of the forces are parallel with $y$-axis. The application points of the forces are $P_1$, $P_2$, and $P_3$, and the distances $AP_1 = d_1$, $P_1P_2 = d_2$, $P_2P_3 = d_3$ and $P_3B = d_4$ are given. 

a) Find the resultant of the system. b) Resolve this resultant into two components at the points $A$ and $B$. Numerical application: $|F_1| = F_1 = 30$ N, $|F_2| = F_2 = 60$ N, $|F_3| = F_3 = 50$ N, $d_1 = 0.1$ m, $d_2 = 0.3$ m, $d_3 = 0.4$ m and $d_4 = 0.4$ m.

![Fig. P2.6 Problem 2.6](image)
2.7 A force $\mathbf{F}$ acts vertically downward, parallel to the $y$-axis, and intersects the $xz$ plane at the point $P_1(x_1, y_1, z_1)$. Resolve this force into three components acting through the points $P_2 = P_2(x_2, y_2, z_2)$, $P_3 = P_3(x_3, y_3, z_3)$ and $P_4 = P_4(x_4, y_4, z_4)$. Numerical application: $|\mathbf{F}| = F = 50$ N, $P_1 = P_1(2, 0, 4)$, $P_2 = P_2(1, 1, 1)$, $P_3 = P_3(6, 0, 0)$, and $P_4 = P_4(0, 0, 3)$. The coordinates are given in meters.

2.8 Determine the resultant of the given system of forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, shown in the Fig. P2.8. The angle between the direction of the force $\mathbf{F}_1$ and the $Ox$ axis is $\theta_1$ and the angle between the direction of the force $\mathbf{F}_2$ with the $x$-axis is $\theta_2$. The $x$ and $y$ components of the force $\mathbf{F}_3 = |\mathbf{F}_3| \hat{i} + |\mathbf{F}_3| \hat{j}$ are given. Numerical application: $|\mathbf{F}_1| = F_1 = 250$ N, $|\mathbf{F}_2| = F_2 = 220$ N, $|\mathbf{F}_3| = F_3 = 50$ N, $|\mathbf{F}_{3y}| = F_{3y} = 120$ N, $\theta_1 = 30^\circ$, and $\theta_2 = 45^\circ$.

2.9 The rectangular plate in Fig. P2.9 is subjected to four parallel forces. Determine the magnitude and direction of a resultant force equivalent to the given force system and locate its point of application on the plate. Numerical application: $F_O = 700$ lb, $F_A = 600$ lb, $F_B = 500$ lb, $F_C = 100$ lb, $a = 8$ ft, and $b = 10$ ft. Hint: the moments about the $x$-axis and $y$-axis of the resultant force, are equal to the sum of the moments about the $x$-axis and $y$-axis of all the forces in the system.

2.10 Three forces $\mathbf{F}_O$, $\mathbf{F}_B$, and $\mathbf{F}_C$, as shown in Fig. P2.10, are acting on a rectangular planar plate ($\mathbf{F}_O||Oz$, $\mathbf{F}_B||Oy$, $\mathbf{F}_C||Ox$). The three forces acting on the plate are replaced by a wrench. Find: a) the resultant force for the wrench; b) the magnitude of couple moment, $M$, for the wrench and the point $Q(y, z)$ where its line of
action intersects the plate. Numerical application: $F_O = 800 \text{ lb}$, $F_B = F_C = 500 \text{ lb}$, $a = OA = 6 \text{ ft}$, and $b = AB = 5 \text{ ft}$.

Fig. P2.10  Problem 2.10