Laws of motion

Consider the motion of a system \( \{S\} \) of \( \nu \) particles \( P_1, ..., P_\nu \) \( \{\{S\} = \{P_1, ..., P_\nu\}\} \) in an inertial reference frame \( (0) \). The equation of motion for the \( i \)th particle is

\[
\mathbf{R}_i = m_i \mathbf{a}_i,
\]

where \( \mathbf{R}_i \) is the resultant of all contact and distance forces acting on \( P_i \); \( m_i \) is the mass of \( P_i \); and \( \mathbf{a}_i \) is the acceleration of \( P_i \) in \( (0) \). Equation (1) is the expression of Newton’s second law.

If the inertia force \( \mathbf{R}_i^* \) for \( P_i \) in \( (0) \) is defined as

\[
\mathbf{R}_i^* = -m_i \mathbf{a}_i,
\]

then Eq. (1) may be written as

\[
\mathbf{R}_i + \mathbf{R}_i^* = 0.
\]

Equation (3) is the expression of D’Alembert’s principle.

If \( \{S\} \) is a holonomic system possessing \( n \) degrees of freedom, then the position vector \( \mathbf{r}_i \) of \( P_i \) relative to a point \( O \) fixed in reference frame \( (0) \) may be expressed as a vector function of \( n \) general coordinates \( q_1, ..., q_n \) and time \( t \)

\[
\mathbf{r}_i = \mathbf{r}_i(q_1, ..., q_n, t).
\]

The velocity \( \mathbf{v}_i \) of \( P_i \) in \( (0) \) may now be written as

\[
\mathbf{v}_i = \sum_{r=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_r} \frac{\partial q_r}{\partial t} + \frac{\partial \mathbf{r}_i}{\partial t} = \sum_{r=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_r} \dot{q}_r + \frac{\partial \mathbf{r}_i}{\partial t},
\]

or as

\[
\mathbf{v}_i = \sum_{r=1}^{n} (\mathbf{v}_i)_r \dot{q}_r + \frac{\partial \mathbf{r}_i}{\partial t}.
\]

where \( (\mathbf{v}_i)_r \) is called the \( r \)th partial velocity of \( P_i \) in \( (0) \) and is defined as

\[
(\mathbf{v}_i)_r = \frac{\partial \mathbf{r}_i}{\partial q_r} = \frac{\partial \mathbf{v}_i}{\partial \dot{q}_r}.
\]

Next, replace Eq. (3) with

\[
\sum_{i=1}^{\nu} (\mathbf{R}_i + \mathbf{R}_i^*) \cdot (\mathbf{v}_i)_r = 0.
\]
If a generalized active force $K_r$ and a generalized inertia force $K^*_r$ are defined as

$$K_r = \sum_{i=1}^{\nu} (v_i)_r \cdot R_i = \sum_{i=1}^{\nu} \frac{\partial r_i}{\partial q_r} \cdot R_i = \sum_{i=1}^{\nu} \frac{\partial v_i}{\partial \dot{q}_r} \cdot R_i,$$

and

$$K^*_r = \sum_{i=1}^{\nu} (v_i)_r \cdot R^*_i = \sum_{i=1}^{\nu} \frac{\partial r_i}{\partial q_r} \cdot R^*_i = \sum_{i=1}^{\nu} \frac{\partial v_i}{\partial \dot{q}_r} \cdot R^*_i,$$

then Eq. (8) may be written as

$$K_r + K^*_r = 0, \quad r = 1, \ldots, n. \tag{11}$$

Equations (11) are Kane’s dynamical equations.

Consider the generalized inertia force $K^*_r$:

$$K^*_r = \sum_{i=1}^{\nu} R^*_i \cdot (v_i)_r = -\sum_{i=1}^{\nu} m_i a_i \cdot (v_i)_r = -\sum_{i=1}^{\nu} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_r} =
-\sum_{i=1}^{\nu} \left[ \frac{d}{dt} \left( m_i \dot{r}_i \cdot \frac{\partial r_i}{\partial \dot{q}_r} \right) - m_i \dot{r}_i \cdot \frac{d}{dt} \left( \frac{\partial r_i}{\partial q_r} \right) \right]. \tag{12}$$

Now

$$\frac{d}{dt} \left( \frac{\partial r_i}{\partial q_r} \right) = \sum_{k=1}^{n} \frac{\partial^2 r_i}{\partial q_k \partial q_k} \dot{q}_k + \frac{\partial^2 r_i}{\partial q_r \partial t} = \frac{\partial v_i}{\partial q_r}, \tag{13}$$

and furthermore using Eq. (5)

$$\frac{\partial v_i}{\partial \dot{q}_r} = \frac{\partial r_i}{\partial q_r}. \tag{14}$$

Substitution of Eq. (13) and Eq. (14) in Eq. (12) leads to

$$K^*_r = -\sum_{i=1}^{\nu} \left[ \frac{d}{dt} \left( m_i v_i \cdot \frac{\partial v_i}{\partial \dot{q}_r} \right) - m_i v_i \cdot \frac{\partial v_i}{\partial q_r} \right] =
- \left[ \frac{d}{dt} \frac{\partial}{\partial \dot{q}_r} \left( \sum_{i=1}^{\nu} \frac{1}{2} m_i v_i \cdot v_i \right) - \frac{\partial}{\partial q_r} \left( \sum_{i=1}^{\nu} \frac{1}{2} m_i v_i \cdot v_i \right) \right]. \tag{15}$$

The kinetic energy $T$ of $\{S\}$ in reference frame (0) is defined as

$$T = \frac{1}{2} \sum_{i=1}^{\nu} m_i v_i \cdot v_i. \tag{16}$$
Therefore, the generalized inertia forces $K_r^*$ can be written as

$$K_r^* = -\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) + \frac{\partial T}{\partial q_r},$$  \hspace{1cm} (17)$$

and Kane’s dynamical equations can be written as

$$K_r - \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) + \frac{\partial T}{\partial q_r} = 0,$$  \hspace{1cm} (18)$$

or

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = K_r,$$  \hspace{1cm} (19)$$

and

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = \sum_{i=1}^{\nu} \frac{\partial r_i}{\partial q_r} \cdot \mathbf{R}_i, \hspace{1cm} r = 1, \ldots, n.$$  \hspace{1cm} (20)$$

Equations (20) are known as Lagrange’s equations of motion of the first kind.