Laws of motion

Consider the motion of a system $\{S\}$ of ν particles $P_1, ..., P_\nu$ ($\{S\} = \{P_1, ..., P_\nu\}$)in an inertial reference frame (0). The equation of motion for the ith particle is

$$\mathbf{R}_i = m_i\,\mathbf{a}_i, \tag{1}$$

where $\mathbf{R}_i$ is the resultant of all contact and distance forces acting on P_i; m_i is the mass of P_i; and $\mathbf{a}_{P_i}$ is the acceleration of P_i in (0). Equation (1) is the expression of *Newton's second law.*

If the *inertia force* $\mathbf{R}_i^*$ for P_i in (0) is defined as

$$\mathbf{R}_i^* = -\,m_i\,\mathbf{a}_i, \tag{2}$$

then Eq. (1) may be written as

$$\mathbf{R}_i + \mathbf{R}_i^* = \mathbf{0}. \tag{3}$$

Equation (3) is the expression of *D'Alembert's principle.*

If $\{S\}$ is a *holonomic* system possessing n degrees of freedom, then the position vector $\mathbf{r}_i$ of P_i relative to a point O fixed in reference frame (0) may be expressed as a vector function of n generalized coordinates $q_1, ..., q_n$ and time t

$$\mathbf{r}_i = \mathbf{r}_i(q_1, ..., q_n, t). \tag{4}$$

The velocity $\mathbf{v}_i$ of P_i in (0) may now be written as

$$\mathbf{v}_i = \sum_{r=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_r}\frac{\partial q_r}{\partial t} + \frac{\partial \mathbf{r}_i}{\partial t} = \sum_{r=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_r}\,\dot{q}_r + \frac{\partial \mathbf{r}_i}{\partial t}, \tag{5}$$

or as

$$\mathbf{v}_i = \sum_{r=1}^{n} (\mathbf{v}_i)_r\,\dot{q}_r + \frac{\partial \mathbf{r}_i}{\partial t}, \tag{6}$$

where $(\mathbf{v}_i)_r$ is called the rth *partial velocity* of P_i in (0) and is defined as

$$(\mathbf{v}_i)_r = \frac{\partial \mathbf{r}_i}{\partial q_r} = \frac{\partial \mathbf{v}_i}{\partial \dot{q}_r}. \tag{7}$$

Next, replace Eq. (3) with

$$\sum_{i=1}^{\nu} (\mathbf{R}_i + \mathbf{R}_i^*) \cdot (\mathbf{v}_i)_r = 0. \tag{8}$$

If a *generalized active force* K_r and a *generalized inertia force* K_r^* are defined as

$$K_r = \sum_{i=1}^{\nu} (\mathbf{v}_i)_r \cdot \mathbf{R}_i = \sum_{i=1}^{\nu} \frac{\partial \mathbf{r}_i}{\partial q_r} \cdot \mathbf{R}_i = \sum_{i=1}^{\nu} \frac{\partial \mathbf{v}_i}{\partial \dot{q}_r} \cdot \mathbf{R}_i, \tag{9}$$

and

$$K_r^* = \sum_{i=1}^{\nu} (\mathbf{v}_i)_r \cdot \mathbf{R}_i^* = \sum_{i=1}^{\nu} \frac{\partial \mathbf{r}_i}{\partial q_r} \cdot \mathbf{R}_i^* = \sum_{i=1}^{\nu} \frac{\partial \mathbf{v}_i}{\partial \dot{q}_r} \cdot \mathbf{R}_i^*, \tag{10}$$

then Eq. (8) may be written as

$$K_r + K_r^* = 0, \;\; r = 1, ..., n. \tag{11}$$

Equations (11) are *Kane's dynamical equations.*

Consider the generalized inertia force K_r^*:

$$\begin{aligned} K_r^* = \sum_{i=1}^{\nu} \mathbf{R}_i^* \cdot (\mathbf{v}_i)_r = -\sum_{i=1}^{\nu} m_i \, \mathbf{a}_i \cdot (\mathbf{v}_i)_r = -\sum_{i=1}^{\nu} m_i \, \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_r} = \\ -\sum_{i=1}^{\nu} \left[\frac{d}{dt} \left(m_i \, \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_r} \right) - m_i \, \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_r} \right) \right]. \end{aligned} \tag{12}$$

Now

$$\frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_r} \right) = \sum_{k=1}^{n} \frac{\partial^2 \mathbf{r}_i}{\partial q_r \, \partial q_k} \dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial q_r \, \partial t} = \frac{\partial \mathbf{v}_i}{\partial q_r}, \tag{13}$$

and furthermore using Eq. (5)

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_r} = \frac{\partial \mathbf{r}_i}{\partial q_r}. \tag{14}$$

Substitution of Eq. (13) and Eq. (14) in Eq. (12) leads to

$$\begin{aligned} K_r^* = -\sum_{i=1}^{\nu} \left[\frac{d}{dt} \left(m_i \, \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_r} \right) - m_i \, \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_r} \right] = \\ -\left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}_r} \left(\sum_{i=1}^{\nu} \frac{1}{2} m_i \, \mathbf{v}_i \cdot \mathbf{v}_i \right) - \frac{\partial}{\partial q_r} \left(\sum_{i=1}^{\nu} \frac{1}{2} m_i \, \mathbf{v}_i \cdot \mathbf{v}_i \right) \right]. \end{aligned} \tag{15}$$

The *kinetic energy* T of $\{S\}$ in reference frame (0) is defined as

$$T = \frac{1}{2} \sum_{i=1}^{\nu} m_i \, \mathbf{v}_i \cdot \mathbf{v}_i. \tag{16}$$

Therefore, the generalized inertia forces K_r^* can be written as

$$K_r^* = -\frac{d}{d\,t}\left(\frac{\partial T}{\partial \dot{q}_r}\right) + \frac{\partial T}{\partial q_r}, \tag{17}$$

and Kane's dynamical equations can be written as

$$K_r - \frac{d}{d\,t}\left(\frac{\partial T}{\partial \dot{q}_r}\right) + \frac{\partial T}{\partial q_r} = 0, \tag{18}$$

or

$$\frac{d}{d\,t}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = K_r, \tag{19}$$

and

$$\frac{d}{d\,t}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = \sum_{i=1}^{\nu} \frac{\partial \mathbf{r}_i}{\partial q_r} \cdot \mathbf{R}_i, \quad r = 1, ..., n. \tag{20}$$

Equations (20) are known as *Lagrange's equations of motion* of the first kind.