1 Kinematics of Rigid Body - Problems

Example 1

Find the velocity of the vertex points A and G of the rectangular prism with the dimensions given in Fig. 1.

The rectangular prism has the length a, height b, and width c, and has a uniform rotation about the diagonal OF with the angular velocity ω.

Solution:

The velocity of one vertex can be written as

$$\mathbf{v} = \mathbf{v}_A + \boldsymbol{\omega}\times\mathbf{r},$$

or

$$\begin{aligned} \mathbf{v} &= \boldsymbol{\omega}\times\mathbf{r} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ \omega\dfrac{a}{\sqrt{a^2+b^2+c^2}} & \omega\dfrac{b}{\sqrt{a^2+b^2+c^2}} & \omega\dfrac{c}{\sqrt{a^2+b^2+c^2}} \\ x & y & z \end{vmatrix} \qquad (1) \\ &= \frac{\boldsymbol{\omega}}{\sqrt{a^2+b^2+c^2}}\left[(bz-cy)\,\mathbf{\imath}+(cx-az)\,\mathbf{\jmath}+(ay-bx)\,\mathbf{k}\right], \end{aligned}$$

where x, y, z are the coordinate of the given vertex, $\mathbf{\imath}$, $\mathbf{\jmath}$, $\mathbf{k}$ are the unity vectors, $\mathbf{v}_A = \mathbf{0}$ and ω is the angular velocity. In the previous equation the values $\omega\dfrac{a}{\sqrt{a^2+b^2+c^2}}$, $\omega\dfrac{b}{\sqrt{a^2+b^2+c^2}}$ and $\omega\dfrac{c}{\sqrt{a^2+b^2+c^2}}$ represent the magnitude of the components of the angular velocity.

One can write

$$\omega = \omega\frac{a}{\sqrt{a^2+b^2+c^2}}\mathbf{\imath} + \omega\frac{b}{\sqrt{a^2+b^2+c^2}}\mathbf{\jmath} + \omega\frac{c}{\sqrt{a^2+b^2+c^2}}\mathbf{k},$$

and

$$\mathbf{r} = x\mathbf{\imath} + y\mathbf{\jmath} + z\mathbf{k}.$$

The velocity of the vertex has the magnitude

$$|\mathbf{v}| = \frac{\omega\sqrt{(bz-cy)^2+(cx-az)^2+(ay-bx)^2}}{\sqrt{a^2+b^2+c^2}}. \qquad (2)$$

For the corners $A\,(a,0,0)$ and $G\,(0,b,c)$, using the previous equation one can obtain

$$v_A = v_G = \omega a \sum_{r=1}^{3} \frac{\sqrt{b^2+c^2}}{\sqrt{a^2+b^2+c^2}}. \tag{3}$$

Example 2

The ends of the rod $AB = l$ slide along two perpendicular axis Ox_1 and Oy_1 (see Fig. 2). The velocity of the end B of the bar is v_B for a given position of the angle α. Find

a) the instantaneous center of rotation of the rod

b) the location of the instantaneous center of rotation of the rod with respect to an fix and mobil reference frame

c) the velocity of the point M, where M is at the middle of the rod

Solution:

a) Denote by I the instantaneous center of rotation.
The coordinate of the point I with respect to the fixed axes Ox_1 and Oy_1 can be written as

$$x_1 = l \sin\alpha, \tag{4}$$

$$y_1 = l \cos\alpha. \tag{5}$$

One can write the coordinate of the point I with respect to the mobile axes Bx and By as

$$x = PI = BI \sin\alpha = y_1 \sin\alpha = l \cos\alpha \sin\alpha, \tag{6}$$

$$y = BP = BI \cos\alpha = y_1 \cos\alpha = l \cos^2\alpha. \tag{7}$$

The axis Bx is along the rod (see Fig. 2).

b) Using Eqs.(4) and (5) the location of the instantaneous center of rotation at any instant of the motion with respect to the fixed axis can be written as

$$x_1^2 + y_1^2 = l^2. \tag{8}$$

Equation(8) represent a circle with the origin at $x_1 = 0$, $y_1 = 0$ and the radius l.

Using Eqs.(6) and (7) the location of the instantaneous center of rotation at any instant of the motion with respect to the mobil reference frame is computed as follows

$$x = l \cos\alpha \sin\alpha = \frac{l}{2} \sin 2\alpha, \tag{9}$$

or

$$\sin 2\alpha = \frac{2x}{l}, \tag{10}$$

and

$$y = l\cos^2\alpha = l\frac{l + \cos 2\alpha}{2}, \tag{11}$$

or

$$\cos 2\alpha = \frac{2y}{l} - 1. \tag{12}$$

Using Eqs.(6) and (7) one can write

$$\begin{aligned}(\sin 2\alpha)^2 + (\cos 2\alpha)^2 &= \left(\frac{2x}{l}\right)^2 + \left(\frac{2y}{l} - 1\right)^2 \\ &= \frac{4x^2}{l^2} + \frac{4y^2}{l^2} - \frac{4y}{l} + 1,\end{aligned}$$

or

$$1 = \frac{4x^2}{l^2} + \frac{4y^2}{l^2} - \frac{4y}{l} + 1.$$

The previous equation is can be written

$$\frac{4x^2}{l^2} + \frac{4y^2}{l^2} - \frac{4y}{l} = 0,$$

or

$$\frac{x^2}{l} + \frac{y^2}{l} = y, \tag{13}$$

because $l \neq 0$.
The Eq.(13) take the form

$$\frac{x^2 + y^2 - ly}{l} = 0,$$

or

$$x^2 + y^2 - ly = 0. \tag{14}$$

Equation(14) represent a circle with the origin at $x_1 = 0$, $y_1 = 0$ and the radius l.

c) The velocity of the point M, where M is the midl of the rod AB, is perpendicular to the normal MI, and can be written as

$$\begin{aligned} v_M &= MI\omega, \\ v_B &= BI\omega, \end{aligned} \tag{15}$$

where

$$BI = 2MI \cos \alpha, \tag{16}$$

using Eqs.(15) and (16) one can write

$$v_M = \frac{v_B}{2 \cos \alpha}.$$

The previous equation is the equation of the circle with diameter AB.

Example 3

An crank-slider mechanism shown in Fig. 3 have an angular velocity $\omega = n$. The length of crank is $OM = r$, and the lenght of the connecting rod is $MP = l$. Find.the velocity and the acceleration of the crank for a given angle, θ.

Solution:

a) Denote by x the distance between the pin joint O to the slider P.

One can write

$$x = r \cos \theta + l \cos \varphi. \tag{17}$$

For the $\triangle OMP$ one can write

$$\frac{r}{\sin \varphi} = \frac{l}{\sin \theta},$$

or

$$\cos \varphi = \sqrt{1 - \frac{r^2}{l^2} \sin^2 \theta}. \tag{18}$$

Using Eqs.(17) and (18) one can write

$$x = r \cos \theta + l \sqrt{1 - \frac{r^2}{l^2} \sin^2 \theta}. \tag{19}$$

Denote $\dfrac{r}{l} = \lambda$ and for $\lambda < \frac{1}{2}$, one can write

$$x = r \left(\frac{1}{\lambda} + \cos \theta - \frac{\lambda}{2} \sin^2 \theta - \frac{\lambda^3}{8} \sin^4 \theta - \frac{\lambda^5}{16} \sin^6 \theta + ... \right). \tag{20}$$

Using the relations

$$\sin^2 \theta = \frac{1}{2} \left(1 - \cos 2\theta\right),$$

$$\sin^4\theta = \frac{1}{8}\left(3 - 4\cos 2\theta + \cos 4\theta\right),$$

$$\sin^6\theta = \frac{1}{32}\left(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta\right),$$

one can write Eq.(20)

$$x = r\left(A_0 + \cos\theta + \frac{1}{4}A_2\cos 2\theta + \frac{1}{16}A_4\cos 4\theta + \frac{1}{36}A_6\cos 6\theta + ...\right), \quad (21)$$

where

$$\begin{aligned} A_0 &= \frac{1}{\lambda} - \frac{1}{4}\lambda - \frac{3}{64}\lambda^3 + ... \\ A_2 &= \frac{1}{4}\lambda^3 + ... \\ A_4 &= \frac{1}{4}\lambda^3 + \frac{3}{16}\lambda^5 + ... \\ A_6 &= \frac{9}{128}\lambda^5 + ... \end{aligned}$$

The computation for diferent values of λ is given in Table1.

$\frac{1}{\lambda}$	2	2.5	3	3.5	4	4.5	5
A_2	0.5312	0.4173.	0.3431	0.2918	0.2540	0.2250	0.2020
A_4	0.03626	0.0182	0.0101	0.0062	0.0041	0.0028	0.0021
A_6	0.0016	0.0009	0.0003	0.0001	0.0001	–	–

Table 1

Using the previous Table one can observe that the first and the second terms gives a good approximation for the computation of x. Using this one can write Eq.(21)

$$x = r\left(A_0 + \cos\theta + \frac{1}{4}A_2\cos 2\theta\right).$$

The velocity of point P is the derivative with respect to time of the position of x

$$v = \dot{x} = -r\omega\left(\sin\theta + \frac{A_2}{2}\sin 2\theta\right).$$

The acceleration of point P is the double derivative with respect to time of the position of x

$$a = \ddot{x} = -r\omega\left(\cos\theta + A_2\cos 2\theta\right).$$

For the values $r = 20$ cm, $l = 80$ cm, $n = 120$ rot/min and $\theta = 60°$ one obtain

$$\begin{aligned} v &= -2.42, \ \text{m/s}, \\ a &= -11.92, \ \text{m/s}^2. \end{aligned}$$

Example 4

Using the graphical method find the velocity of the point P of a crank-slider mechanism shown in Fig. 4 when the angular velocity of the crank is ω=constant.

Solution:

a) The position of the crank is defined by the angle θ. The line OH is a perpendicular to the line OP. The line PM intersects the line OH at H ($OH \perp OP$, $PM \cap OH = H$).

The velocity of the point P is

$$v = OH \cdot \omega. \tag{22}$$

b) Explanation of the graphical method

With respect to the instantaneous center of rotation I, one can write

$$u = r\omega = MI \cdot \Omega, \tag{23}$$

$$v = PI \cdot \Omega. \tag{24}$$

Because the triangles, ΔOHM and ΔIPM are similar ($\Delta OHM \sim \Delta IPM$), one can write

$$\frac{OH}{PI} = \frac{OM}{MI}. \tag{25}$$

Using Eqs.(23) and (24) one can write

$$\frac{v}{u} = \frac{PI}{MI} = \frac{OH}{OM}. \tag{26}$$

Because $OM = r$ and $u = r\omega$, and using Eq.(26) one can write

$$\frac{v}{r\omega} = \frac{OH}{r}, \tag{27}$$

or

$$v = \omega \cdot OH. \tag{28}$$

Example 5

Find the velocity of the the rectangular prism with the shape and the dimensions given in Fig. 5.

The rectangular prism has the length a, height b and width c, and has a uniform rotation abaut his sides with the angular velocites ω_1, ω_2 and ω_3,.

Solution:

Because there is a nonparallel special rotation, one can write

$$\boldsymbol{\omega} = \sum_{k=1}^{3} \boldsymbol{\omega}_i = \omega_1 \mathbf{\imath} + \omega_2 \mathbf{\jmath} + \omega_3 \mathbf{k}, \tag{29}$$

where $\mathbf{\imath}$, $\mathbf{\jmath}$ and $\mathbf{k}$ are the unit vectors, and ω_i, $i = 1, 2, 3$ is the angular velocity along each side.

The angular velocity has the magnitude

$$|\boldsymbol{\omega}| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}. \tag{30}$$

One can write the velocity of the point O as

$$\mathbf{v}_O = \sum_{k=1}^{3} \boldsymbol{\omega}_i \times \mathbf{r}_i = \boldsymbol{\omega}_1 \times \overline{OB} + \boldsymbol{\omega}_2 \times \overline{OC} + \boldsymbol{\omega}_3 \times \overline{OA} \tag{31}$$

$$= -c\omega_2 \mathbf{\imath} - a\omega_3 \mathbf{\jmath} - b\omega_1 \mathbf{k}. \tag{32}$$

The product

$$\boldsymbol{\omega} \mathbf{v}_O = -\left(c\omega_1\omega_2 + a\omega_2\omega_3 + b\omega_1\omega_3\right) \neq 0, \tag{33}$$

is obvious nonezero.

One can write

$$\begin{aligned} v_{0x} &= -c\omega_2, \\ v_{0y} &= -a\omega_3, \\ v_{0z} &= -b\omega_1, \end{aligned} \tag{34}$$

and

$$\begin{aligned} \omega_x &= \omega_1, \\ \omega_y &= \omega_2, \\ \omega_z &= \omega_3. \end{aligned} \tag{35}$$

The translation velocity is given by

$$v_t = \frac{c\omega_1\omega_2 + a\omega_2\omega_3 + b\omega_1\omega_3}{\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}}.$$

For the values $\omega_1 = \omega_2 = 3\text{s}^{-1}$, $\omega_3 = 0$, and $a = b = c = 2$ cm, one can obtain

$$\omega = 3\left(\boldsymbol{\imath} + \boldsymbol{\jmath}\right),$$

thus

$$\omega = 3\sqrt{2},$$

and

$$\mathbf{v}_0 = - - b\omega_1\left(\boldsymbol{\imath} + \mathbf{k}\right),$$

or

$$v_t = -3\sqrt{2}.$$

Then, one can write

$$\frac{-b + 3z}{3} = \frac{-3z}{3} = \frac{-b - 3x + 3y}{0},$$

or

$$\begin{aligned} z &= 1, \\ x - y &= 2. \end{aligned}$$

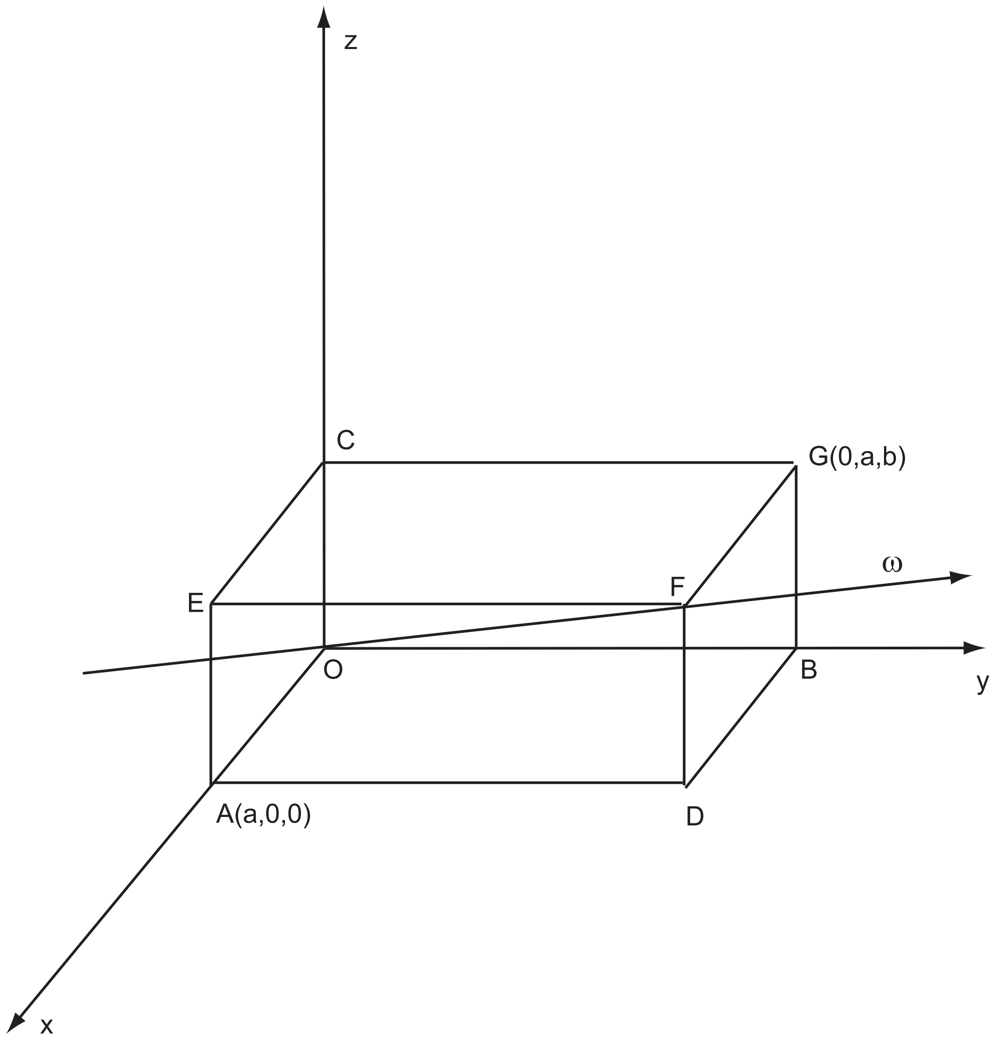

Fig. 1

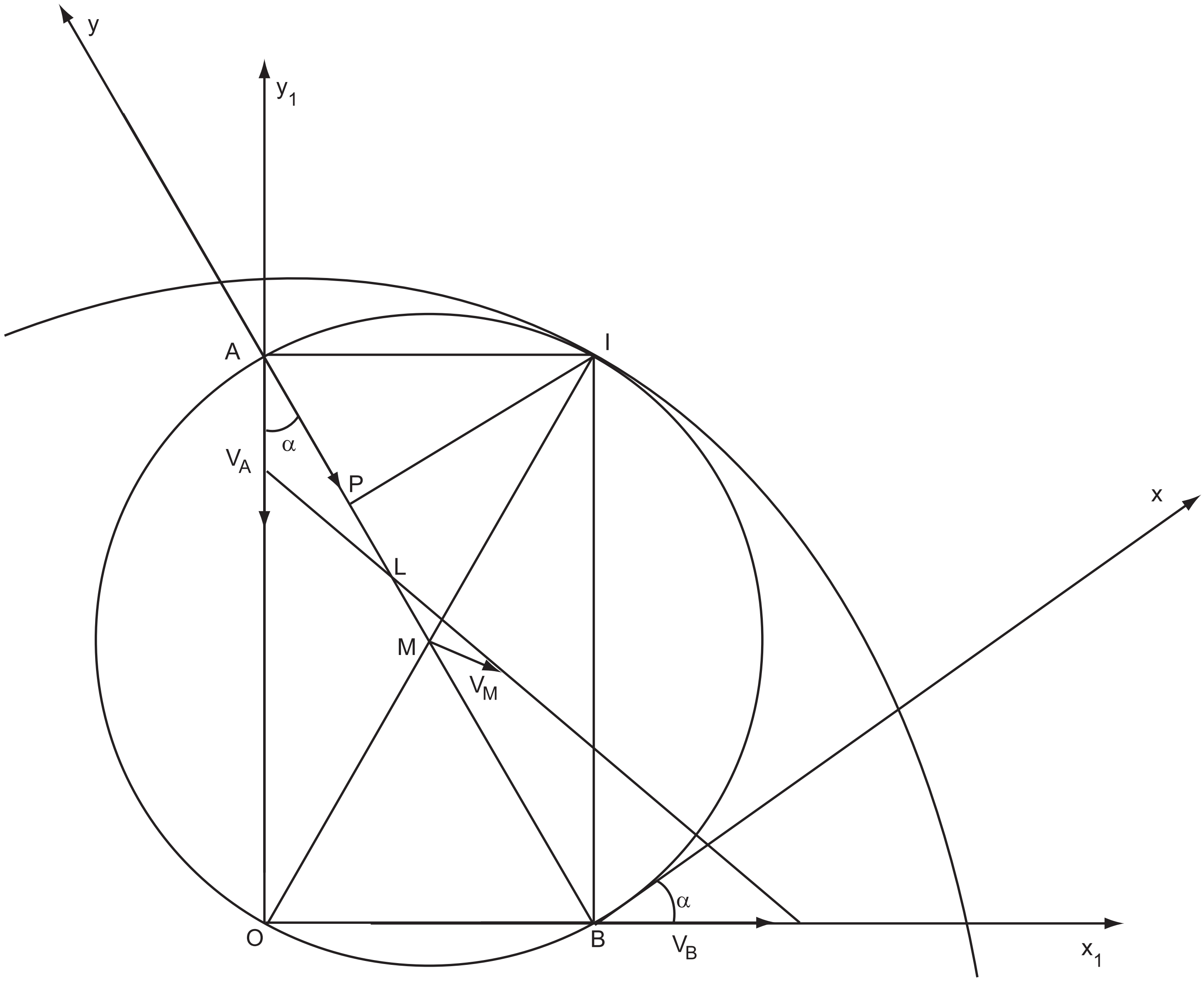

Fig. 2

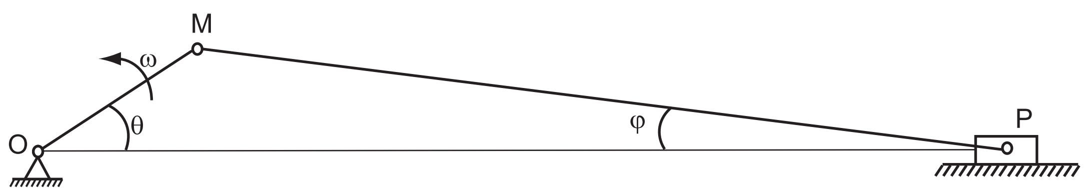

Fig. 3

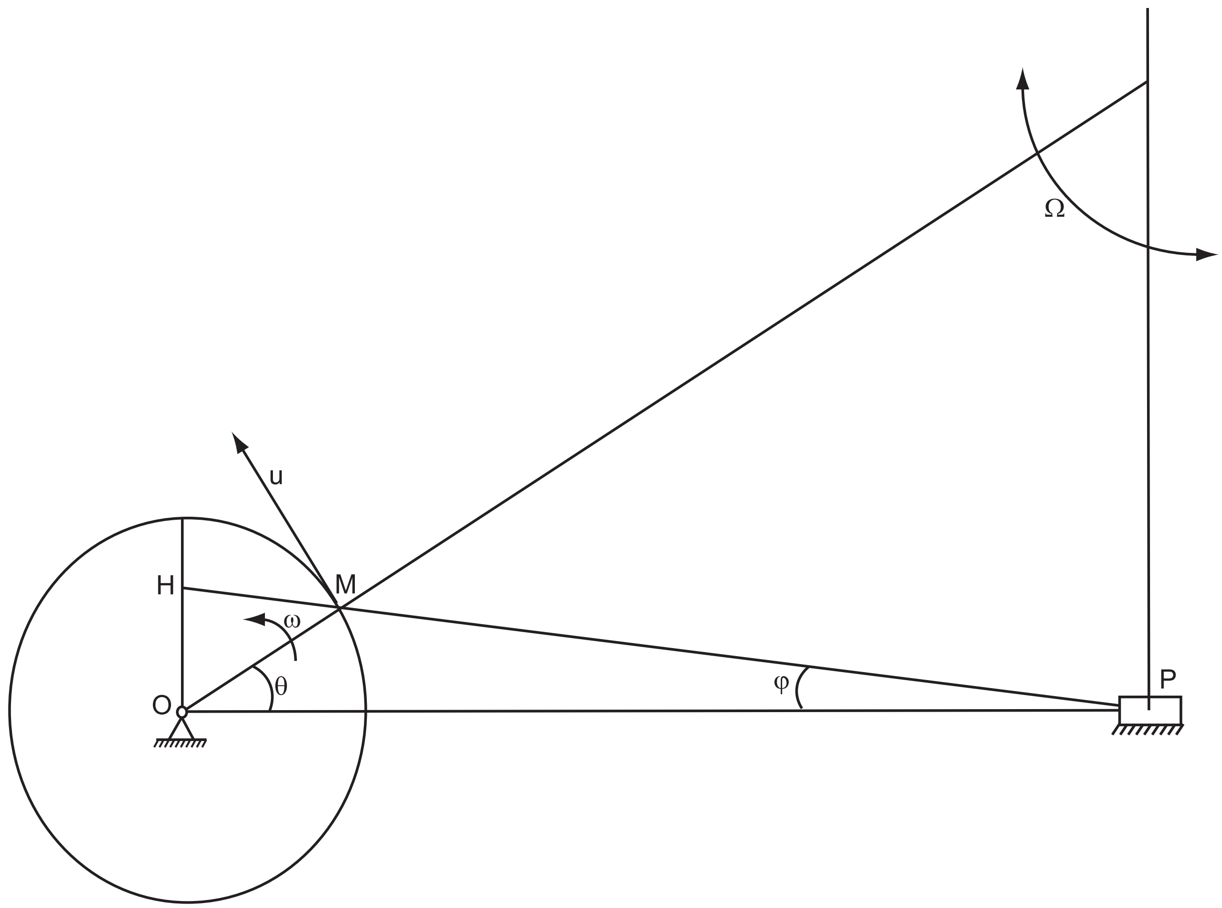

Fig. 4

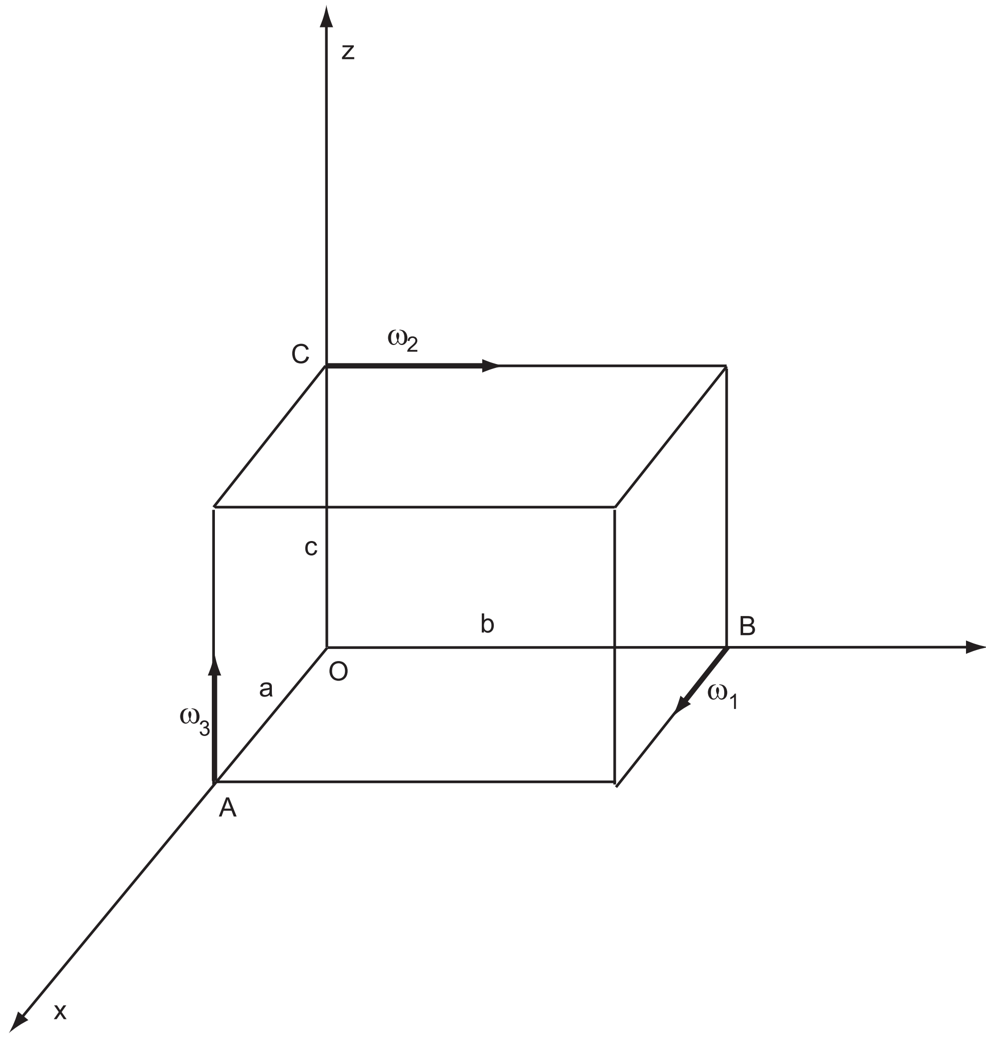

Fig. 5