1 Lagrange equations - Example 2

A double pendulum is considered in Fig. 1.1. The bars 1 and 2 are homogenous and have the lengths $OA = AB = L$ and the masses $m_1 = m_2 = m$. At $O$ and $A$ there are pin joints. The mass centers of links 1 and 2 are $C_1$ and $C_2$.

Find the Lagrange equations of motion if the initial conditions are known.

Link 1 can be rotated at $O$ in a “fixed” cartesian reference frame of unit vectors $[ı, ĵ, k]$ about an axis $k$. To characterize the instantaneous configuration of the system, two generalized coordinates $q_1(t)$ and $q_2(t)$ are employed. The generalized coordinates are quantities associated with the position of the system. The first generalized coordinate $q_1$ denotes the radian measure of the angle between the link 1 and the “fixed” cartesian reference frame. The last generalized coordinate $q_2$ designates also a radian measure of rotation angle between link 1 and link 2.

**Kinematic analysis**

The position vector of the mass center of link 1 is

$$r_{C_1} = 0.5L \sin q_1 \mathbf{ı} + 0.5L \cos q_1 \mathbf{ĵ}, \quad (1)$$

the position vector of the mass center of link 2 is

$$r_{C_2} = (L \sin q_1 + 0.5L \sin q_2) \mathbf{ı} + (L \cos q_1 + 0.5L \cos q_2) \mathbf{Ĵ}. \quad (2)$$

The velocity of $C_1$ is

$$v_{C_1} = \frac{d}{dt} r_{C_1} = \dot{r}_{C_1} = 0.5L \dot{q}_1 \cos q_1 \mathbf{ı} - 0.5L \dot{q}_1 \sin q_1 \mathbf{Ĵ}, \quad (3)$$

and the velocity of $C_2$ is

$$v_{C_2} = \frac{d}{dt} r_{C_2} = \dot{r}_{C_2} =$$

$$(L \dot{q}_1 \cos q_1 + 0.5L \dot{q}_2 \cos q_2) \mathbf{ı} - (L \dot{q}_1 \sin q_1 + 0.5L \dot{q}_2 \sin q_2) \mathbf{Ĵ}. \quad (4)$$

**Kinetic energy**

The kinetic energy of the link 1 which is in rotational motion is

$$T_1 = \frac{1}{2} I_0 \dot{q}_1^2 = \frac{1}{2} \frac{ML^2}{3} \dot{q}_1^2 = \frac{ML^2}{6} \dot{q}_1^2, \quad (5)$$

1
where $I_0$ is the mass moment of inertia about the center of rotation $O$, $I_O = mL^2/3$.

The kinetic energy of the bar 2 is due to the translation and rotation and can be expressed as

$$T_2 = \frac{1}{2} I_{C2} \dot{q}_2^2 + \frac{1}{2} m_2 v_{C2}^2,$$

(6)

where $I_{C2}$ is the mass moment of inertia about the center of mass $C_2$, $I_{C2} = mL^2/12$, and

$$v_{C2}^2 = v_{C2} \cdot v_{C2} = L^2 \dot{q}_1^2 + \frac{1}{4} L^2 \dot{q}_2^2 + L^2 \dot{q}_1 \dot{q}_2 \cos(q_2 - q_1).$$

(7)

Equation (6) becomes

$$T_2 = \frac{1}{2} \frac{mL^2}{12} \dot{q}_2^2 + \frac{1}{2} mL^2 \left[ \dot{q}_1^2 + \frac{1}{4} \dot{q}_2^2 + \dot{q}_1 \dot{q}_2 \cos(q_2 - q_1) \right].$$

(8)

The total kinetic energy of the system is

$$T = T_1 + T_2 = \frac{mL^2}{6} \left[ 4\dot{q}_1^2 + 3\dot{q}_1 \dot{q}_2 \cos(q_2 - q_1) + \dot{q}_2^2 \right].$$

(9)

The left hand sides of Lagrange equations are

$$\frac{\partial T}{\partial \dot{q}_1} = \frac{mL^2}{6} \left[ 8\ddot{q}_1 + 3\ddot{q}_2 \cos(q_2 - q_1) \right],$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) = \frac{mL^2}{6} \left[ 8\ddot{q}_1 + 3\ddot{q}_2 \cos(q_2 - q_1) - 3\dddot{q}_2 (\dddot{q}_2 - \dddot{q}_1) \sin(q_2 - q_1) \right],$$

$$\frac{\partial T}{\partial q_1} = \frac{mL^2}{6} 3 \dot{q}_1 \dot{q}_2 \sin(q_2 - q_1) = \frac{mL^2}{2} \dot{q}_1 \dot{q}_2 \sin(q_2 - q_1);$$

$$\frac{\partial T}{\partial \dot{q}_2} = \frac{mL^2}{6} \left[ 3\ddot{q}_1 \cos(q_2 - q_1) + 2\ddot{q}_2 \right],$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_2} \right) = \frac{mL^2}{6} \left[ 3\ddot{q}_1 \cos(q_2 - q_1) - 3\dddot{q}_1 (\dddot{q}_2 - \dddot{q}_1) \sin(q_2 - q_1) + 2\dddot{q}_2 \right],$$

$$\frac{\partial T}{\partial q_2} = -\frac{mL^2}{6} 3 \dot{q}_1 \dot{q}_2 \sin(q_2 - q_1) = -\frac{mL^2}{2} \dot{q}_1 \dot{q}_2 \sin(q_2 - q_1).$$

(10)
External forces analysis
The gravity forces on links 1 and 2 at the mass centers $C_1$ and $C_2$

$$\mathbf{F}_{C1} = \mathbf{F}_{C2} = m g \mathbf{j}. \tag{11}$$

Generalized forces
There are two generalized forces. The generalized force associated to $q_1$ is

$$Q_1 = \mathbf{F}_{C1} \cdot \frac{\partial \mathbf{r}_{C1}}{\partial q_1} + \mathbf{F}_{C2} \cdot \frac{\partial \mathbf{r}_{C2}}{\partial q_1} =$$

$$m g \mathbf{j} \cdot (0.5 L \cos q_1 \mathbf{j} - 0.5 L \sin q_1 \mathbf{j}) + m g \mathbf{j} \cdot (L \cos q_1 \mathbf{j} - L \sin q_1 \mathbf{j})$$

$$= -1.5 m g L \sin q_1. \tag{12}$$

The generalized force associated to $q_2$ is

$$Q_2 = \mathbf{F}_{C1} \cdot \frac{\partial \mathbf{r}_{C1}}{\partial q_2} + \mathbf{F}_{C2} \cdot \frac{\partial \mathbf{r}_{C2}}{\partial q_2} =$$

$$m g \mathbf{j} \cdot 0 + m g \mathbf{j} \cdot (0.5 L \cos q_2 \mathbf{j} - 0.5 L \sin q_2 \mathbf{j})$$

$$= -0.5 m g L \sin q_2. \tag{13}$$

The two Lagrange equations are

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} = Q_1,$$

$$1.333 m L^2 \ddot{q}_1 + 0.5 m L^2 \ddot{q}_2 \cos(q_2 - q_1) - 0.5 m L^2 \dot{q}_2^2 \sin(q_2 - q_1)$$

$$+ 1.5 m g L \sin q_1 = 0;$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_2} \right) - \frac{\partial T}{\partial q_2} = Q_2,$$

$$0.5 m L^2 \ddot{q}_1 \cos(q_2 - q_1) + 0.333 m L^2 \ddot{q}_2 + 0.5 m L^2 \dot{q}_2^2 \sin(q_2 - q_1)$$

$$+ 0.5 m g L \sin q_2 = 0. \tag{14}$$