1 Lagrange equations - Example 2

A double pendulum is considered in Fig. 1.1. The bars 1 and 2 are homogenuos and have the lengths $OA = AB = L$ and the masses $m_1 = m_2 = m$. At O and A there are pin joints. The mass centers of links 1 and 2 are C_1 and C_2.

Find the Lagrange equations of motion if the initial conditions are known.

Link 1 can be rotated at O in a "fixed" cartesian reference frame of unit vectors $[\mathbf{\imath}, \mathbf{\jmath}, \mathbf{k}]$ about an axis $\mathbf{k}$. To characterize the instantaneous configuration of the system, two generalized coordinates $q_1(t)$ and $q_2(t)$ are employed. The generalized coordinates are quantities associated with the position of the system. The first generalized coordinate q_1 denotes the radian measure of the angle between the link 1 and the "fixed" cartesian reference frame. The last generalized coordinate q_2 designates also a radian measure of rotation angle between link 1 and link 2.

Kinematic analysis

The position vector of the mass center of link 1 is

$$\mathbf{r}_{C1} = 0.5\,L\,\sin q_1\,\mathbf{\imath} + 0.5\,L\,\cos q_1\,\mathbf{\jmath}, \tag{1}$$

the position vector of the mass center of link 2 is

$$\mathbf{r}_{C2} = (L\,\sin q_1 + 0.5\,L\,\sin q_2)\,\mathbf{\imath} + (L\,\cos q_1 + 0.5\,L\,\cos q_2)\,\mathbf{\imath}. \tag{2}$$

The velocity of C_1 is

$$\mathbf{v}_{C1} = \frac{d\,\mathbf{r}_{C1}}{d\,t} = \dot{\mathbf{r}}_{C1} = 0.5\,L\,\dot{q}_1\,\cos q_1\,\mathbf{\imath} - 0.5\,L\,\dot{q}_1\,\sin q_1\,\mathbf{\jmath}, \tag{3}$$

and the velocity of C_2 is

$$\mathbf{v}_{C2} = \frac{d\,\mathbf{r}_{C2}}{d\,t} = \dot{\mathbf{r}}_{C2} =$$
$$(L\,\dot{q}_1\,\cos q_1 + 0.5\,L\,\dot{q}_2\,\cos q_2)\,\mathbf{\imath} - (L\,\dot{q}_1\,\sin q_1 + 0.5\,L\,\dot{q}_2\,\sin q_2)\,\mathbf{\jmath}. \tag{4}$$

Kinetic energy

The kinetic energy of the link 1 which is in rotational motion is

$$T_1 = \frac{1}{2}\,I_0\,\dot{q}_1^2 = \frac{1}{2}\,\frac{ML^2}{3}\,\dot{q}_1^2 = \frac{ML^2}{6}\,\dot{q}_1^2, \tag{5}$$

where I_0 is the mass moment of inertia about the center of rotation O, $I_O = mL^2/3$.
The kinetic energy of the bar 2 is due to the translation and rotation and can be expressed as

$$T_2 = \frac{1}{2} I_{C2} \dot{q}_2^2 + \frac{1}{2} m_2 \mathbf{v}_{C2}^2, \tag{6}$$

where I_{C2} is the mass moment of inertia about the center of mass C_2, $I_{C2} = mL^2/12$, and

$$\mathbf{v}_{C2}^2 = \mathbf{v}_{C2} \cdot \mathbf{v}_{C2} = L^2 \, \dot{q}_1^2 + \frac{1}{4} \, L^2 \, \dot{q}_2^2 + L^2 \, \dot{q}_1 \, \dot{q}_2 \cos(q_2 - q_1). \tag{7}$$

Equation (6) becomes

$$T_2 = \frac{1}{2} \, \frac{mL^2}{12} \, \dot{q}_2^2 + \frac{1}{2} mL^2 \left[\dot{q}_1^2 + \frac{1}{4} \, \dot{q}_2^2 + \dot{q}_1 \, \dot{q}_2 \cos(q_2 - q_1) \right]. \tag{8}$$

The total kinetic energy of the system is

$$T = T_1 + T_2 = \frac{mL^2}{6} \, \left[4\dot{q}_1^2 + 3 \, \dot{q}_1 \, \dot{q}_2 \, \cos(q_2 - q_1) + \dot{q}_2^2 \right]. \tag{9}$$

The left hand sides of Lagrange equations are

$$\begin{aligned}
\frac{\partial T}{\partial \dot{q}_1} &= \frac{mL^2}{6} \, \left[8\dot{q}_1 + 3\dot{q}_2 \, \cos(q_2 - q_1) \right], \\
\frac{d}{d\,t} \left(\frac{\partial T}{\partial \dot{q}_1} \right) &= \frac{mL^2}{6} \, \left[8\ddot{q}_1 + 3\ddot{q}_2 \, \cos(q_2 - q_1) - 3\dot{q}_2 \, (\dot{q}_2 - \dot{q}_1) \, \sin(q_2 - q_1) \right], \\
\frac{\partial T}{\partial q_1} &= \frac{mL^2}{6} \, 3 \, \dot{q}_1 \, \dot{q}_2 \, \sin(q_2 - q_1) = \frac{mL^2}{2} \, \dot{q}_1 \, \dot{q}_2 \, \sin(q_2 - q_1); \\
\frac{\partial T}{\partial \dot{q}_2} &= \frac{mL^2}{6} \, \left[3\dot{q}_1 \, \cos(q_2 - q_1) + 2\dot{q}_2 \right], \\
\frac{d}{d\,t} \left(\frac{\partial T}{\partial \dot{q}_2} \right) &= \frac{mL^2}{6} \, \left[3\ddot{q}_1 \, \cos(q_2 - q_1) - 3\dot{q}_1 \, (\dot{q}_2 - \dot{q}_1) \, \sin(q_2 - q_1) + 2\ddot{q}_2 \right], \\
\frac{\partial T}{\partial q_2} &= -\frac{mL^2}{6} \, 3 \, \dot{q}_1 \, \dot{q}_2 \, \sin(q_2 - q_1) = -\frac{mL^2}{2} \, \dot{q}_1 \, \dot{q}_2 \, \sin(q_2 - q_1).
\end{aligned} \tag{10}$$

External forces analysis

The gravity forces on links 1 and 2 at the mass centers C_1 and C_2

$$\mathbf{F}_{C1} = \mathbf{F}_{C2} = mg\,\mathbf{j}. \tag{11}$$

Generalized forces

There are two generalized forces. The generalized force associated to q_1 is

$$Q_1 = \mathbf{F}_{C1} \cdot \frac{\partial \mathbf{r}_{C1}}{\partial q_1} + \mathbf{F}_{C2} \cdot \frac{\partial \mathbf{r}_{C2}}{\partial q_1} =$$
$$mg\,\mathbf{j} \cdot (0.5\,L\,\cos q_1\,\mathbf{i} - 0.5\,L\,\sin q_1\,\mathbf{j}) + mg\,\mathbf{j} \cdot (L\,\cos q_1\,\mathbf{i} - L\,\sin q_1\,\mathbf{j})$$
$$= -1.5mgL\,\sin q_1. \tag{12}$$

The generalized force associated to q_2 is

$$Q_2 = \mathbf{F}_{C1} \cdot \frac{\partial \mathbf{r}_{C1}}{\partial q_2} + \mathbf{F}_{C2} \cdot \frac{\partial \mathbf{r}_{C2}}{\partial q_2} =$$
$$mg\,\mathbf{j} \cdot \mathbf{0} + mg\,\mathbf{j} \cdot (0.5\,L\,\cos q_2\,\mathbf{i} - 0.5\,L\,\sin q_2\,\mathbf{j})$$
$$= -0.5mgL\,\sin q_2. \tag{13}$$

The two Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = Q_1,$$
$$1.333mL^2\ddot{q}_1 + 0.5mL^2\ddot{q}_2\cos(q_2 - q_1) - 0.5mL^2\dot{q}_2^2\sin(q_2 - q_1)$$
$$+1.5mgL\sin q_1 = 0;$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} = Q_2,$$
$$0.5mL^2\ddot{q}_1\cos(q_2 - q_1) + 0.333mL^2\ddot{q}_2 + 0.5mL^2\dot{q}_1^2\sin(q_2 - q_1)$$
$$+0.5mgL\sin q_2 = 0. \tag{14}$$

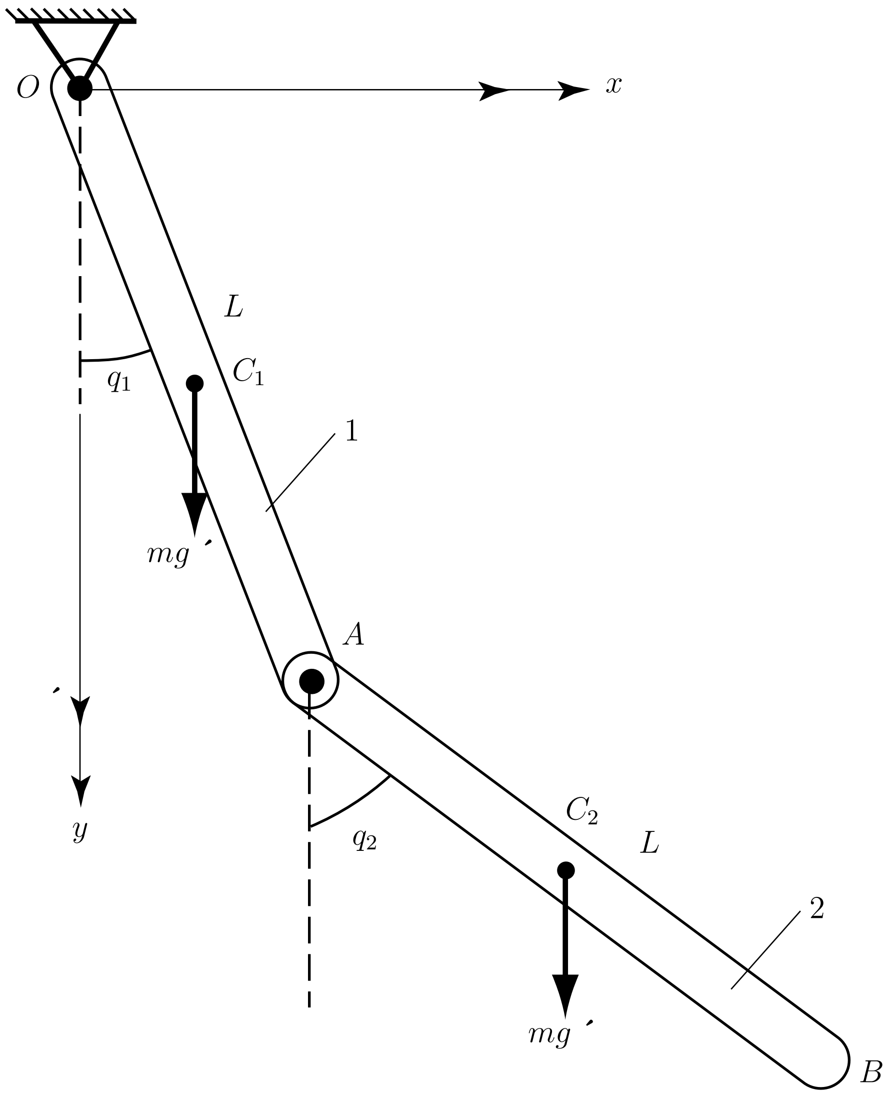

Figure 1