8 KINEMATIC CHAINS WITH CONTINUOUS FLEXIBLE LINKS

8.1 Transverse vibrations of a flexible link

Often a kinematic chain consists in part of continuous elastic components supported by rigid bodies. Small motions of the components relative to the rigid bodies generally are governed by partial differential equations. In such cases these equations cannot be solved by the method of separation of variables.

Figure 8.1 shows a cantilever beam $B$ of length $L$, constant flexural rigidity $EI$ and constant mass per unit length $\rho$. When $B$ is supported by a rigid body fixed in a frame, small flexural vibrations of $B$ are governed by the equation

$$EI \frac{\partial^4 y(x,t)}{\partial x^4} + \rho \frac{\partial^2 y(x,t)}{\partial t^2} = 0,$$

(8.1)

and by the boundary conditions

$$y(0,t) = y'(0,t) = y''(L,t) = y'''(L,t) = 0.$$  

(8.2)

The general solution of Eq. (8.1) that satisfies Eq. (8.2) can be expressed as

$$y(x,t) = \sum_{i=1}^{\infty} \Phi_i(x) q_i(t),$$

(8.3)

where $\Phi_i(x)$ and $q_i(t)$ are functions of $x$ and $t$, respectively, defined as

$$\Phi_i = \cosh \frac{\lambda_i x}{L} - \cos \frac{\lambda_i x}{L} - \cosh \lambda_i + \cos \lambda_i \left( \sinh \frac{\lambda_i x}{L} - \sin \frac{\lambda_i x}{L} \right),$$

(8.4)

and

$$q_i = \alpha_i \cos p_i t + \beta_i \sin p_i t,$$

(8.5)

where $\lambda_i, i = 1, \ldots, \infty$ are the consecutive roots of the transcendental equation

$$\cos \lambda \cosh \lambda + 1 = 0,$$

(8.6)

while

$$p_i = \left( \frac{\lambda_i}{L} \right)^2 \left( \frac{EI}{\rho} \right)^{1/2},$$

(8.7)
Figure 8.1
and \( \alpha_i \) and \( \beta_i \) are constants that depends upon initial conditions. The functions \( \Phi_i(x) \) satisfy the orthogonality relations

\[
\int_0^L \Phi_i \Phi_j \rho \, dx = m \delta_{ij} \quad (i, j = 1, \ldots, \infty),
\]

and

\[
EI \int_0^L \Phi_i'' \Phi_j'' \, dx = p_i^2 m \delta_{ij} \quad (i, j = 1, \ldots, \infty),
\]

where \( m \) is the mass of the beam and \( \delta_{ij} \) is the Kronecker delta.

### 8.2 Equations of motion for a flexible link

In Fig. 8.2, a schematic representation of a kinematic chain is given. The system is formed by a rigid body \( RB \) that supports a uniform cantilever beam \( B \) of length \( L \), flexural rigidity \( EI \), and mass per unit length \( \rho \). Only planar motions of the kinematic chain in a fixed reference frame \((0)\) of unit vectors \([\hat{ı}_0, \hat{\jmath}_0, \hat{k}_0]\) will be considered.

To characterize the instantaneous configuration of the rigid body \( RB \), generalized coordinates \( q_1, q_2, q_3 \) are employed. The first generalized coordinate \( q_1 \) denotes the distance from \( C_R \), the mass center of \( RB \), to the horizontal axis of the reference frame \((0)\). The generalized coordinate \( q_2 \) denotes the distance from \( C_R \), to the vertical axis of \((0)\). The last generalized coordinate \( q_3 \), \( (s_3 = \sin q_3, c_3 = \cos q_3) \), designates the radian measure of the rotation angle between \( RB \) and the horizontal axis.

Generalized speeds \( u_1, u_2, \) and \( u_3, \) used to characterize the motion of \( RB \) in \((0)\), are defined as

\[
u_1 = \mathbf{v}_{CR} \cdot \hat{ı}, \quad u_2 = \mathbf{v}_{CR} \cdot \hat{\jmath}, \quad u_3 = \mathbf{\omega}_{R0} \cdot \hat{k},
\]

where \( \mathbf{v}_{CR} \) is the velocity in \((0)\) of the mass center \( C_R \) of \( RB \), \( \mathbf{\omega}_{R0} \) is the angular velocity of \( RB \) in \((0)\), and \([\hat{ı}, \hat{\jmath}, \hat{k}]\) form a dextral set of mutually perpendicular unit vectors fixed in \( RB \) and directed as shown in Fig. 8.2. It follows immediately that

\[
\mathbf{v}_{CR} = u_1 \hat{ı} + u_2 \hat{\jmath}, \quad \mathbf{\omega}_{R0} = u_3 \hat{k}.
\]

The unit vectors \( \hat{ı}_0, \hat{\jmath}_0, \hat{k}_0 \) can be expressed as

\[
\hat{ı}_0 = c_3 \hat{ı} - s_3 \hat{\jmath}, \quad \hat{\jmath}_0 = s_3 \hat{ı} + c_3 \hat{\jmath}, \quad \hat{k}_0 = \hat{k}.
\]
Figure 8.2
The velocity of $C_R$ in (0) is

$$v_{C_R} = \dot{q}_1 \mathbf{i}_0 + \dot{q}_2 \mathbf{j}_0$$

$$= (\dot{q}_1 c_3 + \dot{q}_2 s_3) \mathbf{i} + (-\dot{q}_1 s_3 + \dot{q}_2 c_3) \mathbf{j}. \quad (8.13)$$

From Eqs. (8.11) and (8.13) follows that

$$u_1 = \dot{q}_1 c_3 + \dot{q}_2 s_3,$$

$$u_2 = -\dot{q}_1 s_3 + \dot{q}_2 c_3,$$

$$u_3 = \dot{q}_3. \quad (8.14)$$

Equation (8.14) can be solved uniquely for $\dot{q}_1, \dot{q}_2, \dot{q}_3$, and thus $u_1, u_2, u_3$ form a set of generalized speeds for the $RB$.

**Kinematics**

Deformations of the cantilever beam $B$ can be discussed in terms of the displacement $y(x,t)$ of a generic point $P$ on the beam $B$. The point $P$ is situated at a distance $x$ from the point $Q$, the point at which $B$ is attached to $RB$. The displacement $y$ can be expressed as

$$y(x,t) = \sum_{i=1}^{n} \Phi_i(x) q_{3+i}(t), \quad (8.15)$$

where $\Phi_i(x)$ is a totally unrestricted function of $x$, $q_{3+i}(t)$ is an equally unrestricted function of $t$, and $n$ is any positive integer. Generalized speeds $u_{3+i}, i = 1, \ldots, n$ are introduced as

$$u_{3+i} = \dot{q}_{3+i}, \quad i = 1, \ldots, n. \quad (8.16)$$

The velocity of $Q$ in (0) is

$$v_Q = v_{C_R} + \omega_{R0} \times b \mathbf{1}$$

$$= u_1 \mathbf{i} + u_2 \mathbf{j} + \begin{vmatrix} 1 & \mathbf{j} & \mathbf{k} \\ 0 & 0 & u_3 \\ b & 0 & 0 \end{vmatrix} = u_1 \mathbf{i} + (u_2 + bu_3) \mathbf{j}, \quad (8.17)$$

where $b$ is the distance from $C_R$ to $Q$.

The velocity of any point $P$ of the elastic link $B$ in (0) is

$$v_P = v_{C_R} + \frac{\partial}{\partial t} \left[ (b + x) \mathbf{i} + y \mathbf{j} \right] + \omega_{R0} \times \left[ (b + x) \mathbf{i} + y \mathbf{j} \right]$$
\[
\begin{align*}
\mathbf{v}_{C_B} &= \mathbf{v}_P(x = \frac{L}{2}, t) = \left( u_1 - u_3 \sum_{i=1}^{n} \Phi_i q_{3+i} \right) \mathbf{i} \\
&\quad + \left[ u_2 + (b + 0.5 L) u_3 + \sum_{i=1}^{n} \Phi_i u_{3+i} \right] \mathbf{j}. \tag{8.19}
\end{align*}
\]

The angular acceleration of \( RB \) in the reference frame \((0)\) is

\[
\mathbf{a}_{C_R} = \frac{\partial}{\partial t} \mathbf{v}_{C_R} + \omega_{R0} \times \mathbf{v}_{C_R} = \left( \dot{u}_1 \mathbf{i} + \dot{u}_2 \mathbf{j} + \left| \begin{array}{ccc}
1 & \mathbf{j} & \mathbf{k} \\
0 & 0 & u_3 \\
u_1 & u_2 & 0
\end{array} \right| \right) \mathbf{i} \\
&\quad + \left[ \dot{u}_2 + u_3 u_1 + (b + x) \dot{u}_3 + \sum_{i=1}^{n} \Phi_i (\dot{u}_{3+i} - u_{3+i}^2) \right] \mathbf{j}. \tag{8.21}
\]

The acceleration of point \( P \) in the reference frame \((0)\) is

\[
\mathbf{a}_P = \frac{\partial}{\partial t} \mathbf{v}_P + \omega_{R0} \times \mathbf{v}_P = \left[ \dot{u}_1 - u_2 u_3 - (b + x) u_3^2 - \sum_{i=1}^{n} \Phi_i (\dot{u}_{3+i} - u_{3+i}^2) \right] \mathbf{i} \\
&\quad + \left[ \dot{u}_2 + u_3 u_1 + (b + x) \dot{u}_3 + \sum_{i=1}^{n} \Phi_i (\dot{u}_{3+i} - u_{3+i}^2) \right] \mathbf{j}. \tag{8.21}
\]
Generalized inertia forces

If \( m_R \) and \( I_z \) are the mass of \( RB \) and the moment of inertia of \( RB \) about a line passing through \( C_R \) and parallel to \( k \), then the generalized inertia force \( F^*_r \) is given by

\[
F^*_r = \frac{\partial \omega_{R0}}{\partial u_r} \cdot (-I_z \alpha_{R0}) + \frac{\partial \mathbf{v}_{C_R}}{\partial u_r} \cdot (-m_R \mathbf{a}_{C_R}) + \int_0^L \frac{\partial \mathbf{v}_P}{\partial u_r} \cdot (-\mathbf{a}_P) \, \rho \, dx, \quad r = 1, \ldots, 3 + n.
\] (8.22)

The constants \( m_B, e_B, I_B, E_i, F_i, \) and \( G_{ij} \) are defined as

\[
m_B = \int_0^L \rho \, dx, \quad e_B = \int_0^L x \rho \, dx, \quad I_B = \int_0^L x^2 \rho \, dx,
\]

\[
E_i = \int_0^L \Phi_i \rho \, dx, \quad F_i = \int_0^L x \Phi_i \rho \, dx, \quad G_{ij} = \int_0^L \Phi_i \Phi_j \rho \, dx,
\]

\[i, j = 1, \ldots, n.\]

Equation (8.22) then leads to

\[
F_1^* = -(m_R + m_B)(\dot{u}_1 - u_2 u_3) + \dot{u}_3 \sum_{i=1}^n E_i q_{3+i}
\]

\[+ 2u_3 \sum_{i=1}^n E_i u_{3+i} + u_3^2 (b m_B + e_B),\]

\[F_2^* = -(m_R + m_B)(\dot{u}_2 + u_3 u_1) - \sum_{i=1}^n E_i \dot{u}_{3+i}\]

\[- \dot{u}_3 (b m_B + e_B) + u_3^2 \sum_{i=1}^n E_i q_{3+i},\]

\[F_3^* = (\dot{u}_1 - u_2 u_3) \sum_{i=1}^n E_i q_{3+i} - (\dot{u}_2 + u_3 u_1)(b m_B + e_B)\]

\[- \dot{u}_3 (b^2 m_B + 2 b e_B + I_B + I_z) - \sum_{i=1}^n \dot{u}_{3+i} (b E_i + F_i)\]

\[-2u_3 \sum_{i=1}^n \sum_{k=1}^n G_{ik} q_{3+i} u_{3+k} - \dot{u}_3 \sum_{i=1}^n \sum_{k=1}^n G_{ik} q_{3+i} q_{3+k},\]

\[F_{3+j}^* = - \dot{u}_2 E_j - \sum_{i=1}^n G_{ij} \dot{u}_{3+i} - \dot{u}_3 (b E_j + F_j) - u_3 u_1 E_j\]

\[+ u_3^2 \sum_{i=1}^n G_{ij} q_{3+i}, \quad j = 1, \ldots, n.\]

(8.23)
Generalized active forces

The contributions to the generalized active forces are made by the internal forces, and by the gravitational forces exerted. The internal forces are considered first. The force $d\mathbf{F}$ is the force exerted on a generic differential element of $B$

$$d\mathbf{F} = -\frac{\partial V(x,t)}{\partial x} \, dx \, \mathbf{j}, \quad (8.24)$$

where $V(x,t)$ is the shear at point $P$. If rotatory inertia is neglected, then $V(x,t)$ may be expressed in terms of the bending moment $M(x,t)$ as

$$V(x,t) = \frac{\partial M(x,t)}{\partial x}. \quad (8.25)$$

Since

$$M = EI \frac{\partial^2 y}{\partial x^2}, \quad (8.26)$$

Eqs. (8.24),(8.26) yield

$$d\mathbf{F} = -\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 y}{\partial x^2} \right) \, dx \, \mathbf{j}. \quad (8.27)$$

The system of forces exerted on the rigid body $RB$ by the elastic beam $B$ is equivalent to a couple of torque $M(0,t)\mathbf{k}$ together with a force $-V(0,t)\mathbf{j}$ applied at point $Q$. Hence, $(F_r)_I$, the contribution of the internal forces to the generalized active force $F_r$, is given by

$$(F_r)_I = \frac{\partial \omega_{R0}}{\partial u_r} \cdot M(0,t)\mathbf{k} - \frac{\partial \mathbf{v}_Q}{\partial u_r} \cdot V(0,t)\mathbf{j} - \int_0^L \frac{\partial \mathbf{v}_P}{\partial u_r} \cdot \mathbf{j} \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 y}{\partial x^2} \right) \, dx$$

$$= EI \left( \frac{\partial \omega_{R0}}{\partial u_r} \cdot \mathbf{k} \frac{\partial^2 y(0,t)}{\partial x^2} - \frac{\partial \mathbf{v}_Q}{\partial u_r} \cdot \mathbf{j} \frac{\partial^3 y(0,t)}{\partial x^3} \right) - \int_0^L \frac{\partial \mathbf{v}_P}{\partial u_r} \cdot \mathbf{j} \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 y(x,t)}{\partial x^2} \right) \, dx, \quad r = 1, \ldots, 3 + n. \quad (8.28)$$

which leads to

$$(F_1)_I = 0,$$

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\[(F_2)_I = - \sum_{i=1}^{n} q_{3+i} \left[ (EI\Phi_i'')_{x=0} + \int_{0}^{L} (EI\Phi_i'')' \, dx \right], \]
\[(F_3)_I = b (F_2)_I + \sum_{i=1}^{n} q_{3+i} \left[ (EI\Phi_i'')_{x=0} - \int_{0}^{L} x (EI\Phi_i'')' \, dx \right], \]
\[(F_{3+j})_I = - \sum_{i=1}^{n} q_{3+i} \int_{0}^{L} \Phi_j (EI\Phi_i'')' \, dx, \quad j = 1, \ldots, n. \quad (8.29)\]

The restrictions on \(\Phi_i\) to ensure that \(y\) and \(y'\) vanish at \(x = 0\) while \(M\) and \(V\) vanish at \(x = L\) are
\[\Phi_i(0) = \Phi_i'(0) = \Phi_i''(L) = \Phi_i'''(L) = 0, \quad i = 1, \ldots, n. \quad (8.30)\]

When the integrations are carried out, the following expressions result for the contribution of the internal forces to the generalized active forces
\[(F_1)_I = (F_2)_I = (F_3)_I = 0, \]
\[(F_{3+j})_I = - \sum_{i=1}^{n} H_{ij} q_{3+i}, \quad j = 1, \ldots, n, \quad (8.31)\]

where \(H_{ij}\) is defined as
\[H_{ij} = \int_{0}^{L} EI\Phi_i'' \Phi_j'' \, dx, \quad i, j = 1, \ldots, n. \quad (8.32)\]

The gravitational forces exerted on \(RB\) and \(B\) by the Earth, are denoted by \(G_R\), and \(G_B\), and can be expressed as
\[G_R = -m_R g J_0 = -m_R g (s_3 \mathbf{i} + c_3 \mathbf{j}), \]
\[G_B = -m_B g J_0 = -m_B g (s_3 \mathbf{i} + c_3 \mathbf{j}). \quad (8.33)\]

The contribution to \(F_r\) of the gravitational forces is
\[(F_r)_G = \frac{\partial \mathbf{v}_{CR}}{\partial u_r} \cdot \mathbf{G}_R + \frac{\partial \mathbf{v}_{CB}}{\partial u_r} \cdot \mathbf{G}_B, \quad r = 1, \ldots, 3 + n. \quad (8.34)\]

The generalized active forces are
\[F_r = (F_r)_I + (F_r)_G, \quad r = 1, \ldots, 3 + n. \quad (8.35)\]

To arrive at the dynamical equations governing the system, all that remains to be done is to substitute into Kane’s dynamical equations, namely,
\[F_r^* + F_r = 0, \quad r = 1, \ldots, 3 + n. \quad (8.36)\]
FLEXIBLE BEAM CANTILEVERED IN A RIGID BODY FALLING UNDER GRAVITY

Apply [Clear, Names "Global" ];

(0) - fixed reference frame attached to the ground with the unit vectors \( \{i_0, j_0, k_0\} \)
(1) - mobile reference frame attached to the rigid body base RB with the unit vectors \( \{i, j, k\} \)

\[
\text{Cross}[xx_-, yy_] := (xx[[2]] yy[[3]] - xx[[3]] yy[[2]],
xx[[3]] yy[[1]] - xx[[1]] yy[[3]],
xx[[1]] yy[[2]] - xx[[2]] yy[[1]]);
\]

Initial data

\[
L = 1.;
b = 0.1;
c = 2 b;
d = 0.3;
mR = 1.;
lam1 = 1.875;
ro = 0.21991;
mB = L ro;
EI = 35.3;
Iz = \frac{1}{12} mR (c^2 + d^2);
g = 9.81;
\]

Transformation matrix from (0) to (1)

\[
R_{01} = \begin{bmatrix}
\cos q3[t] & -\sin q3[t] & 0 \\
\sin q3[t] & \cos q3[t] & 0 \\
0 & 0 & 1
\end{bmatrix};
\]

Kinematic eqs. in generalized speeds

\[
\text{rule} = \{q1'[t] -> u1[t] \cos q3[t] - u2[t] \sin q3[t],
q2'[t] -> u1[t] \sin q3[t] + u2[t] \cos q3[t],
q3'[t] -> u3[t],
q4'[t] -> u4[t];
\]

Angular velocity of RB in (0)

\[
\omega_{R0} = \{0, 0, u3[t]\};
\]

Linear velocity of mass center CR of RB in (0) expressed in terms of (1) \( \{i, j, k\} \)

\[
v_{CR} = \{u1[t], u2[t], 0\};
\]

Linear velocity vQ of point Q on RB in (0) expressed in terms of (1) \( \{i, j, k\} \)

\[
rQ = \{b, 0, 0\};
vQ = v_{CR} + \text{Cross}[\omega_{R0}, rQ];
F11 = (\cosh[lam1 x/L] \cdot \cos[lam1 x/L] \\
(\cosh[lam1] + \cos[lam1]) / (\sinh[lam1] + \sin[lam1]) \\
(\sinh[lam1 x/L] \cdot \sin[lam1 x/L]));
\]

Elastic displacement of an arbitrary point P on the
elastic link B in (l) expressed in terms of (l) \{(i,j,k)\}
yP1 = q4[t] Fi1;

(' linear velocity of an arbitrary point P on the elastic link in (0) expressed in terms of (l) \{(i,j,k)\}'

rP = (b + x, yP1, 0);
FiCB = Fi1 /. x -> L/2.;

(' elastic displacement of the midpoint CB on the elastic link B in (l) expressed in terms of (l) \{(i,j,k)\}'
yCB = q4[t] FiCB;

(' linear velocity of a midpoint CB on the elastic link B in (0) expressed in terms of (l) \{(i,j,k)\}'
rCB = (b + L/2, yCB, 0);
vCB = vCR + D[rCB, t] + Cross[wR, rCB] /. rule;

(' angular acceleration of RB in (0) expressed in terms of (l) \{(i,j,k)\}'
alphaR = D[wR, t];

(' linear acceleration of CR of RB in (0) expressed in terms of (l) \{(i,j,k)\}'

(' linear acceleration of an arbitrary point P on link B in (0) expressed in terms of (l) \{(i,j,k)\}'

(' generalized inertia forces ')
inte1 = ExpandAll[D[vP, u1[t]] . (aP) ro];
inte2 = ExpandAll[D[vP, u2[t]] . (aP) ro];
inte3 = ExpandAll[D[vP, u3[t]] . (aP) ro];
inte4 = ExpandAll[D[vP, u4[t]] . (aP) ro];

Integrala1 = Chop[Integrate[inte1], \{x, 0, L\}]];
Integrala2 = Chop[Integrate[inte2], \{x, 0, L\}]];
Integrala3 = Chop[Integrate[inte3], \{x, 0, L\}]];
Integrala4 = Chop[Integrate[inte4], \{x, 0, L\}]];

Fin1 = D[wR, u1[t]]. (Iz alphaR) +
D[vCR, u1[t]]. (mR aCR) + Integrala1;
Fin2 = D[wR, u2[t]]. (Iz alphaR) +
D[vCR, u2[t]]. (mR aCR) + Integrala2;
Fin3 = D[wR, u3[t]]. (Iz alphaR) +
D[vCR, u3[t]]. (mR aCR) + Integrala3;
Fin4 = D[wR, u4[t]]. (Iz alphaR) +
D[vCR, u4[t]]. (mR aCR) + Integrala4;

(' generalized active forces ')

(' gravitational forces ')}
\[ \text{GR} = \{0, \cdot \text{mR g}, 0\} \cdot \text{R01}; \]
\[ \text{GB} = \{0, \cdot \text{mB g}, 0\} \cdot \text{R01}; \]
\[ \text{F1G} = D[\text{vCR}, u1[t]] \cdot \text{GR} + D[\text{vCB}, u1[t]] \cdot \text{GB}; \]
\[ \text{F2G} = D[\text{vCR}, u2[t]] \cdot \text{GR} + D[\text{vCB}, u2[t]] \cdot \text{GB}; \]
\[ \text{F3G} = D[\text{vCR}, u3[t]] \cdot \text{GR} + D[\text{vCB}, u3[t]] \cdot \text{GB}; \]
\[ \text{F4G} = D[\text{vCR}, u4[t]] \cdot \text{GR} + D[\text{vCB}, u4[t]] \cdot \text{GB}; \]

\[ \text{M} = \text{EI D}[\text{yP1, x}, x]; \]
\[ \text{M0} = \{0, 0\}, \text{M}\cdot x\rightarrow 0; \]
\[ \text{V} = D[\text{M}, x]; \]
\[ \text{V0} = \{0, V\cdot x\rightarrow 0, 0\}; \]
\[ \text{F} = \{0, D[\text{M}, x], 0\}; \]
\[ \text{integ1} = \text{ExpandAll}[D[\text{vP}, u1[t]] \cdot \text{F}]; \]
\[ \text{integ2} = \text{ExpandAll}[D[\text{vP}, u2[t]] \cdot \text{F}]; \]
\[ \text{integ3} = \text{ExpandAll}[D[\text{vP}, u3[t]] \cdot \text{F}]; \]
\[ \text{integ4} = \text{ExpandAll}[D[\text{vP}, u4[t]] \cdot \text{F}]; \]

\[ \text{Integral1} = \text{Chop}\left[\text{Integrate}[\text{integ1}, x, 0, L]\right]; \]
\[ \text{Integral2} = \text{Chop}\left[\text{Integrate}[\text{integ2}, x, 0, L]\right]; \]
\[ \text{Integral3} = \text{Chop}\left[\text{Integrate}[\text{integ3}, x, 0, L]\right]; \]
\[ \text{Integral4} = \text{Chop}\left[\text{Integrate}[\text{integ4}, x, 0, L]\right]; \]

\[ \text{F1I} = D[\text{wR, u1[t]}] \cdot \text{M0} - D[\text{vQ, u1[t]}] \cdot \text{V0} - \text{Integral1}; \]
\[ \text{F2I} = D[\text{wR, u2[t]}] \cdot \text{M0} - D[\text{vQ, u2[t]}] \cdot \text{V0} - \text{Integral2}; \]
\[ \text{F3I} = D[\text{wR, u3[t]}] \cdot \text{M0} - D[\text{vQ, u3[t]}] \cdot \text{V0} - \text{Integral3}; \]
\[ \text{F4I} = D[\text{wR, u4[t]}] \cdot \text{M0} - D[\text{vQ, u4[t]}] \cdot \text{V0} - \text{Integral4}; \]

\[ \text{e1} = \text{Fin1} + \text{F1I} + \text{F1G}; \]
\[ \text{e2} = \text{Fin2} + \text{F2I} + \text{F2G}; \]
\[ \text{e3} = \text{Fin3} + \text{F3I} + \text{F3G}; \]
\[ \text{e4} = \text{Fin4} + \text{F4I} + \text{F4G}; \]
\[ \text{e5} = \cdot \text{u3[t]} + q3'[t]; \]
\[ \text{e6} = \cdot \text{u4[t]} + q4'[t]; \]

(* numerical simulation *)

\[ \text{kane} = \text{NDSolve}\left[ \begin{align*} 
\text{e1} &= 0, \\
\text{e2} &= 0, \\
\text{e3} &= 0, \\
\text{e4} &= 0, \\
\text{e5} &= 0, \\
\text{e6} &= 0, \\
\text{u1[t]} &= 0.0, \\
\text{u2[t]} &= 0.0, \\
\text{u3[t]} &= 0.0, \\
\text{u4[t]} &= 0.0, \\
\text{q3[t]} &= 1.0, \\
\text{q4[t]} &= 0.0001, \\
\{\text{u1[t]}, \text{u2[t]}, \text{u3[t]}, \text{u4[t]}, \text{q3[t]}, \text{q4[t]}\}, \{t, 0.0, 1.0\}\right]; \]

\[ \text{Plot}\left[\text{Evaluate}[\text{u1[t]} /. \text{kane}], \{t, 0.0, 1.0\}, \right. \]
\[ \text{AxesLabel} \rightarrow \{"t[s]", "u1[m/s]"\}; \]
\[ \text{Plot}\left[\text{Evaluate}[\text{u2[t]} /. \text{kane}], \{t, 0.0, 1.0\}, \right. \]
\[ \text{AxesLabel} \rightarrow \{"t[s]", "u2[m/s]"\}; \]
\[ \text{Plot}\left[\text{Evaluate}[\text{u3[t]} /. \text{kane}], \{t, 0.0, 1.0\}, \right. \]
\[ \text{AxesLabel} \rightarrow \{"t[s]", "u3[rad/s]"\}; \]
\[ \text{Plot}\left[\text{Evaluate}[\text{u4[t]} /. \text{kane}], \{t, 0.0, 1.0\}, \right. \]
\[ \text{AxesLabel} \rightarrow \{"t[s]", "u4[m/s]"\}; \]
\[ \text{Plot}\left[\text{Evaluate}[\text{q3[t]} /. \text{kane}], \{t, 0.0, 1.0\}, \right. \]
\[ \text{AxesLabel} \rightarrow \{"t[s]", "u4[m/s]"\}; \]

(* numerical simulation *)
PlotRange -> {All, All},
AxesLabel -> {"t[s]", "q3[m]"};
Plot[Evaluate[q4[t] /. kane], {t, 0.0, 1.0},
PlotRange -> {All, All},
AxesLabel -> {"t[s]", "q4[m]"};