1 OPEN KINEMATIC CHAINS

In this chapter Kane’s approach is used to formulate the equations of motion for open kinematic chains. A detailed dynamic analysis of a three DOF open kinematic chain is presented [Kane and Levinson].

1.1 Kinematics of open kinematic chains

Figure 1.1(a) is a schematic representation of an open kinematic chain (robot arm) consisting of three elements 1, 2, and 3. The last link 3 holds rigidly a rigid body $RB$. Body 1 can be rotated at $A$ in a “fixed” reference frame $(0)$ of unit vectors $[\vec{i}_0, \vec{j}_0, \vec{k}_0]$ about a vertical axis $\vec{i}_0$. The unit vector $\vec{i}_0$ is fixed in 1. At the pin joint $B$ the link 1 is connected to link 2. The element 2 rotates relative to 1 about a horizontal axis fixed in both 1 and 2, passing through $B$, and perpendicular to the axis of 1. The last link 3 is connected to 2 by means of a slider joint. The mass centers of links 1, 2, and 3 are $C_1$, $C_2$, and $C_3$, respectively. The distances $L_1 = AC_1$, $L_2 = BC_2$, and $L_B = AB$ are indicated in Fig. 1.1(a). The reference frame (1) of unit vectors $[\vec{i}_1, \vec{j}_1, \vec{k}_1]$ is attached to link 1, and the reference frame (2) of unit vectors $[\vec{i}_2, \vec{j}_2, \vec{k}_2]$ is attached to link 2, as shown in Fig. 1.1.

Let

$$p_x = r_{C_3C_R} \cdot \vec{i}_2, \quad p_y = r_{C_3C_R} \cdot \vec{j}_2, \quad p_z = r_{C_3C_R} \cdot \vec{k}_2,$$

where $r_{C_3C_R}$ is the position vector from $C_3$ to $C_R$, where $C_R$ is the mass center of $RB$.

To characterize the instantaneous configuration of the arm, generalized coordinates $q_1(t), q_2(t), q_3(t)$ are employed. The generalized coordinates are quantities associated with the position of the system. The first generalized coordinate $q_1$ denotes the radian measure of the angle between the axes of 1 and 2 ($s_1 = \sin q_1, c_1 = \cos q_1$), and $q_2$ is the distance from $C_2$ to $C_3$. The last generalized coordinate $q_3$ ($s_3 = \sin q_3, c_3 = \cos q_3$), designates also a radian measure of rotation angle between 1 and 0.

As important as generalized coordinates are generalized speeds, these being quantities associated with the motion of a system. The generalized speeds $u_1(t), ..., u_n(t)$, where $n$ is the number of generalized coordinates can be introduced as

$$u_r = \sum_{s=1}^{n} A_{rs} \dot{q}_s + B_r, \quad r = 1, ..., n,$$

(1.1)
Figure 1

Schematic representation of the robotic arm in 3D

(a)

(b)
where $A_{rs}$ and $B_{r}$ are functions of $q_{1}, ..., q_{n}$, and the time $t$; $A_{rs}$ and $B_{r}$ $(r, s = 1, ..., n)$ are chosen such that Eq.(1.1) can be solved uniquely for $q_{1}, ..., q_{n}$. The generalized speeds $u_{1}, ..., u_{n}$ serve as variables on an equal footing with the generalized coordinates $q_{1}, ..., q_{n}$. Their introduction can enable one to take advantage of special features of a given physical system to bring equations of motion into a particularly simple form. Generally, this is accomplished by taking $u_{r}$ to be an angular velocity measure number, a velocity measure number, or simply $\dot{q}_{r}$. Considering, for example, the robotic arm, one can introduce $u_{1}, u_{2}, u_{3}$ as

$$u_{1} = \omega_{10} \cdot \mathbf{i}_{1}, \quad u_{2} = \omega_{21} \cdot \mathbf{j}_{2}, \quad u_{3} = (2) \mathbf{v}_{C_{3}} \cdot \mathbf{k}_{2},$$

(1.2)

where $\omega_{10}$ is the angular velocity of 1 in the fixed reference frame (0), $\omega_{21}$ is the angular velocity of 2 with respect to reference frame (1), and $(2)\mathbf{v}_{C_{3}}$ is the velocity of $C_{3}$ in reference frame (2).

In the case of the three DOF robot arm, $\dot{q}_{3} = u_{1}$, $\dot{q}_{1} = u_{2}$, and $\dot{q}_{2} = u_{3}$ or

$$u_{1} = \dot{q}_{3}, \quad u_{2} = \dot{q}_{1}, \quad u_{3} = \dot{q}_{2}. \quad (1.3)$$

Equation (1.3) can be solved uniquely for $\dot{q}_{1}, \dot{q}_{2}, \dot{q}_{3}$.

### 1.1.1 Angular velocities

Next the angular velocity of 1, 2, and 3 will be expressed in the fixed reference frame (0). One can express the angular velocity of 1 in (0) as

$$\omega_{10} = \dot{q}_{3} \mathbf{i}_{1} = u_{1} \mathbf{i}_{1}. \quad (1.4)$$

The angular velocity of link 2 with respect to (1) is

$$\omega_{21} = \dot{q}_{1} \mathbf{j}_{2}, \quad (1.5)$$

and the angular velocity of link 2 with respect to the fixed reference frame (0) is

$$\omega_{20} = \omega_{10} + \omega_{21} = \dot{q}_{3} \mathbf{i}_{1} + \dot{q}_{1} \mathbf{j}_{2}. \quad (1.6)$$

The unit vector $\mathbf{i}_{0}$ can be expressed as Fig. 1.1(b)

$$\mathbf{i}_{0} = \mathbf{i}_{1} = c_{1} \mathbf{i}_{2} + s_{1} \mathbf{k}_{2}. \quad (1.7)$$
The angular velocity of link 2 in (0) written in terms of the reference frame (2) is

$$\mathbf{\omega}_{20} = u_1 (c_1 \mathbf{i}_2 + s_1 \mathbf{k}_2) + u_2 \mathbf{j}_2 = u_1 c_1 \mathbf{i}_2 + u_2 \mathbf{j}_2 + u_1 s_1 \mathbf{k}_2,$$

(1.8)
or

$$\mathbf{\omega}_{20} = Z_1 \mathbf{i}_2 + u_2 \mathbf{j}_2 + Z_2 \mathbf{k}_2,$$

(1.9)

where $Z_1 = u_1 c_1$ and $Z_2 = u_1 s_1$. The quantities $Z_i$ are introduced to minimize the writing. The link 3 and the rigid body $RB$ have the same rotational motion as link 2, i.e. $\mathbf{\omega}_{30} = \mathbf{\omega}_{R0} = \mathbf{\omega}_{20}$.

### 1.1.2 Angular accelerations

The angular acceleration of 1 in (0) can be expressed as

$$\mathbf{\alpha}_{10} = \ddot{q}_3 \mathbf{i}_1 = \dot{u}_1 \mathbf{i}_1.$$  

(1.10)

The angular velocity of link 2 with respect to (0) is

$$\mathbf{\alpha}_{20} = \frac{d}{dt} \mathbf{\omega}_{20} = \frac{(2)}{dt} \mathbf{\omega}_{20} = (\dot{u}_1 c_1 - u_1 \dot{q}_1 s_1) \mathbf{i}_2 + \dot{u}_2 \mathbf{j}_2 + (\dot{u}_1 s_1 + u_1 \dot{q}_1 c_1) \mathbf{k}_2$$

$$= (\dot{u}_1 c_1 - u_1 u_2 s_1) \mathbf{i}_2 + \dot{u}_2 \mathbf{j}_2 + (\dot{u}_1 s_1 + u_1 u_2 c_2) \mathbf{k}_2$$

$$= (\dot{u}_1 c_1 + Z_3) \mathbf{i}_2 + \dot{u}_2 \mathbf{j}_2 + (\dot{u}_1 s_1 + Z_4) \mathbf{k}_2,$$

(1.11)

where $\frac{(2)}{dt}$ represents the partial derivative with respect to time in reference frame (2), $[\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2]$, $Z_3 = -Z_1 u_2$, and $Z_4 = Z_1 u_2$. The link 3 and the rigid body $RB$ have the same angular acceleration as link 2, i.e. $\mathbf{\alpha}_{30} = \mathbf{\alpha}_{20}$.

### 1.1.3 Linear velocities

The position vector of $C_1$, the mass center of link 1, is

$$\mathbf{r}_{C_1} = L_1 \mathbf{k}_1,$$

(1.12)

and the velocity of $C_1$ in (0) is

$$\mathbf{v}_{C_1} = \frac{d}{dt} \mathbf{r}_{C_1} = \frac{(1)}{dt} \mathbf{r}_1 + \mathbf{\omega}_{10} \times \mathbf{r}_{C_1}$$

$$= 0 + \begin{vmatrix} \mathbf{i}_1 & \mathbf{J}_1 & \mathbf{k}_1 \end{vmatrix} = u_1 L_1 \mathbf{j}_1 = Z_5 \mathbf{j}_1,$$

(1.13)
where $Z_5 = -u_1 L_1$.
The position vector of $C_2$, the mass center of link 2, is

$$
\mathbf{r}_{C_2} = L_B \mathbf{k}_1 + L_2 \mathbf{k}_2 = L_B (-s_1 \mathbf{i}_2 + c_1 \mathbf{k}_2) + L_2 \mathbf{k}_2
$$

$$
= -L_B s_1 \mathbf{i}_2 + (L_B c_1 + L_2) \mathbf{k}_2.
$$

(1.14)

The velocity of $C_2$ in (0) is

$$
\mathbf{v}_{C_2} = \frac{d}{dt} \mathbf{r}_{C_2} = \frac{(2)\partial}{\partial t} \mathbf{r}_{C_2} + \mathbf{\omega}_{20} \times \mathbf{r}_{C_2}
$$

$$
= -L_B c_1 u_2 \mathbf{i}_2 - L_B c_1 u_2 \mathbf{k}_2 + \begin{vmatrix}
\mathbf{i}_2 & \mathbf{j}_2 & \mathbf{k}_2 \\
u_1 c_1 & u_2 & u_1 s_1 \\
-L_B s_1 & 0 & L_B c_1 + L_2
\end{vmatrix}
$$

$$
= L_2 u_2 \mathbf{i}_2 - (L_B + L_2 c_1) u_1 \mathbf{j}_2 = L_2 u_2 \mathbf{i}_2 + Z_6 u_1 \mathbf{j}_2
$$

$$
= Z_7 \mathbf{i}_2 + Z_8 \mathbf{j}_2,
$$

(1.15)

where $Z_6 = -(L_B + L_2 c_1)$, $Z_7 = L_2 u_2$, and $Z_8 = Z_6 u_1$.
The position vector of $C_3$ with respect to reference frame (0) is

$$
\mathbf{r}_{C_3} = \mathbf{r}_{C_2} + q_2 \mathbf{k}_2
$$

$$
= -L_B s_1 \mathbf{i}_2 + (L_B c_1 + L_2 + q_2) \mathbf{k}_2,
$$

(1.16)

and the velocity of this mass center in (0) is

$$
\mathbf{v}_{C_3} = \frac{d}{dt} \mathbf{r}_{C_3} = \frac{(2)\partial}{\partial t} \mathbf{r}_{C_3} + \mathbf{\omega}_{20} \times \mathbf{r}_{C_3}
$$

$$
= -L_B c_1 u_2 \mathbf{i}_2 - (L_B c_1 u_2 + u_3) \mathbf{k}_2 + \begin{vmatrix}
\mathbf{i}_2 & \mathbf{j}_2 & \mathbf{k}_2 \\
u_1 c_1 & u_2 & u_1 s_1 \\
-L_B s_1 & 0 & L_B c_1 + L_2 + q_2
\end{vmatrix}
$$

$$
= (L_2 + q_2) u_2 \mathbf{i}_2 - (L_B + L_2 c_1 + c_1 q_2) u_1 \mathbf{j}_2 + u_3 \mathbf{k}_2
$$

$$
= u_2 Z_9 \mathbf{i}_2 + u_1 Z_{10} \mathbf{j}_2 + u_3 \mathbf{k}_2
$$

$$
= Z_{11} \mathbf{i}_2 + Z_{12} \mathbf{j}_2 + u_3 \mathbf{k}_2,
$$

(1.17)

where $Z_9 = L_2 + q_2$, $Z_{10} = Z_6 + q_2 c_1$, $Z_{11} = u_2 Z_9$, and $Z_{12} = Z_{10} u_1$. The position vector of the mass center $C_R$ of the rigid body $RB$ is

$$
\mathbf{r}_{C_R} = \mathbf{r}_{C_3} + \mathbf{r}_{C_3 C_R}
$$

$$
= (p_x - L_B s_1) \mathbf{i}_2 + p_y \mathbf{j}_2 + (L_B c_1 + L_2 + q_2 + p_z) \mathbf{k}_2.
$$

(1.18)
The linear acceleration of the mass center where

\[ \vec{a}_{C_1} = \frac{d}{dt} \vec{v}_{C_1} = (1) \frac{\partial}{\partial t} \vec{v}_{C_1} + \omega_{20} \times \vec{v}_{C_1} \]

\[ = -L_B c_1 u_2 \vec{r}_2 - (L_B c_1 u_2 + u_3) \vec{k}_2 + \begin{vmatrix} \vec{r}_2 & \vec{J}_2 & \vec{k}_2 \\ u_1 c_1 & u_2 & u_1 s_1 \\ p_x - L_B s_1 & p_y & L_B c_1 + L_2 + q_2 + p_z \end{vmatrix} \]

\[ = (u_1 Z_{13} + u_2 Z_{14}) \vec{r}_2 + u_1 Z_{15} \vec{J}_2 + (Z_1 6 u_1 - u_2 p_x + u_3) \vec{k}_2 \]

\[ = Z_{17} \vec{r}_2 + Z_{18} \vec{J}_2 + Z_{19} \vec{k}_2, \]  

(1.19)

where \( Z_{13} = -s_1 p_y, Z_{14} = Z_9 + p_z, Z_{15} = Z_1 0 + s_1 p_x - c_1 p_z, Z_{16} = c_1 p_y, Z_{17} = u_1 Z_{13} + u_2 Z_{14}, Z_{18} = u_1 Z_{15}, \) and \( Z_{19} = Z_{16} u_1 - u_2 p_x + u_3. \)

### 1.1.4 Linear accelerations

The acceleration of \( C_1 \) is

\[ \vec{a}_{C_2} = \frac{d}{dt} \vec{v}_{C_2} = \frac{(2)}{\partial t} \vec{v}_{C_2} + \omega_{20} \times \vec{v}_{C_2} \]

\[ = (\dot{u}_2 L_2 - Z_2 Z_8) \vec{r}_2 + (Z_6 \dot{u}_1 + L_2 s_1 u_2 u_1 + Z_2 Z_7) \vec{J}_2 + (Z_1 Z_8 - u_2 Z_7) \vec{k}_2 \]

\[ = (\dot{u}_2 L_2 + Z_{21}) \vec{r}_2 + (Z_6 \dot{u}_1 + Z_{23}) \vec{J}_2 + Z_{24} \vec{k}_2, \]  

(1.21)

where \( Z_{21} = L_2 s_1 u_2, Z_{22} = -Z_2 Z_8, Z_{23} = Z_{21} u_1 + Z_2 Z_7, \) and \( Z_{24} = Z_1 Z_8 - u_2 Z_7. \)

The linear acceleration of the mass center \( C_2 \) is

The acceleration of \( C_3 \) is

\[ \vec{a}_{C_3} = \frac{d}{dt} \vec{v}_{C_3} = \frac{(2)}{\partial t} \vec{v}_{C_3} + \omega_{20} \times \vec{v}_{C_3} \]

\[ = (\dot{u}_2 Z_9 + Z_{26}) \vec{r}_2 + (\dot{u}_1 Z_{10} + Z_{27}) \vec{J}_2 + (\dot{u}_3 + Z_{28}) \vec{k}_2, \]  

(1.22)
where \( Z_{25} = Z_{21} - u_3 c_1 + q_2 s_1 u_2 \), \( Z_{26} = 2u_2 u_3 - Z_2 Z_{12} \), \( Z_{27} = Z_{25} u_1 + Z_2 Z_{11} - Z_1 u_3 \), and \( Z_{28} = Z_1 Z_{12} - u_2 Z_{11} \).

The acceleration of \( C_R \) is

\[
\mathbf{a}_{C_R} = \frac{d}{dt} \mathbf{v}_{C_R} = (\mathbf{2}) \frac{\partial}{\partial t} \mathbf{v}_{C_R} + \omega \times \mathbf{v}_{C_R} = (\dot{u}_1 Z_{13} + \dot{u}_2 Z_{14} + Z_{32}) \mathbf{i}_2 \\
+ (\dot{u}_1 Z_{15} + Z_{33}) \mathbf{j}_2 + (\dot{u}_1 Z_{16} - p_x \dot{u}_2 + \dot{u}_3 + Z_{34}) \mathbf{k}_2,
\]

(1.23)

where \( Z_{29} = -Z_{16} u_2 \), \( Z_{30} = Z_{25} + u_2 (c_1 p_x + s_1 p_y) \), \( Z_{31} = Z_{13} u_2 \), \( Z_{32} = Z_{29} u_1 + u_2 (u_3 + Z_{19}) - Z_2 Z_{18} \), \( Z_{33} = Z_{30} u_1 + Z_2 Z_{17} - Z_1 Z_{19} \), and \( Z_{34} = Z_{31} u_1 + Z_1 Z_{18} - u_2 Z_{17} \).

### 1.2 Generalized inertia forces

To explain what the generalized inertia forces are, a system \( \{S\} \) formed by \( \nu \) particles \( P_1, ..., P_\nu \) and having masses \( m_1, ..., m_\nu \) is considered. Suppose that \( n \) generalized speeds \( u_r \) have been introduced. Let \( \mathbf{v}_{P_j} \) and \( \mathbf{a}_{P_j} \) denote, respectively, the velocity of \( P_j \) and the acceleration of \( P_j \) in a reference frame \( (0) \).

Define \( F_{in,j} \), called the inertia force for \( P_j \), as

\[
F_{in,j} = -m_j \mathbf{a}_{P_j},
\]

(1.24)

The quantities \( F_{1}^*, ..., F_{n}^* \), defined as

\[
F_{r}^* = \sum_{j=1}^{\nu} \frac{\partial \mathbf{v}_{P_j}}{\partial u_r} \cdot \mathbf{a}_{P_j}, \quad r = 1, ..., n,
\]

(1.25)

are called generalized inertia forces for \( \{S\} \).

The contribution to \( F_{r}^* \), made by the particles of a rigid body \( RB \) belonging to \( \{S\} \), are

\[
(F_r^*)_R = \frac{\partial \mathbf{v}_C}{\partial u_r} \cdot \mathbf{F}_{in} + \frac{\partial \omega}{\partial u_r} \cdot \mathbf{T}_{in}, \quad r = 1, ..., n,
\]

(1.26)

where \( \mathbf{v}_C \) is the velocity of the center of gravity of \( RB \) in \( (0) \), and \( \omega = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k} \) is the angular velocity of \( RB \) in \( (0) \).

The inertia force for the rigid body \( RB \) is

\[
\mathbf{F}_{in} = -m \mathbf{a}_C,
\]

(1.27)
where \( m \) is the mass of \( RB \), and \( \mathbf{a}_C \) is the acceleration of the mass center of \( RB \) in the fixed reference frame. The inertia torque \( \mathbf{T}_{\text{in}} \) for \( RB \) is

\[
\mathbf{T}_{\text{in}} = -\mathbf{\alpha} \cdot \mathbf{I} - \mathbf{\omega} \times \mathbf{I} \cdot \mathbf{\omega},
\]

where \( \mathbf{\alpha} = \dot{\mathbf{\omega}} = \alpha_x \mathbf{i} + \alpha_y \mathbf{j} + \alpha_z \mathbf{k} \) is the angular acceleration of \( RB \) in (0), and \( \mathbf{I} = (I_x \mathbf{i} + (I_y \mathbf{j}) \mathbf{j} + (I_z \mathbf{k}) \mathbf{k} \) is the central inertia dyadic of \( RB \). The central principal axes of \( RB \) are parallel to \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) and the associated moments of inertia have the values \( I_x, I_y, I_z \), respectively. The inertia matrix associated to \( \mathbf{I} \) is

\[
\mathbf{I} \rightarrow \begin{bmatrix}
I_x & 0 & 0 \\
0 & I_y & 0 \\
0 & 0 & I_z
\end{bmatrix}.
\]

The dot product of the vector \( \mathbf{\alpha} \) with the dyadic \( \mathbf{I} \) is

\[
\mathbf{\alpha} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{\alpha} = \alpha_x I_x \mathbf{i} + \alpha_y I_y \mathbf{j} + \alpha_z I_z \mathbf{k},
\]

and the cross product between a vector and a dyadic is

\[
\mathbf{\omega} \times (\mathbf{I} \cdot \mathbf{\omega}) = \begin{vmatrix}
1 & \mathbf{J} & \mathbf{k} \\
\mathbf{J} & \mathbf{\omega}_x & \mathbf{\omega}_y \\
\mathbf{\omega}_y & \mathbf{\omega}_x & \mathbf{\omega}_z
\end{vmatrix} = -(\mathbf{\omega}_y \mathbf{\omega}_z (I_y - I_z) \mathbf{j} - \mathbf{\omega}_z \mathbf{\omega}_x (I_z - I_x) \mathbf{k} - \mathbf{\omega}_x \mathbf{\omega}_y (I_x - I_y) \mathbf{k}).
\]

Referring to the three DOF robot arm, let \( m_1, m_2, m_3, m_R \) be the masses of 1, 2, 3, \( RB \), respectively. The links 1, 2, 3, and the rigid body \( RB \) have the following mass distribution properties. The central principal axes of 1 are parallel to \( \mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1 \), Fig. 1.1(a), and the associated moments of inertia have the values \( A_x, A_y, A_z \), respectively. The central inertia dyadic of 1 is

\[
\mathbf{I}_1 = (A_x \mathbf{i}_1) \mathbf{i}_1 + (A_y \mathbf{j}_1) \mathbf{j}_1 + (A_z \mathbf{k}_1) \mathbf{k}_1.
\]

The central principal axes of 2 and 3 are parallel to \( \mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2 \) and the associated moments of inertia have values \( B_x, B_y, B_z, \) and \( C_x, C_y, C_z \) respectively. The central inertia dyadic of 2 is

\[
\mathbf{I}_2 = (B_x \mathbf{i}_2) \mathbf{i}_2 + (B_y \mathbf{j}_2) \mathbf{j}_2 + (B_z \mathbf{k}_2) \mathbf{k}_2,
\]

and the central inertia dyadic of 3 is

\[
\mathbf{I}_3 = (C_x \mathbf{i}_2) \mathbf{i}_2 + (C_y \mathbf{j}_2) \mathbf{j}_2 + (C_z \mathbf{k}_2) \mathbf{k}_2.
\]
The central inertia dyadic of the rigid body $RB$ is

$$
\tilde{I}_R = (D_{11}J_2 + D_{12}J_2 + D_{13}k_2)J_2 + (D_{21}J_2 + D_{22}J_2 + D_{23}k_2)J_2 + (D_{31}J_2 + D_{32}J_2 + D_{33}k_2)k_2.
$$

(1.35)

To facilitate the writing of results, the following notation is introduced

$$k_1 = B_y - B_z, \; k_2 = B_z - B_x, \; k_3 = B_x - B_y, \; k_4 = C_y - C_z, \; k_5 = C_z - C_x, \; k_6 = C_x - C_y, \; k_7 = D_{33} - D_{22}, \; k_8 = D_{11} - D_{33}, \; k_9 = D_{22} - D_{11}, \; k_{10} = B_x + k_1, \; k_{11} = B_z - k_3, \; k_{12} = C_x + k_4, \; k_{13} = C_z - k_6, \; k_{14} = D_{11} - k_7, \; k_{15} = D_{31} + D_{13}, \; k_{16} = D_{33} + k_9.$$

The inertia torque of 1 in (0) can be written as

$$T_{in\,1} = -\alpha_{10} \cdot \tilde{I}_1 - \omega_{10} \times (\tilde{I}_1 \cdot \omega_{10}) = -A_x \ddot{q}_3 \mathbf{i}_1 = -A_x \ddot{u}_1 \mathbf{i}_1.
$$

(1.36)

The inertia torque of 2 in (0) is

$$T_{in\,2} = -\alpha_{20} \cdot \tilde{I}_2 - \omega_{20} \times (\tilde{I}_2 \cdot \omega_{20}),$$

or

$$T_{in\,2} = -(\dot{u}_1 Z_{35} + Z_{36})J_2 - (\dot{u}_2 B_y + Z_{38})J_2 - (\dot{u}_1 Z_{39} + Z_{40})k_2,
$$

(1.37)

where $Z_{35} = c_1 B_x$, $Z_{36} = Z_3 k_{10}$, $Z_{37} = Z_2 Z_1$, $Z_{38} = -Z_3 k_2$, $Z_{39} = s_1 B_z$, $Z_{40} = Z_4 k_{11}$.

Similarly the inertia torque of 3 in (0) is

$$T_{in\,3} = -(\dot{u}_1 Z_{41} + Z_{42})J_2 - (\dot{u}_2 C_y + Z_{43})J_2 - (\dot{u}_1 Z_{44} + Z_{45})k_2,
$$

(1.38)

where $Z_{40} = Z_4 k_{11}$, $Z_{41} = c_1 C_x$, $Z_{42} = Z_3 k_{12}$, $Z_{43} = -Z_3 k_5$, $Z_{44} = s_1 C_z$, $Z_{45} = Z_4 k_{13}$.

The inertia torque of $RB$ in (0) is

$$T_{in\,R} = -(\dot{u}_1 Z_{49} + \dot{u}_2 D_{12} + Z_{50})J_2 - (\dot{u}_1 Z_{51} + \dot{u}_2 D_{22} + Z_{52})J_2 - (\dot{u}_1 Z_{53} + \dot{u}_2 D_{32} + Z_{54})k_2,$n

(1.39)

where $Z_{46} = Z_1^2$, $Z_{47} = u_2^2$, $Z_{48} = Z_2^2$, $Z_{49} = D_{11} c_1 + D_{13} s_1$, $Z_{50} = k_{14} Z_3 + k_{15} Z_4 - D_{12} Z_{37} + D_{23} (Z_{47} - Z_{48})$, $Z_{51} = D_{23} s_1 + D_{21} c_1$, $Z_{52} = D_{31} (Z_{48} - Z_{46}) + k_8 Z_{37}$, $Z_{53} = D_{33} s_1 + D_{31} c_1$, $Z_{54} = k_{15} Z_3 + k_{16} Z_4 + D_{23} Z_{37} + D_{12} (Z_{46} - Z_{47})$.

The inertia force for link $j = 1, 2, 3$ is

$$F_{in\,j} = -m_j \mathbf{a}_{C_j},
$$

(1.40)
and the inertia force for the rigid body $RB$ is

$$F_{in R} = -m_R a_{CR}. \quad (1.41)$$

The contribution of link $j = 1, 2, 3$ to the generalized inertial force $F^*_r$ is

$$(F^*_r)_j = \frac{\partial v_{Cj}}{\partial u_r} \cdot F_{in j} + \frac{\partial \omega_{j0}}{\partial u_r} \cdot T_{in j}, \ r = 1, 2, 3, \quad (1.42)$$

and the contribution of the rigid body $RB$ to the generalized inertial force $F^*_r$ is

$$(F^*_r)_R = \frac{\partial v_{CR}}{\partial u_r} \cdot F_{in R} + \frac{\partial \omega_{20}}{\partial u_r} \cdot T_{in R}. \quad (1.43)$$

The three generalized inertia forces are computed with

$$F^*_r = \sum_{j=1}^{3} (F^*_r)_j + (F^*_r)_R =$$

$$\sum_{j=1}^{3} \left( \frac{\partial v_{Cj}}{\partial u_r} \cdot F_{in j} + \frac{\partial \omega_{j0}}{\partial u_r} \cdot T_{in j} \right) +$$

$$\frac{\partial v_{CR}}{\partial u_r} \cdot F_{in R} + \frac{\partial \omega_{20}}{\partial u_r} \cdot T_{in R}, \ r = 1, 2, 3. \quad (1.44)$$

### 1.3 Generalized active forces

To explain what these are, again the system $\{S\}$ of $\nu$ particles is consider and let $R_i$ be the resultant of all contact and body forces acting on a generic particle $P_i$ of $\{S\}$, and define $F_r$ as

$$F_r = \sum_{i=1}^{\nu} \left( \frac{\partial v_{P_i}}{\partial u_i} \cdot R_i \right), \ r = 1, ..., n, \quad (1.45)$$

where $F_r$ is called the $r$th generalized active force for $\{S\}$.

The task of constructing expressions for $F_r$ frequently is facilitated by the following facts. Many forces that contribute to $R_i$ make no contributions to $F_r$. For example, if $RB$ is a rigid body belonging to $\{S\}$, the total contribution to $F_r$ of all gravitational forces exerted by particles of $RB$ on each other is equal to zero. Furthermore, if a set of contact and/or body forces acting on $RB$ is equivalent to a couple of torque $T$ together with force $R$ applied
at a point \( Q \) of \( RB \), then \((F_r)_R\), the contribution of this set of forces to \( F_r \), is given by

\[
(F_r)_R = \frac{\partial \omega}{\partial u_r} \cdot T + \frac{\partial v_Q}{\partial u_r} \cdot R, \quad r = 1, \ldots, n, \tag{1.46}
\]

where \( \omega \) is the angular velocity of \( RB \) in \((0)\), and \( v_Q \) is the velocity of \( Q \) in \((0)\). In the case of the robot arm, there are two kinds of forces that contribute to the generalized active forces \( F_1, F_2, F_3 \) namely, contact forces applied in order to drive 1, 2, 3 and \( RB \), and gravitational forces exerted on 1, 2, 3, and \( RB \) by the Earth. Considering, first, the contact forces, Figure 1.1(a) the set of such forces transmitted from 0 to 1 (through bearings and by means of motor) is replaced with a couple of torque \( T_{01} \) together with a force \( F_{01} \) applied to 1 at \( A \). Similarly, the set of contact forces transmitted from 1 to 2 is replaced with a couple of torque \( T_{12} \) together with a force \( F_{12} \) applied to 2 at \( B \). The law of action and reaction then guarantees that the set of contact forces transmitted from 1 to 2 is equivalent to a couple of torque \(-T_{12}\) together with the force \(-F_{12}\) applied to 1 and \( B \). Next, the set of contact forces exerted on 2 by 3 is replaced with a couple of torque \( T_{23} \) together with a force \( F_{23} \) applied to 3 at \( C_3 \). The law of action and reaction guarantees that the set of contact forces transmitted from 3 to 2 is equivalent to a couple of torque \(-T_{23}\) together with the force \(-F_{23}\) applied to 2 and \( C_3 \), \((C_{32} \in \text{link2})\) the point of instantaneously coinciding with \( C_3 \), \((C_3 \in \text{link3})\).

The expressions \( T_{01}, F_{01}, T_{12}, F_{12}, T_{23}, \) and \( F_{23} \) are

\[
T_{01} = T_{01x}1 + T_{01y}j_1 + T_{01z}k_1, \quad F_{01} = F_{01x}1 + F_{01y}j_1 + F_{01z}k_1, \\
T_{12} = T_{12x}1 + T_{12y}j_2 + T_{12z}k_2, \quad F_{12} = F_{12x}1 + F_{12y}j_2 + F_{12z}k_2, \\
T_{23} = T_{23x}1 + T_{23y}j_2 + T_{23z}k_2, \quad F_{23} = F_{23x}1 + F_{23y}j_2 + F_{23z}k_2. \tag{1.47}
\]

As for gravitational forces exerted on 1, 2, 3, and \( RB \) by the Earth, these are denoted by \( G_1, G_2, G_3, G_R \), respectively, and can be expressed as

\[
G_1 = -m_1 g j_1, \\
G_2 = -m_2 g j_1 = -m_2 g (c_1 j_2 + s_1 k_2), \\
G_3 = -m_3 g j_1 = -m_3 g (c_1 j_2 + s_1 k_2), \\
G_R = -m_R g j_1 = -m_R g (c_1 j_2 + s_1 k_2). \tag{1.48}
\]

The reason for replacing \( j_1 \) with \( c_1 j_2 + s_1 k_2 \) in connection with \( G_2, G_3, \) and \( G_R \), is that they are soon to be dot-multiplied with \( \frac{\partial v_{C_2}}{\partial u_r}, \frac{\partial v_{C_3}}{\partial u_r}, \) and \( \frac{\partial v_{C_B}}{\partial u_r} \).
which have been expressed in terms of $I_2, J_2, k_2$.

One can express $(F_r)_1$, the contribution to the generalized active force $F_r$ of all forces and torques acting on particles of body 1, as

$$(F_r)_1 = \frac{\partial \omega_{10}}{\partial u_r} \cdot (T_{01} - T_{12}) + \frac{\partial \nu_{C1}}{\partial u_r} \cdot G_1 + \frac{\partial \nu_B}{\partial u_r} \cdot (-F_{12}) , \ r = 1, 2, 3. \ (1.49)$$

The contribution to the generalized active force of all forces and torques acting on link 2 is

$$(F_r)_2 = \frac{\partial \omega_{20}}{\partial u_r} \cdot (T_{12} - T_{23}) + \frac{\partial \nu_B}{\partial u_r} \cdot F_{12} + \frac{\partial \nu_{C2}}{\partial u_r} \cdot G_2 + \frac{\partial \nu_{C32}}{\partial u_r} \cdot (-F_{23}) , \ r = 1, 2, 3. \ (1.50)$$

The contribution to the generalized active force of all forces and torques acting on link 3 is

$$(F_r)_3 = \frac{\partial \omega_{20}}{\partial u_r} \cdot T_{23} + \frac{\partial \nu_{C3}}{\partial u_r} \cdot G_3 + \frac{\partial \nu_{C3}}{\partial u_r} \cdot F_{23} , \ r = 1, 2, 3. \ (1.51)$$

The contribution to the generalized active force of all forces and torques acting on rigid body $RB$ is

$$(F_r)_R = \frac{\partial \nu_{CR}}{\partial u_r} \cdot G_R , \ r = 1, 2, 3. \ (1.52)$$

The generalized active force of all forces and torques acting on 1, 2, 3, and $RB$ are

$$F_r = (F_r)_1 + (F_r)_2 + (F_r)_3 + (F_r)_R , \ r = 1, 2, 3, \ (1.53)$$

or

$$F_1 = T_{12x} ,$$

$$F_2 = T_{12y} - g \left[ (m_2 L_2 + m_3 Z_9 + m_r Z_{14}) \ c_1 - m_R p_x s_1 \right] ,$$

$$F_3 = F_{23z} - g \left( m_3 + m_R \right) s_1. \ (1.54)$$

To arrive at the dynamical equations governing the robot arm, all that remains to be done is to substitute into Kane’s dynamical equations, namely,

$$F^*_r + F_r = 0 , \ r = 1, 2, 3, \ (1.55)$$
or

\[ X_{11} \ddot{u}_1 + X_{12} \ddot{u}_2 + X_{13} \ddot{u}_3 = Y_1, \]
\[ X_{21} \ddot{u}_1 + X_{22} \ddot{u}_2 + X_{23} \ddot{u}_3 = Y_2, \]
\[ X_{31} \ddot{u}_1 + X_{32} \ddot{u}_2 + X_{33} \ddot{u}_3 = Y_3, \]  \hspace{1cm} (1.56)

where

\[
X_{11} = -[A_x + c_1(Z_{35} + Z_{41} + Z_{49}) + s_1(Z_{39} + Z_{44} + Z_{53}) + \\
\quad m_1L_x^2 + m_2Z_6^2 + m_3Z_{10}^2 + m_R(Z_{13}^2 + Z_{15}^2 + Z_{16}^2)], \\
X_{12} = X_{21} = -[Z_{31} + m_R(Z_{13}Z_{14} - Z_{16}p_x)], \\
X_{13} = X_{31} = -m_RZ_{16}, \\
Y_1 = c_1(Z_{36} + Z_{42} + Z_{50}) + s_1(Z_{40} + Z_{45} + Z_{54}) + m_2Z_6Z_{23} + \\
\quad m_3Z_{16}Z_{27} + m_R(Z_{13}Z_{32} + Z_{15}Z_{33} + Z_{16}Z_{31}) - T_{01x}, \\
X_{22} = -[B_y + C_y + D_{22} + m_2L_x^2 + m_3Z_9 + m_R(Z_{14}^{32} + p_x)], \\
X_{23} = X_{32} = m_Rp_x, \\
Y_2 = Z_{38} + Z_{42} + Z_{52} + m_2L_xZ_{22} + m_3Z_9Z_{26} + m_R(Z_{14}Z_{42} - \\
\quad p_xZ_{34}) - T_{12x} + g[m_2L_x + m_3Z_9 + m_RZ_{14}]c_1 - m_Rp_x s_1], \\
X_{33} = -(m_3 + m_R), \\
Y_3 = m_3Z_{28} + m_RZ_{34} - F_{23x} + g(m_3 + m_R)s_2.
\]

### 1.4 Numerical simulation

The robot arm is characterized by the following geometry [Kane and Levinson]:

- \( L_1 = 0.3 \) m, \( L_2 = 0.5 \) m, \( L_B = 1.1 \) m, \( p_x = 0.2 \) m, \( p_y = 0.4 \) m, \( p_z = 0.6 \) m,
- \( A_x = 11 \) kg m\(^2\), \( B_x = 7 \) kg m\(^2\), \( B_y = 6 \) kg m\(^2\), \( B_z = 2 \) kg m\(^2\), \( C_x = 5 \) kg m\(^2\),
- \( C_y = 4 \) kg m\(^2\), \( C_z = 1 \) kg m\(^2\), \( D_{11} = 2 \) kg m\(^2\), \( D_{22} = 2.5 \) kg m\(^2\), \( D_{33} = 1.3 \) kg m\(^2\),
- \( D_{12} = D_{21} = 0.6 \) kg m\(^2\), \( D_{13} = D_{31} = -1.1 \) kg m\(^2\), \( D_{32} = D_{23} = 0.75 \) kg m\(^2\). The masses of the rigid bodies are \( m_1 = 87 \) kg, \( m_2 = 63 \) kg, \( m_3 = 42 \) kg, \( m_R = 50 \) kg, and the gravitational acceleration is \( g = 9.81 \) m/s\(^2\).

The initial conditions, at \( t = 0 \) s, are \( q_1(0) = \pi / 6 \) rad, \( q_2(0) = 0.1 \) m,
\( q_3(0) = \pi / 18 \) rad, and \( \dot{q}_1(0) = \dot{q}_2(0) = \dot{q}_3(0) = 0 \).

The robot arm can be brought from an initial state of rest in reference frame (0) to a final state of rest in (0) such that \( q_1, q_2, \) and \( q_3 \) have specified values \( q_{1f}, q_{2f}, \) and \( q_{3f}, \) respectively, by using the following feedback control laws:

\[
T_{01x} = -\beta_{01}\dot{q}_3 - \gamma_{01}(q_3 - q_{3f}),
\]

\[ q_3 = q_{3f}, \]
\[
T_{12y} = -\beta_{12}\dot{q}_1 - \gamma_{12}(q_1 - q_{1f}) + g[(m_2L_2 + m_3Z_9 + m_RZ_{14})c_1 - m_Rp_ks_1], \\
F_{23z} = -\beta_{23}\dot{q}_2 - \gamma_{23}(q_2 - q_{2f}) + g(m_3 + m_R)s_3.
\]

The constant gains are \(\beta_{01} = 464\) N m s/rad, \(\gamma_{01} = 306\) N m/rad, \(\beta_{12} = 216\) N m s/rad, \(\gamma_{12} = 285\) N m/rad, \(\beta_{23} = 169\) N m s, \(\gamma_{23} = 56\) N/m, and the specified values for the generalized coordinates are \(q_{1f} = \pi/3\) rad, \(q_{2f} = 0.4\) m, and \(q_{3f} = 7\pi/18\) rad. Fig. 1.2 represents the values of \(q_1, q_2,\) and \(q_3\) from \(t = 0\) to \(t = 30\) s and the \texttt{Mathematica}\textsuperscript{TM} program is given in Appendix 9.

### 1.5 Kinetic energy

The total kinetic energy of the robot arm in (0) is

\[
T = \sum_{i=1}^{3} T_i + T_R. \tag{1.57}
\]

The kinetic energy of link \(i, i = 1, 2, 3,\) is

\[
T_i = \frac{1}{2}m_i\mathbf{v}_{C_i} \cdot \mathbf{v}_{C_i} + \frac{1}{2}\mathbf{\omega}_{i0} \cdot (\bar{I}_i \cdot \mathbf{\omega}_{i0}). \tag{1.58}
\]

The kinetic energy of rigid body RB is

\[
T_R = \frac{1}{2}m_R\mathbf{v}_R \cdot \mathbf{v}_R + \frac{1}{2}\mathbf{\omega}_{20} \cdot (\bar{I}_R \cdot \mathbf{\omega}_{20}). \tag{1.59}
\]

The generalized inertia forces can be computed also with the formula

\[
F^*_r = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r}, \quad r = 1, 2, 3. \tag{1.60}
\]
Figure 2