

# 1 OPEN KINEMATIC CHAINS

In this chapter Kane's approach is used to formulate the equations of motion for open kinematic chains. A detailed dynamic analysis of a three DOF open kinematic chain is presented [Kane and Levinson].

## 1.1 Kinematics of open kinematic chains

Figure 1.1(a) is a schematic representation of an open kinematic chain (robot arm) consisting of three elements 1, 2, and 3. The last link 3 holds rigidly a rigid body  $RB$ . Body 1 can be rotated at  $A$  in a "fixed" reference frame (0) of unit vectors  $[\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0]$  about a vertical axis  $\mathbf{i}_0$ . The unit vector  $\mathbf{i}_0$  is fixed in 1. At the pin joint  $B$  the link 1 is connected to link 2. The element 2 rotates relative to 1 about a horizontal axis fixed in both 1 and 2, passing through  $B$ , and perpendicular to the axis of 1. The last link 3 is connected to 2 by means of a slider joint. The mass centers of links 1, 2, and 3 are  $C_1$ ,  $C_2$ , and  $C_3$ , respectively. The distances  $L_1 = AC_1$ ,  $L_2 = BC_2$ , and  $L_B = AB$  are indicated in Fig. 1.1(a). The reference frame (1) of unit vectors  $[\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1]$  is attached to link 1, and the reference frame (2) of unit vectors  $[\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2]$  is attached to link 2, as shown in Fig. 1.1.

Let

$$p_x = \mathbf{r}_{C_3C_R} \cdot \mathbf{i}_2, \quad p_y = \mathbf{r}_{C_3C_R} \cdot \mathbf{j}_2, \quad p_z = \mathbf{r}_{C_3C_R} \cdot \mathbf{k}_2,$$

where  $\mathbf{r}_{C_3C_R}$  is the position vector from  $C_3$  to  $C_R$ , where  $C_R$  is the mass center of  $RB$ .

To characterize the instantaneous configuration of the arm, *generalized coordinates*  $q_1(t)$ ,  $q_2(t)$ ,  $q_3(t)$  are employed. The generalized coordinates are quantities associated with the position of the system. The first generalized coordinate  $q_1$  denotes the radian measure of the angle between the axes of 1 and 2 ( $s_1 = \sin q_1$ ,  $c_1 = \cos q_1$ ), and  $q_2$  is the distance from  $C_2$  to  $C_3$ . The last generalized coordinate  $q_3$ , ( $s_3 = \sin q_3$ ,  $c_3 = \cos q_3$ ), designates also a radian measure of rotation angle between 1 and 0.

As important as generalized coordinates are *generalized speeds*, these being quantities associated with the motion of a system. The generalized speeds  $u_1(t), \dots, u_n(t)$ , where  $n$  is the number of generalized coordinates can be introduced as

$$u_r = \sum_{s=1}^n A_{rs} \dot{q}_s + B_r, \quad r = 1, \dots, n, \quad (1.1)$$



where  $A_{rs}$  and  $B_r$  are functions of  $q_1, \dots, q_n$ , and the time  $t$ ;  $A_{rs}$  and  $B_r$  ( $r, s = 1, \dots, n$ ) are chosen such that Eq.(1.1) can be solved uniquely for  $q_1, \dots, q_n$ . The generalized speeds  $u_1, \dots, u_n$  serve as variables on an equal footing with the generalized coordinates  $q_1, \dots, q_n$ . Their introduction can enable one to take advantage of special features of a given physical system to bring equations of motion into a particularly simple form. Generally, this is accomplished by taking  $u_r$  to be an angular velocity measure number, a velocity measure number, or simply  $\dot{q}_r$ . Considering, for example, the robotic arm, one can introduce  $u_1, u_2, u_3$  as

$$u_1 = \boldsymbol{\omega}_{10} \cdot \mathbf{1}_1, \quad u_2 = \boldsymbol{\omega}_{21} \cdot \mathbf{J}_2, \quad u_3 = {}^{(2)}\mathbf{v}_{C_3} \cdot \mathbf{k}_2, \quad (1.2)$$

where  $\boldsymbol{\omega}_{10}$  is the angular velocity of 1 in the fixed reference frame (0),  $\boldsymbol{\omega}_{21}$  is the angular velocity of 2 with respect to reference frame (1), and  ${}^{(2)}\mathbf{v}_{C_3}$  is the velocity of  $C_3$  in reference frame (2).

In the case of the three DOF robot arm,  $\dot{q}_3 = u_1$ ,  $\dot{q}_1 = u_2$ , and  $\dot{q}_2 = u_3$  or

$$u_1 = \dot{q}_3, \quad u_2 = \dot{q}_1, \quad u_3 = \dot{q}_2. \quad (1.3)$$

Equation (1.3) can be solved uniquely for  $\dot{q}_1, \dot{q}_2, \dot{q}_3$ .

### 1.1.1 Angular velocities

Next the angular velocity of 1, 2, and 3 will be expressed in the fixed reference frame (0). One can express the angular velocity of 1 in (0) as

$$\boldsymbol{\omega}_{10} = \dot{q}_3 \mathbf{1}_1 = u_1 \mathbf{1}_1. \quad (1.4)$$

The angular velocity of link 2 with respect to (1) is

$$\boldsymbol{\omega}_{21} = \dot{q}_1 \mathbf{J}_2, \quad (1.5)$$

and the angular velocity of link 2 with respect to the fixed reference frame (0) is

$$\boldsymbol{\omega}_{20} = \boldsymbol{\omega}_{10} + \boldsymbol{\omega}_{21} = \dot{q}_3 \mathbf{1}_1 + \dot{q}_1 \mathbf{J}_2. \quad (1.6)$$

The unit vector  $\mathbf{1}_0$  can be expressed as Fig. 1.1(b)

$$\mathbf{1}_0 = \mathbf{1}_1 = c_1 \mathbf{1}_2 + s_1 \mathbf{k}_2. \quad (1.7)$$

The angular velocity of link 2 in (0) written in terms of the reference frame (2) is

$$\boldsymbol{\omega}_{20} = u_1(c_1 \mathbf{1}_2 + s_1 \mathbf{k}_2) + u_2 \mathbf{J}_2 = u_1 c_1 \mathbf{1}_2 + u_2 \mathbf{J}_2 + u_1 s_1 \mathbf{k}_2, \quad (1.8)$$

or

$$\boldsymbol{\omega}_{20} = Z_1 \mathbf{1}_2 + u_2 \mathbf{J}_2 + Z_2 \mathbf{k}_2, \quad (1.9)$$

where  $Z_1 = u_1 c_1$  and  $Z_2 = u_1 s_1$ . The quantities  $Z_i$  are introduced to minimize the writing. The link 3 and the rigid body  $RB$  have the same rotational motion as link 2, i.e.  $\boldsymbol{\omega}_{30} = \boldsymbol{\omega}_{R0} = \boldsymbol{\omega}_{20}$ .

### 1.1.2 Angular accelerations

The angular acceleration of 1 in (0) can be expressed as

$$\boldsymbol{\alpha}_{10} = \ddot{q}_3 \mathbf{1}_1 = \dot{u}_1 \mathbf{1}_1. \quad (1.10)$$

The angular velocity of link 2 with respect to (0) is

$$\begin{aligned} \boldsymbol{\alpha}_{20} &= \frac{d}{dt} \boldsymbol{\omega}_{20} = \frac{{}^{(2)}\partial}{\partial t} \boldsymbol{\omega}_{20} = (\dot{u}_1 c_1 - u_1 \dot{q}_1 s_1) \mathbf{1}_2 + \dot{u}_2 \mathbf{J}_2 + (\dot{u}_1 s_1 + u_1 \dot{q}_1 c_1) \mathbf{k}_2 \\ &= (\dot{u}_1 c_1 - u_1 u_2 s_1) \mathbf{1}_2 + \dot{u}_2 \mathbf{J}_2 + (\dot{u}_1 s_1 + u_1 u_2 c_2) \mathbf{k}_2 \\ &= (\dot{u}_1 c_1 + Z_3) \mathbf{1}_2 + \dot{u}_2 \mathbf{J}_2 + (\dot{u}_1 s_1 + Z_4) \mathbf{k}_2, \end{aligned} \quad (1.11)$$

where  $\frac{{}^{(2)}\partial}{\partial t}$  represents the partial derivative with respect to time in reference frame (2),  $[\mathbf{1}_2, \mathbf{J}_2, \mathbf{k}_2]$ ,  $Z_3 = -Z_1 u_2$ , and  $Z_4 = Z_1 u_2$ . The link 3 and the rigid body  $RB$  have the same angular acceleration as link 2, i.e.  $\boldsymbol{\alpha}_{30} = \boldsymbol{\alpha}_{20}$ .

### 1.1.3 Linear velocities

The position vector of  $C_1$ , the mass center of link 1, is

$$\mathbf{r}_{C_1} = L_1 \mathbf{k}_1, \quad (1.12)$$

and the velocity of  $C_1$  in (0) is

$$\begin{aligned} \mathbf{v}_{C_1} &= \frac{d}{dt} \mathbf{r}_{C_1} = \frac{{}^{(1)}\partial}{\partial t} \mathbf{r}_1 + \boldsymbol{\omega}_{10} \times \mathbf{r}_{C_1} \\ &= \mathbf{0} + \begin{vmatrix} \mathbf{1}_1 & \mathbf{J}_1 & \mathbf{k}_1 \\ u_1 & 0 & 0 \\ 0 & 0 & L_1 \end{vmatrix} = -u_1 L_1 \mathbf{J}_1 = Z_5 \mathbf{J}_1, \end{aligned} \quad (1.13)$$

where  $Z_5 = -u_1 L_1$ .

The position vector of  $C_2$ , the mass center of link 2, is

$$\begin{aligned}\mathbf{r}_{C_2} &= L_B \mathbf{k}_1 + L_2 \mathbf{k}_2 = L_B(-s_1 \mathbf{i}_2 + c_1 \mathbf{k}_2) + L_2 \mathbf{k}_2 \\ &= -L_B s_1 \mathbf{i}_2 + (L_B c_1 + L_2) \mathbf{k}_2.\end{aligned}\quad (1.14)$$

The velocity of  $C_2$  in (0) is

$$\begin{aligned}\mathbf{v}_{C_2} &= \frac{d}{dt} \mathbf{r}_{C_2} = \frac{{}^{(2)}\partial}{\partial t} \mathbf{r}_{C_2} + \boldsymbol{\omega}_{20} \times \mathbf{r}_{C_2} \\ &= -L_B c_1 u_2 \mathbf{i}_2 - L_B c_1 u_2 \mathbf{k}_2 + \begin{vmatrix} \mathbf{i}_2 & \mathbf{j}_2 & \mathbf{k}_2 \\ u_1 c_1 & u_2 & u_1 s_1 \\ -L_B s_1 & 0 & L_B c_1 + L_2 \end{vmatrix} \\ &= L_2 u_2 \mathbf{i}_2 - (L_B + L_2 c_1) u_1 \mathbf{j}_2 = L_2 u_2 \mathbf{i}_2 + Z_6 u_1 \mathbf{j}_2 \\ &= Z_7 \mathbf{i}_2 + Z_8 \mathbf{j}_2,\end{aligned}\quad (1.15)$$

where  $Z_6 = -(L_B + L_2 c_1)$ ,  $Z_7 = L_2 u_2$ , and  $Z_8 = Z_6 u_1$ .

The position vector of  $C_3$  with respect to reference frame (0) is

$$\begin{aligned}\mathbf{r}_{C_3} &= \mathbf{r}_{C_2} + q_2 \mathbf{k}_2 \\ &= -L_B s_1 \mathbf{i}_2 + (L_B c_1 + L_2 + q_2) \mathbf{k}_2,\end{aligned}\quad (1.16)$$

and the velocity of this mass center in (0) is

$$\begin{aligned}\mathbf{v}_{C_3} &= \frac{d}{dt} \mathbf{r}_{C_3} = \frac{{}^{(2)}\partial}{\partial t} \mathbf{r}_{C_3} + \boldsymbol{\omega}_{20} \times \mathbf{r}_{C_3} \\ &= -L_B c_1 u_2 \mathbf{i}_2 - (L_B c_1 u_2 + u_3) \mathbf{k}_2 + \begin{vmatrix} \mathbf{i}_2 & \mathbf{j}_2 & \mathbf{k}_2 \\ u_1 c_1 & u_2 & u_1 s_1 \\ -L_B s_1 & 0 & L_B c_1 + L_2 + q_2 \end{vmatrix} \\ &= (L_2 + q_2) u_2 \mathbf{i}_2 - (L_B + L_2 c_1 + c_1 q_2) u_1 \mathbf{j}_2 + u_3 \mathbf{k}_2 \\ &= u_2 Z_9 \mathbf{i}_2 + u_1 Z_{10} \mathbf{j}_2 + u_3 \mathbf{k}_2 \\ &= Z_{11} \mathbf{i}_2 + Z_{12} \mathbf{j}_2 + u_3 \mathbf{k}_2,\end{aligned}\quad (1.17)$$

where  $Z_9 = L_2 + q_2$ ,  $Z_{10} = Z_6 + q_2 c_1$ ,  $Z_{11} = u_2 Z_9$ , and  $Z_{12} = Z_{10} u_1$ . The position vector of the mass center  $C_R$  of the rigid body  $RB$  is

$$\begin{aligned}\mathbf{r}_{C_R} &= \mathbf{r}_{C_3} + \mathbf{r}_{C_3 C_R} \\ &= (p_x - L_B s_1) \mathbf{i}_2 + p_y \mathbf{j}_2 + (L_B c_1 + L_2 + q_2 + p_z) \mathbf{k}_2.\end{aligned}\quad (1.18)$$

The velocity of  $C_R$  in (0) is

$$\begin{aligned}
\mathbf{v}_{C_R} &= \frac{d}{dt} \mathbf{r}_{C_R} = \frac{{}^{(2)}\partial}{\partial t} \mathbf{r}_{C_R} + \boldsymbol{\omega}_{20} \times \mathbf{r}_{C_R} \\
&= -L_B c_1 u_2 \mathbf{i}_2 - (L_B c_1 u_2 + u_3) \mathbf{k}_2 + \begin{vmatrix} \mathbf{i}_2 & \mathbf{J}_2 & \mathbf{k}_2 \\ u_1 c_1 & u_2 & u_1 s_1 \\ p_x - L_B s_1 & p_y & L_B c_1 + L_2 + q_2 + p_z \end{vmatrix} \\
&= (u_1 Z_{13} + u_2 Z_{14}) \mathbf{i}_2 + u_1 Z_{15} \mathbf{J}_2 + (Z_1 6u_1 - u_2 p_x + u_3) \mathbf{k}_2 \\
&= Z_{17} \mathbf{i}_2 + Z_{18} \mathbf{J}_2 + Z_{19} \mathbf{k}_2, \tag{1.19}
\end{aligned}$$

where  $Z_{13} = -s_1 p_y$ ,  $Z_{14} = Z_9 + p_z$ ,  $Z_{15} = Z_{10} + s_1 p_x - c_1 p_z$ ,  $Z_{16} = c_1 p_y$ ,  $Z_{17} = u_1 Z_{13} + u_2 Z_{14}$ ,  $Z_{18} = u_1 Z_{15}$ , and  $Z_{19} = Z_{16} u_1 - u_2 p_x + u_3$ .

#### 1.1.4 Linear accelerations

The acceleration of  $C_1$  is

$$\begin{aligned}
\mathbf{a}_{C_1} &= \frac{d}{dt} \mathbf{v}_{C_1} = \frac{{}^{(1)}\partial}{\partial t} \mathbf{v}_{C_1} + \boldsymbol{\omega}_{10} \times \mathbf{v}_{C_1} \\
&= -L_1 \dot{u}_1 \mathbf{J} + \begin{vmatrix} \mathbf{i}_1 & \mathbf{J}_1 & \mathbf{k}_1 \\ u_1 & 0 & 0 \\ 0 & -L_1 u_1 & 0 \end{vmatrix} \\
&= -L_1 \dot{u}_1 \mathbf{J}_1 - L_1 u_1^2 \mathbf{k}_1 \\
&= -L_1 \dot{u}_1 \mathbf{J}_1 + Z_{20} \mathbf{k}_1, \tag{1.20}
\end{aligned}$$

where  $Z_{20} = -L_1 u_1^2 = u_1 Z_5$ .

The linear acceleration of the mass center  $C_2$  is

$$\begin{aligned}
\mathbf{a}_{C_2} &= \frac{d}{dt} \mathbf{v}_{C_2} = \frac{{}^{(2)}\partial}{\partial t} \mathbf{v}_{C_2} + \boldsymbol{\omega}_{20} \times \mathbf{v}_{C_2} \\
&= (\dot{u}_2 L_2 - Z_2 Z_8) \mathbf{i}_2 + (Z_6 \dot{u}_1 + L_2 s_1 u_2 u_1 + Z_2 Z_7) \mathbf{J}_2 + (Z_1 Z_8 - u_2 Z_7) \mathbf{k}_2 \\
&= (\dot{u}_2 L_2 + Z_{22}) \mathbf{i}_2 + (Z_6 \dot{u}_1 + Z_{23}) \mathbf{J}_2 + Z_{24} \mathbf{k}_2, \tag{1.21}
\end{aligned}$$

where  $Z_{21} = L_2 s_1 u_2$ ,  $Z_{22} = -Z_2 Z_8$ ,  $Z_{23} = Z_{21} u_1 + Z_2 Z_7$ , and  $Z_{24} = Z_1 Z_8 - u_2 Z_7$ .

The acceleration of  $C_3$  is

$$\begin{aligned}
\mathbf{a}_{C_3} &= \frac{d}{dt} \mathbf{v}_{C_3} = \frac{{}^{(2)}\partial}{\partial t} \mathbf{v}_{C_3} + \boldsymbol{\omega}_{20} \times \mathbf{v}_{C_3} \\
&= (\dot{u}_2 Z_9 + Z_{26}) \mathbf{i}_2 + (\dot{u}_1 Z_{10} + Z_{27}) \mathbf{J}_2 + (\dot{u}_3 + Z_{28}) \mathbf{k}_2, \tag{1.22}
\end{aligned}$$

where  $Z_{25} = Z_{21} - u_3 c_1 + q_2 s_1 u_2$ ,  $Z_{26} = 2u_2 u_3 - Z_2 Z_{12}$ ,  $Z_{27} = Z_{25} u_1 + Z_2 Z_{11} - Z_1 u_3$ , and  $Z_{28} = Z_1 Z_{12} - u_2 Z_{11}$ .

The acceleration of  $C_R$  is

$$\begin{aligned} \mathbf{a}_{C_R} &= \frac{d}{dt} \mathbf{v}_{C_R} = \frac{{}^{(2)}\partial}{\partial t} \mathbf{v}_{C_R} + \boldsymbol{\omega}_{20} \times \mathbf{v}_{C_R} = (\dot{u}_1 Z_{13} + \dot{u}_2 Z_{14} + Z_{32}) \mathbf{i}_2 \\ &\quad + (\dot{u}_1 Z_{15} + Z_{33}) \mathbf{j}_2 + (\dot{u}_1 Z_{16} - p_x \dot{u}_2 + \dot{u}_3 + Z_{34}) \mathbf{k}_2, \end{aligned} \quad (1.23)$$

where  $Z_{29} = -Z_{16} u_2$ ,  $Z_{30} = Z_{25} + u_2 (c_1 p_x + s_1 p_z)$ ,  $Z_{31} = Z_{13} u_2$ ,  $Z_{32} = Z_{29} u_1 + u_2 (u_3 + Z_{19}) - Z_2 Z_{18}$ ,  $Z_{33} = Z_{30} u_1 + Z_2 Z_{17} - Z_1 Z_{19}$ , and  $Z_{34} = Z_{31} u_1 + Z_1 Z_{18} - u_2 Z_{17}$ .

## 1.2 Generalized inertia forces

To explain what the *generalized inertia forces* are, a system  $\{S\}$  formed by  $\nu$  particles  $P_1, \dots, P_\nu$  and having masses  $m_1, \dots, m_\nu$  is considered. Suppose that  $n$  generalized speeds  $u_r$  have been introduced. Let  $\mathbf{v}_{P_j}$  and  $\mathbf{a}_{P_j}$  denote, respectively, the velocity of  $P_j$  and the acceleration of  $P_j$  in a reference frame (0).

Define  $\mathbf{F}_{\text{in } j}$ , called the *inertia force* for  $P_j$ , as

$$\mathbf{F}_{\text{in } j} = -m_j \mathbf{a}_{P_j}. \quad (1.24)$$

The quantities  $F_1^*, \dots, F_n^*$ , defined as

$$F_r^* = \sum_{j=1}^{\nu} \frac{\partial \mathbf{v}_{P_j}}{\partial u_r} \cdot \mathbf{a}_{P_j}, \quad r = 1, \dots, n, \quad (1.25)$$

are called *generalized inertia forces* for  $\{S\}$ .

The contribution to  $F_r^*$ , made by the particles of a rigid body  $RB$  belonging to  $\{S\}$ , are

$$(F_r^*)_R = \frac{\partial \mathbf{v}_C}{\partial u_r} \cdot \mathbf{F}_{\text{in}} + \frac{\partial \boldsymbol{\omega}}{\partial u_r} \cdot \mathbf{T}_{\text{in}}, \quad r = 1, \dots, n, \quad (1.26)$$

where  $\mathbf{v}_C$  is the velocity of the center of gravity of  $RB$  in (0), and  $\boldsymbol{\omega} = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$  is the angular velocity of  $RB$  in (0).

The inertia force for the rigid body  $RB$  is

$$\mathbf{F}_{\text{in}} = -m \mathbf{a}_C, \quad (1.27)$$

where  $m$  is the mass of  $RB$ , and  $\mathbf{a}_C$  is the acceleration of the mass center of  $RB$  in the fixed reference frame. The inertia torque  $\mathbf{T}_{\text{in}}$  for  $RB$  is

$$\mathbf{T}_{\text{in}} = -\boldsymbol{\alpha} \cdot \bar{I} - \boldsymbol{\omega} \times \bar{I} \cdot \boldsymbol{\omega}, \quad (1.28)$$

where  $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \alpha_x \mathbf{i} + \alpha_y \mathbf{j} + \alpha_z \mathbf{k}$  is the angular acceleration of  $RB$  in (0), and  $\bar{I} = (I_x \mathbf{i}\mathbf{i}) + (I_y \mathbf{j}\mathbf{j}) + (I_z \mathbf{k}\mathbf{k})$  is the central inertia dyadic of  $RB$ . The central principal axes of  $RB$  are parallel to  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  and the associated moments of inertia have the values  $I_x, I_y, I_z$ , respectively. The inertia matrix associated to  $\bar{I}$  is

$$\bar{I} \rightarrow \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}. \quad (1.29)$$

The dot product of the vector  $\boldsymbol{\alpha}$  with the dyadic  $\bar{I}$  is

$$\boldsymbol{\alpha} \cdot \bar{I} = \bar{I} \cdot \boldsymbol{\alpha} = \alpha_x I_x \mathbf{i} + \alpha_y I_y \mathbf{j} + \alpha_z I_z \mathbf{k}, \quad (1.30)$$

and the cross product between a vector and a dyadic is

$$\begin{aligned} \boldsymbol{\omega} \times (\bar{I} \cdot \boldsymbol{\omega}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ \omega_x I_x & \omega_y I_y & \omega_z I_z \end{vmatrix} = \\ &= -\omega_y \omega_z (I_y - I_z) \mathbf{i} - \omega_z \omega_x (I_z - I_x) \mathbf{j} - \omega_x \omega_y (I_x - I_y) \mathbf{k}. \end{aligned} \quad (1.31)$$

Referring to the three DOF robot arm, let  $m_1, m_2, m_3, m_R$  be the masses of 1, 2, 3,  $RB$ , respectively. The links 1, 2, 3, and the rigid body  $RB$  have the following mass distribution properties. The central principal axes of 1 are parallel to  $\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1$ , Fig. 1.1(a), and the associated moments of inertia have the values  $A_x, A_y, A_z$ , respectively. The central inertia dyadic of 1 is

$$\bar{I}_1 = (A_x \mathbf{i}_1 \mathbf{i}_1) + (A_y \mathbf{j}_1 \mathbf{j}_1) + (A_z \mathbf{k}_1 \mathbf{k}_1). \quad (1.32)$$

The central principal axes of 2 and 3 are parallel to  $\mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2$  and the associated moments of inertia have values  $B_x, B_y, B_z$ , and  $C_x, C_y, C_z$  respectively. The central inertia dyadic of 2 is

$$\bar{I}_2 = (B_x \mathbf{i}_2 \mathbf{i}_2) + (B_y \mathbf{j}_2 \mathbf{j}_2) + (B_z \mathbf{k}_2 \mathbf{k}_2), \quad (1.33)$$

and the central inertia dyadic of 3 is

$$\bar{I}_3 = (C_x \mathbf{i}_2 \mathbf{i}_2) + (C_y \mathbf{j}_2 \mathbf{j}_2) + (C_z \mathbf{k}_2 \mathbf{k}_2), \quad (1.34)$$

The central inertia dyadic of the rigid body  $RB$  is

$$\begin{aligned}\bar{I}_R = & (D_{11}\mathbf{1}_2 + D_{12}\mathbf{J}_2 + D_{13}\mathbf{k}_2)\mathbf{1}_2 + (D_{21}\mathbf{1}_2 + D_{22}\mathbf{J}_2 + D_{23}\mathbf{k}_2)\mathbf{J}_2 + \\ & (D_{31}\mathbf{1}_2 + D_{32}\mathbf{J}_2 + D_{33}\mathbf{k}_2)\mathbf{k}_2.\end{aligned}\quad (1.35)$$

To facilitate the writing of results, the following notation is introduced

$$\begin{aligned}k_1 = B_y - B_z, k_2 = B_z - B_x, k_3 = B_x - B_y, k_4 = C_y - C_z, k_5 = C_z - C_x, k_6 = \\ C_x - C_y, k_7 = D_{33} - D_{22}, k_8 = D_{11} - D_{33}, k_9 = D_{22} - D_{11}, k_{10} = B_x + k_1, k_{11} = \\ B_z - k_3, k_{12} = C_x + k_4, k_{13} = C_z - k_6, k_{14} = D_{11} - k_7, k_{15} = D_{31} + D_{13}, k_{16} = \\ D_{33} + k_9.\end{aligned}$$

The inertia torque of 1 in (0) can be written as

$$\mathbf{T}_{\text{in } 1} = -\boldsymbol{\alpha}_{10} \cdot \bar{I}_1 - \boldsymbol{\omega}_{10} \times (\bar{I}_1 \cdot \boldsymbol{\omega}_{10}) = -A_x \ddot{q}_3 \mathbf{1}_1 = -A_x \dot{u}_1 \mathbf{1}_1. \quad (1.36)$$

The inertia torque of 2 in (0) is

$$\mathbf{T}_{\text{in } 2} = -\boldsymbol{\alpha}_{20} \cdot \bar{I}_2 - \boldsymbol{\omega}_{20} \times (\bar{I}_2 \cdot \boldsymbol{\omega}_{20}),$$

or

$$\mathbf{T}_{\text{in } 2} = -(\dot{u}_1 Z_{35} + Z_{36})\mathbf{1}_2 - (\dot{u}_2 B_y + Z_{38})\mathbf{J}_2 - (\dot{u}_1 Z_{39} + Z_{40})\mathbf{k}_2, \quad (1.37)$$

where  $Z_{35} = c_1 B_x$ ,  $Z_{36} = Z_3 k_{10}$ ,  $Z_{37} = Z_2 Z_1$ ,  $Z_{38} = -Z_{37} k_2$ ,  $Z_{39} = s_1 B_z$ ,  $Z_{40} = Z_4 k_{11}$ .

Similarly the inertia torque of 3 in (0) is

$$\mathbf{T}_{\text{in } 3} = -(\dot{u}_1 Z_{41} + Z_{42})\mathbf{1}_2 - (\dot{u}_2 C_y + Z_{43})\mathbf{J}_2 - (\dot{u}_1 Z_{44} + Z_{45})\mathbf{k}_2, \quad (1.38)$$

where  $Z_{40} = Z_4 k_{11}$ ,  $Z_{41} = c_1 C_x$ ,  $Z_{42} = Z_3 k_{12}$ ,  $Z_{43} = -Z_{37} k_5$ ,  $Z_{44} = s_1 C_z$ ,  $Z_{45} = Z_4 k_{13}$ .

The inertia torque of  $RB$  in (0) is

$$\begin{aligned}\mathbf{T}_{\text{in } R} = & -(\dot{u}_1 Z_{49} + \dot{u}_2 D_{12} + Z_{50})\mathbf{1}_2 - (\dot{u}_1 Z_{51} + \dot{u}_2 D_{22} + Z_{52})\mathbf{J}_2 \\ & -(\dot{u}_1 Z_{53} + \dot{u}_2 D_{32} + Z_{54})\mathbf{k}_2,\end{aligned}\quad (1.39)$$

where  $Z_{46} = Z_1^2$ ,  $Z_{47} = u_2^2$ ,  $Z_{48} = Z_2^2$ ,  $Z_{49} = D_{11}c_1 + D_{13}s_1$ ,  $Z_{50} = k_{14}Z_3 + k_{15}Z_4 - D_{12}Z_{37} + D_{23}(Z_{47} - Z_{48})$ ,  $Z_{51} = D_{23}s_1 + D_{21}c_1$ ,  $Z_{52} = D_{31}(Z_{48} - Z_{46}) + k_8 Z_{37}$ ,  $Z_{53} = D_{33}s_1 + D_{31}c_1$ ,  $Z_{54} = k_{15}Z_3 + k_{16}Z_4 + D_{23}Z_{37} + D_{12}(Z_{46} - Z_{47})$ .

The inertia force for link  $j = 1, 2, 3$  is

$$\mathbf{F}_{\text{in } j} = -m_j \mathbf{a}_{C_j}, \quad (1.40)$$

and the inertia force for the rigid body  $RB$  is

$$\mathbf{F}_{\text{in } R} = -m_R \mathbf{a}_{C_R}. \quad (1.41)$$

The contribution of link  $j = 1, 2, 3$  to the generalized inertial force  $F_r^*$  is

$$(F_r^*)_j = \frac{\partial \mathbf{v}_{C_j}}{\partial u_r} \cdot \mathbf{F}_{\text{in } j} + \frac{\partial \boldsymbol{\omega}_{j0}}{\partial u_r} \cdot \mathbf{T}_{\text{in } j}, \quad r = 1, 2, 3, \quad (1.42)$$

and the contribution of the rigid body  $RB$  to the generalized inertial force  $F_r^*$  is

$$(F_r^*)_R = \frac{\partial \mathbf{v}_{C_R}}{\partial u_r} \cdot \mathbf{F}_{\text{in } R} + \frac{\partial \boldsymbol{\omega}_{20}}{\partial u_r} \cdot \mathbf{T}_{\text{in } R}. \quad (1.43)$$

The three generalized inertia forces are computed with

$$\begin{aligned} F_r^* &= \sum_{j=1}^3 (F_r^*)_j + (F_r^*)_R = \\ &= \sum_{j=1}^3 \left( \frac{\partial \mathbf{v}_{C_j}}{\partial u_r} \cdot \mathbf{F}_{\text{in } j} + \frac{\partial \boldsymbol{\omega}_{j0}}{\partial u_r} \cdot \mathbf{T}_{\text{in } j} \right) + \\ &= \frac{\partial \mathbf{v}_{C_R}}{\partial u_r} \cdot \mathbf{F}_{\text{in } R} + \frac{\partial \boldsymbol{\omega}_{20}}{\partial u_r} \cdot \mathbf{T}_{\text{in } R}, \quad r = 1, 2, 3. \end{aligned} \quad (1.44)$$

### 1.3 Generalized active forces

To explain what these are, again the system  $\{S\}$  of  $\nu$  particles is considered and let  $\mathbf{R}_i$  be the resultant of all contact and body forces acting on a generic particle  $P_i$  of  $\{S\}$ , and define  $F_r$  as

$$F_r = \sum_{i=1}^{\nu} \left( \frac{\partial \mathbf{v}_{P_i}}{\partial u_i} \cdot \mathbf{R}_i \right), \quad r = 1, \dots, n, \quad (1.45)$$

where  $F_r$  is called the  $r$ th *generalized active force* for  $\{S\}$ .

The task of constructing expressions for  $F_r$  frequently is facilitated by the following facts. Many forces that contribute to  $\mathbf{R}_i$  make no contributions to  $F_r$ . For example, if  $RB$  is a rigid body belonging to  $\{S\}$ , the total contribution to  $F_r$  of all gravitational forces exerted by particles of  $RB$  on each other is equal to zero. Furthermore, if a set of contact and/or body forces acting on  $RB$  is equivalent to a couple of torque  $\mathbf{T}$  together with force  $\mathbf{R}$  applied

at a point  $Q$  of  $RB$ , then  $(F_r)_R$ , the contribution of this set of forces to  $F_r$ , is given by

$$(F_r)_R = \frac{\partial \boldsymbol{\omega}}{\partial u_r} \cdot \mathbf{T} + \frac{\partial \mathbf{v}_Q}{\partial u_r} \cdot \mathbf{R}, \quad r = 1, \dots, n, \quad (1.46)$$

where  $\boldsymbol{\omega}$  is the angular velocity of  $RB$  in (0), and  $\mathbf{v}_Q$  is the velocity of  $Q$  in (0). In the case of the robot arm, there are two kinds of forces that contribute to the generalized active forces  $F_1, F_2, F_3$  namely, contact forces applied in order to drive 1, 2, 3 and  $RB$ , and gravitational forces exerted on 1, 2, 3, and  $RB$  by the Earth. Considering, first, the contact forces, Figure 1.1(a) the set of such forces transmitted from 0 to 1 (through bearings and by means of motor) is replaced with a couple of torque  $\mathbf{T}_{01}$  together with a force  $\mathbf{F}_{01}$  applied to 1 at  $A$ . Similarly, the set of contact forces transmitted from 1 to 2 is replaced with a couple of torque  $\mathbf{T}_{12}$  together with a force  $\mathbf{F}_{12}$  applied to 2 at  $B$ . The law of action and reaction then guarantees that the set of contact forces transmitted from 1 to 2 is equivalent to a couple of torque  $-\mathbf{T}_{12}$  together with the force  $-\mathbf{F}_{12}$  applied to 1 and  $B$ . Next, the set of contact forces exerted on 2 by 3 is replaced with a couple of torque  $\mathbf{T}_{23}$  together with a force  $\mathbf{F}_{23}$  applied to 3 at  $C_3$ . The law of action and reaction guarantees that the set of contact forces transmitted from 3 to 2 is equivalent to a couple of torque  $-\mathbf{T}_{23}$  together with the force  $-\mathbf{F}_{23}$  applied to 2 and  $C_{32}$ , ( $C_{32} \in \text{link2}$ ) the point of instantaneously coinciding with  $C_3$ , ( $C_3 \in \text{link3}$ ). The expressions  $\mathbf{T}_{01}$ ,  $\mathbf{F}_{01}$ ,  $\mathbf{T}_{12}$ ,  $\mathbf{F}_{12}$ ,  $\mathbf{T}_{23}$ , and  $\mathbf{F}_{23}$  are

$$\begin{aligned} \mathbf{T}_{01} &= T_{01x}\mathbf{1}_1 + T_{01y}\mathbf{J}_1 + T_{01z}\mathbf{k}_1, & \mathbf{F}_{01} &= F_{01x}\mathbf{1}_1 + F_{01y}\mathbf{J}_1 + F_{01z}\mathbf{k}_1, \\ \mathbf{T}_{12} &= T_{12x}\mathbf{1}_2 + T_{12y}\mathbf{J}_2 + T_{12z}\mathbf{k}_2, & \mathbf{F}_{12} &= F_{12x}\mathbf{1}_2 + F_{12y}\mathbf{J}_2 + F_{12z}\mathbf{k}_2, \\ \mathbf{T}_{23} &= T_{23x}\mathbf{1}_2 + T_{23y}\mathbf{J}_2 + T_{23z}\mathbf{k}_2, & \mathbf{F}_{23} &= F_{23x}\mathbf{1}_2 + F_{23y}\mathbf{J}_2 + F_{23z}\mathbf{k}_2. \end{aligned} \quad (1.47)$$

As for gravitational forces exerted on 1, 2, 3, and  $RB$  by the Earth, these are denoted by  $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3, \mathbf{G}_R$ , respectively, and can be expressed as

$$\begin{aligned} \mathbf{G}_1 &= -m_1 g \mathbf{1}_1, \\ \mathbf{G}_2 &= -m_2 g \mathbf{1}_1 = -m_2 g (c_1 \mathbf{1}_2 + s_1 \mathbf{k}_2), \\ \mathbf{G}_3 &= -m_3 g \mathbf{1}_1 = -m_3 g (c_1 \mathbf{1}_2 + s_1 \mathbf{k}_2), \\ \mathbf{G}_R &= -m_R g \mathbf{1}_1 = -m_R g (c_1 \mathbf{1}_2 + s_1 \mathbf{k}_2). \end{aligned} \quad (1.48)$$

The reason for replacing  $\mathbf{1}_1$  with  $c_1 \mathbf{1}_2 + s_1 \mathbf{k}_2$  in connection with  $\mathbf{G}_2, \mathbf{G}_3$ , and  $\mathbf{G}_R$ , is that they are soon to be dot-multiplied with  $\frac{\partial \mathbf{v}_{C_2}}{\partial u_r}$ ,  $\frac{\partial \mathbf{v}_{C_3}}{\partial u_r}$ , and  $\frac{\partial \mathbf{v}_{C_R}}{\partial u_r}$

which have been expressed in terms of  $\mathbf{1}_2, \mathbf{J}_2, \mathbf{k}_2$ .

One can express  $(F_r)_1$ , the contribution to the generalized active force  $F_r$  of all forces and torques acting on particles of body 1, as

$$(F_r)_1 = \frac{\partial \boldsymbol{\omega}_{10}}{\partial u_r} \cdot (\mathbf{T}_{01} - \mathbf{T}_{12}) + \frac{\partial \mathbf{v}_{C_1}}{\partial u_r} \cdot \mathbf{G}_1 + \frac{\partial \mathbf{v}_B}{\partial u_r} \cdot (-\mathbf{F}_{12}), \quad r = 1, 2, 3. \quad (1.49)$$

The contribution to the generalized active force of all forces and torques acting on link 2 is

$$(F_r)_2 = \frac{\partial \boldsymbol{\omega}_{20}}{\partial u_r} \cdot (\mathbf{T}_{12} - \mathbf{T}_{23}) + \frac{\partial \mathbf{v}_B}{\partial u_r} \cdot \mathbf{F}_{12} + \frac{\partial \mathbf{v}_{C_2}}{\partial u_r} \cdot \mathbf{G}_2 + \frac{\partial \mathbf{v}_{C_{32}}}{\partial u_r} \cdot (-\mathbf{F}_{23}), \quad r = 1, 2, 3. \quad (1.50)$$

The contribution to the generalized active force of all forces and torques acting on link 3 is

$$(F_r)_3 = \frac{\partial \boldsymbol{\omega}_{20}}{\partial u_r} \cdot \mathbf{T}_{23} + \frac{\partial \mathbf{v}_{C_3}}{\partial u_r} \cdot \mathbf{G}_3 + \frac{\partial \mathbf{v}_{C_3}}{\partial u_r} \cdot \mathbf{F}_{23}, \quad r = 1, 2, 3. \quad (1.51)$$

The contribution to the generalized active force of all forces and torques acting on rigid body  $RB$  is

$$(F_r)_R = \frac{\partial \mathbf{v}_{C_R}}{\partial u_r} \cdot \mathbf{G}_R, \quad r = 1, 2, 3. \quad (1.52)$$

The generalized active force of all forces and torques acting on 1, 2, 3, and  $RB$  are

$$F_r = (F_r)_1 + (F_r)_2 + (F_r)_3 + (F_r)_R, \quad r = 1, 2, 3, \quad (1.53)$$

or

$$\begin{aligned} F_1 &= T_{12x}, \\ F_2 &= T_{12y} - g [(m_2 L_2 + m_3 Z_9 + m_r Z_{14}) c_1 - m_R p_x s_1], \\ F_3 &= F_{23z} - g (m_3 + m_R) s_1. \end{aligned} \quad (1.54)$$

To arrive at the dynamical equations governing the robot arm, all that remains to be done is to substitute into Kane's dynamical equations, namely,

$$F_r^* + F_r = 0, \quad r = 1, 2, 3, \quad (1.55)$$

or

$$\begin{aligned}
X_{11}\dot{u}_1 + X_{12}\dot{u}_2 + X_{13}\dot{u}_3 &= Y_1, \\
X_{21}\dot{u}_1 + X_{22}\dot{u}_2 + X_{23}\dot{u}_3 &= Y_2, \\
X_{31}\dot{u}_1 + X_{32}\dot{u}_2 + X_{33}\dot{u}_3 &= Y_3,
\end{aligned} \tag{1.56}$$

where

$$\begin{aligned}
X_{11} &= -[A_x + c_1(Z_{35} + Z_{41} + Z_{49}) + s_1(Z_{39} + Z_{44} + Z_{53}) + \\
& m_1L_1^2 + m_2Z_6^2 + m_3Z_{10}^2 + m_R(Z_{13}^2 + Z_{15}^2 + Z_{16}^2)], \\
X_{12} = X_{21} &= -[Z_{51} + m_R(Z_{13}Z_{14} - Z_{16}p_x)], \\
X_{13} = X_{31} &= -m_RZ_{16}, \\
Y_1 &= c_1(Z_{36} + Z_{42} + Z_{50}) + s_1(Z_{40} + Z_{45} + Z_{54}) + m_2Z_6Z_{23} + \\
& m_3Z_{10}Z_{27} + m_R(Z_{13}Z_{32} + Z_{15}Z_{33} + Z_{16}Z_{34}) - T_{01x}, \\
X_{22} &= -[B_y + C_y + D_{22} + m_2L_2^2 + m_3Z_9^2 + m_R(Z_{14}^2 + p_x^2)], \\
X_{23} = X_{32} &= m_Rp_x, \\
Y_2 &= Z_{38} + Z_{42} + Z_{52} + m_2L_2Z_{22} + m_3Z_9Z_{26} + m_R(Z_{14}Z_{32} - \\
& p_xZ_{34}) - T_{12y} + g[m_2L_2 + m_3Z_9 + m_RZ_{14}]c_1 - m_Rp_x s_1, \\
X_{33} &= -(m_3 + m_R), \\
Y_3 &= m_3Z_{28} + m_RZ_{34} - F_{23z} + g(m_3 + m_R)s_2.
\end{aligned}$$

## 1.4 Numerical simulation

The robot arm is characterized by the following geometry [Kane and Levinson]:  $L_1=0.3$  m,  $L_2=0.5$  m,  $L_B=1.1$  m,  $p_x=0.2$  m,  $p_y=0.4$  m,  $p_z=0.6$  m,  $A_x=11$  kg m<sup>2</sup>,  $B_x=7$  kg m<sup>2</sup>,  $B_y=6$  kg m<sup>2</sup>,  $B_z=2$  kg m<sup>2</sup>,  $C_x=5$  kg m<sup>2</sup>,  $C_y=4$  kg m<sup>2</sup>,  $C_z=1$  kg m<sup>2</sup>,  $D_{11}=2$  kg m<sup>2</sup>,  $D_{22}=2.5$  kg m<sup>2</sup>,  $D_{33}=1.3$  kg m<sup>2</sup>,  $D_{12} = D_{21}=0.6$  kg m<sup>2</sup>,  $D_{13} = D_{31}=-1.1$  kg m<sup>2</sup>,  $D_{32} = D_{23}=0.75$  kg m<sup>2</sup>. The masses of the rigid bodies are  $m_1=87$  kg,  $m_2=63$  kg,  $m_3=42$  kg,  $m_R=50$  kg, and the gravitational acceleration is  $g=9.81$  m/s<sup>2</sup>.

The initial conditions, at  $t=0$  s, are  $q_1(0) = \pi/6$  rad,  $q_2(0) = 0.1$  m,  $q_3(0) = \pi/18$  rad, and  $\dot{q}_1(0) = \dot{q}_2(0) = \dot{q}_3(0) = 0$ .

The robot arm can be brought from an initial state of rest in reference frame (0) to a final state of rest in (0) such that  $q_1$ ,  $q_2$ , and  $q_3$  have specified values  $q_{1f}$ ,  $q_{2f}$ , and  $q_{3f}$ , respectively, by using the following feedback control laws

$$T_{01x} = -\beta_{01}\dot{q}_3 - \gamma_{01}(q_3 - q_{3f}),$$

$$\begin{aligned}
T_{12y} &= -\beta_{12}\dot{q}_1 - \gamma_{12}(q_1 - q_{1f}) + g[(m_2L_2 + m_3Z_9 + \\
&\quad m_RZ_{14})c_1 - m_Rp_x s_1], \\
F_{23z} &= -\beta_{23}\dot{q}_2 - \gamma_{23}(q_2 - q_{2f}) + g(m_3 + m_R)s_3.
\end{aligned}$$

The constant gains are  $\beta_{01}=464$  N m s/rad,  $\gamma_{01}=306$  N m/rad,  $\beta_{12}=216$  N m s/rad,  $\gamma_{12}=285$  N m/rad,  $\beta_{23}=169$  N s/m,  $\gamma_{23}=56$  N/m, and the specified values for the generalized coordinates are  $q_{1f} = \pi/3$  rad,  $q_{2f} = 0.4$  m, and  $q_{3f} = 7\pi/18$  rad. Fig. 1.2 represents the values of  $q_1$ ,  $q_2$ , and  $q_3$  from  $t = 0$  to  $t = 30$  s and the *Mathematica*<sup>TM</sup> program is given in Appendix 9.

## 1.5 Kinetic energy

The total kinetic energy of the robot arm in (0) is

$$T = \sum_{i=1}^3 T_i + T_R. \quad (1.57)$$

The kinetic energy of link  $i$ ,  $i = 1, 2, 3$ , is

$$T_i = \frac{1}{2}m_i\mathbf{v}_{C_i} \cdot \mathbf{v}_{C_i} + \frac{1}{2}\boldsymbol{\omega}_{i0} \cdot (\bar{I}_i \cdot \boldsymbol{\omega}_{i0}). \quad (1.58)$$

The kinetic energy of rigid body  $RB$  is

$$T_R = \frac{1}{2}m_R\mathbf{v}_R \cdot \mathbf{v}_R + \frac{1}{2}\boldsymbol{\omega}_{20} \cdot (\bar{I}_R \cdot \boldsymbol{\omega}_{20}). \quad (1.59)$$

The generalized inertia forces can be computed also with the formula

$$F_r^* = \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r}, \quad r = 1, 2, 3. \quad (1.60)$$