Abstract

In the design of asphalt paving mixtures, aggregates from n stockpiles (generally n = 2, 3, 4, ..., 10) are blended at selected proportions to satisfy specified gradation ranges and other criteria. The mean and variance of any characteristic (such as %-passing through a given sieve) of each aggregate is generally known. In a laboratory environment, the proportion \( w_i \) (i = 2, 3, ..., 10) from each stockpile is controlled by the mix designer so that the n proportions (or weights) can be assumed fixed. This is unlike a plant operation where aggregates are fed from n bins into a mixing-drum so that the proportion of each aggregate, \( W_i \), from each bin is a random variable. The statistical analyses of the former case is well-know and is first repeated herein in sections 1 and 2, while less accurate information is available about the latter. So, the main objective of this article is to provide more accurate statistical information for the case when aggregate proportions are random variables.

1. Introduction

Suppose \( X_1, X_2, ..., X_n \) are random variables with known process means \( \mu_1, \mu_2, ..., \mu_n \), known process variances \( \sigma_{11}, \sigma_{22}, ..., \sigma_{nn} \), respectively, and known covariances \( \text{COV}(X_i, X_j) = \sigma_{ij}(i \neq j) = \rho_{ij} \sigma_i \sigma_j \), where \( \rho_{ij} \) is the correlation coefficient between the inputs \( X_i \) and \( X_j \). In the field of Statistics, \( \mu_i \)'s are also referred to as the population first origin moments, and \( \sigma_n \)'s are called the population second central moments. Let the characteristic of a mixture (such as %-passing through a sieve) having n aggregates be denoted by \( Y_n = \sum_{i=1}^{n} W_i X_i \), where the proportions \( W_i \)'s are random variables with also known first two moments. The primary objective of this paper is to obtain the first two moments of the output \( Y_n \) under all different scenarios based on the nature of \( W_i \)'s and their relationships to \( X_i \)'s. At least half of the following developments have been well known in statistical literature for many years, but are repeated herein only for the sake of completeness. Further, the developments are presented in the order of the simplest to the most complicated, where \( W_i \)'s and \( X_i \)'s are correlated variates and also pair-wise correlated together.
2. The Trivial Case Where \( W_i \)'s = \( w_i \)'s Are Known Constants.

In this case, the output \( Y_n \) reduces to \( \sum_{i=1}^{n} w_i X_i \) and is referred to as a linear combination (LC). Thus, we have complete information about the first two moments of the \( n \) inputs \( X_i \)'s, and the objective is to use them to compute the first two moments of the linear output \( Y_n \). Such LCs occur frequently in industrial applications and also in the field of Statistics (the simplest of all examples is the case of sample mean \( Y_n = \bar{X} \), which is a LC with each \( w_i = 1/n \)). In this simple case, complete information about the first two moments of \( Y_n \) has been known in statistical literature since the 18th century and has been reported in numerous sources, and are listed below.

\[ \mu(Y_n) = \mathbb{E}(Y_n) = \sum_{i=1}^{n} w_i \mu_i, \quad (1a) \]

where \( \mathbb{E} \) represents the linear Expected-Value operator throughout this paper. Eq. (1a) shows that the mean of the mixture \( \mathbb{E}(Y_n) \) is the same LC of \( \mu_i \)'s as \( Y_n \) is of \( X_i \)'s. The variance \( \mathbb{V}(Y_n) = \sigma^2(Y_n) \), whose expression is also given in numerous sources, can be computed by applying the nonlinear Variance-Operator \( \mathbb{V} \) and is also provided below.

\[ \mathbb{V}(Y_n) = \sigma^2(Y_n) = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j>i}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_i \sigma_j \rho_{ij}, \quad (1b) \]

where \( \sigma_{ij} = \mathbb{E} \left[ (X_i - \mu_i)(X_j - \mu_j) \right] \), \( \rho_{ij} = \sigma_{ij} / (\sigma_i \sigma_j) \), and variances \( \mathbb{V}(X_i) = \sigma_{ii} = \sigma_i^2 \), \( i = 1, 2, ..., n \) are all known parameters. If \( X_1, X_2, ..., X_n \) are stochastically independent, then \( \sigma_{ij} \) in equation (1b) for all \( i \neq j \) is identically zero, and as a result the \( \mathbb{V}(Y_n) \) reduces to \( \sum_{i=1}^{n} w_i^2 \sigma_i^2 \).

For example, if \( Y \) is the mean of a random sample from an infinite population, then this last formula yields the very well-known expression for the variance of the mean as \( \mathbb{V}(\bar{X}) = \sigma^2/n \), where \( \sigma^2 \) is the variance of individuals in the target population. Further, if \( X_i \)'s are also normally distributed (besides being jointly independent), then \( Y_n \) is also normally distributed and expressed as \( N(\sum_{i=1}^{n} w_i \mu_i, \sum_{i=1}^{n} w_i^2 \sigma_i^2) \). However, if \( X_i \)'s are correlated (i.e., \( \sigma_{ij} \neq 0 \) for \( i \neq j \)) and are also normally distributed, then from statistical theory the linear combination \( Y_n = \)
\[
\sum_{i=1}^{n} w_i X_i \text{ is still normally distributed (or Laplace-Gaussian) with } E(Y_n) = \sum_{i=1}^{n} w_i \mu_i \text{ and } V(Y_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}.
\]

The simple LC, \(Y_n = \sum_{i=1}^{n} w_i X_i\), occurs frequently in many industrial applications, and specifically in the design and production of asphalt paving mixtures, where a laboratory combines aggregates from \(n\) stockpile samples in such a manner that the blend of aggregates meets certain design specifications. The aggregate characteristics, \(X_i's\), from each stockpile have known mean and variance based on quality control tests on samples obtained from the stockpiles. A specific example is provided below.

**Example 1.** Suppose an asphalt mixture (or a job-mix formula) is designed to contain aggregates from \(n = 3\) stockpiles, having means \(E(X_1) = \mu_1 = 35.1, \mu_2 = E(X_2) = 46.4\), and \(\mu_3 = 62.8\%\) passing through the 2.36mm-sieve, and variances \(V(X_1) = \sigma_{11} = 8.60, V(X_2) = \sigma_{22} = 16.40\), and \(V(X_3) = \sigma_{33} = 12.96\%^2\). The percentages passing certain sieve sizes are key characteristics used in the design and control of asphalt paving mixtures. For the design of asphalt paving mixtures in a laboratory environment, the proportion \(w_i (i = 1, 2, 3)\) from each of the 3 stockpiles can be exactly controlled by the mix designer. Thus, for a laboratory Job Mix Formula (JMF), unlike a plant operation, we can assume that the proportions from each stockpile can be controlled and are not random variables. Assuming that \(w_1 = 0.40, w_2 = 0.35,\) and \(w_3 = 0.25\), the characteristic of interest for the combined blend is obtained from the LC: \(Y_3 = 0.40X_1 + 0.35X_2 + 0.25X_3\). Then, for the combined \% passing the 2.36mm-sieve, equations (1a & b) yield the mean \(E(Y_3) = 0.40\times35.1 + 0.35\times46.4 + 0.25\times62.8 = 45.98\%,\) and variance \(V(Y_3) = 0.40^2\times8.60 + 0.35^2\times16.40 + 0.25^2\times12.96 = 4.195 \rightarrow \sigma(Y_3) = 2.05\% \rightarrow\) The coefficient of variation (or variation coefficient) of \(Y_3\) is given by \(CV(Y_3) = 2.05/45.98 = 4.45\%.\) Thus, assuming that the \(X_i's\) are normally distributed, then the overall \% passing the 2.36mm-sieve has a sampling distribution which is normal with process mean 45.98\% and process variance 4.195\%\(^2\), designated as \(N(45.98, 4.195)\).

It should be highlighted that the output, \(Y_n\), is this paper is not the same as in the classical mixture experiments, where \(X_i's\) themselves are proportion of \(n\) ingredients that
constitute a mixture so that \( \sum_{i=1}^{n} X_i \) is constrained to equal 1 (or 100%). The mere objective in statistical mixture designs is to identify \((n-1)\) of the \(X_i\)'s in such a manner that some characteristic of the final mixture is optimized. Cornell (2002) provides an example of a mixture experiment where \(n = 3\) ingredients \(X_1 = \text{proportion of Polyethylene}, X_2 = \text{proportion of Polystyrene},\) and \(X_3 = \text{proportion of Propylene},\) were blended to form fiber that would be spun into yarn for draperies. The objective of this mixture experiment was to determine the approximate values of \(X_1, X_2,\) (and by necessity \(X_3\)) such that the resulting yarn elongation, measured in kilograms of force applied, was maximized, bearing in mind that the constraint \(X_1 + X_2 + X_3 \equiv 1\) among the 3 ingredients must be satisfied. While in this paper, as illustrated in Example 1 above, the output \(Y_n\) represents the overall characteristic of a mixture, comprised of aggregates with differing known proportions from \(n\) stockpiles with constraint \(\sum_{i=1}^{n} w_i \equiv 1.\)

(3) The Case Where \(W_i\)'s Are Correlated Random Variables With Known First Two Moments but Independent of Correlated \(X_i\)'s

Before we formulate results for the sum of products of \(n\) random variables given by \(Y_n = \sum_{i=1}^{n} W_i X_i\), we first obtain the mean and variance of the product of two independent random variables, which again has been known in statistical literature for well over 60 years [e.g., see Kapur and Lamberson (1977, p. 99)] but repeated below for completeness.

Let \(W_1\) and \(X_1\) be two independent random variables with known process means \(\xi_1, \mu_1,\) and known process variances \(\omega_{11} = \omega_1^2\) and \(\sigma_{11} = \sigma_1^2,\) respectively. Let \(Y_1 = W_1 X_1;\) our objective is to obtain the expected-value (or mean) and the variance of the random variable \(Y_1,\) assuming that \(W_1\) and \(X_1\) are independent.

\[
E(Y_1) = E(W_1 X_1) = E(W_1)E(X_1) = \xi_1 \mu_1 \tag{2a}
\]

\[
V(Y_1) = E[(W_1 X_1 - \xi_1 \mu_1)^2] = E[(W_1 X_1 - W_1 \mu_1 + W_1 \mu_1 - \xi_1 \mu_1)^2] = E[W_1 (X_1 - \mu_1) + \mu_1 (W_1 - \xi_1)]^2 \\
= E(W_1^2 (X_1 - \mu_1)^2) + \mu_1^2 E(W_1 - \xi_1)^2 + 2 \mu_1 E[W_1 (X_1 - \mu_1)(W_1 - \xi_1)] \\
= E(W_1^2) \sigma_1^2 + \mu_1^2 \omega_1^2 + 2 \mu_1 E(X_1 - \mu_1) E[W_1 (W_1 - \xi_1)] \\
= (\omega_1^2 + \xi_1^2) \sigma_1^2 + \mu_1^2 \omega_1^2 + 0 \\
= \omega_1^2 \sigma_1^2 + \xi_1^2 \sigma_1^2 + \mu_1^2 \omega_1^2 \tag{2b}
\]
In the above developments leading to Eqs. (2a & b), we have used the well-known fact that the expected-value of a product of two independent random variables is equal to the product of their expectations, i.e., \( E(W_1X_1) = E(W_1)E(X_1) \). However, the converse of this last statement is not necessarily true; in other words, the equality \( E(W_1X_1) = E(W_1)E(X_1) \) may hold, but still the two random variables may not be stochastically independent. In the case of a bivariate normal random vector \([W_1 \quad X_1]'\), the equality \( E(W_1X_1) = E(W_1)E(X_1) \) does guarantee that the random components \( W_1 \) and \( X_1 \) are independent. The probability density function (pdf) of \( Y_1 = W_1X_1 \), when the two variates are independent and normal, is provided by Springer (1979, p.136) and also has been studied by other authors such as C. C. Craig (1936) and L. A. Aroian (1947). For the product \( Y_1 = W_1X_1 \), the density function can be obtained by inserting \( n = 2 \) into the formula (4.6.30) of Springer (1979, p. 136), but it does not simplify to an easily applicable form. Thus, the reader must be cognizant of the fact the product of two independent normal random variables is not at all normally distributed, and its exact pdf has a very complicated form as provided by Springer (1979, p. 136).

Now consider the general nonlinear combination, NLC, \( Y_n = \sum_{i=1}^{n} W_iX_i \), where \( W_i \)'s have known covariances \( \omega_{ij} \) (i \( \neq \) j) but are stochastically independent of all \( X_i \)'s, and \( X_i \)'s have also known covariances \( \sigma_{ij} \) (i \( \neq \) j). Further, the 1st two moments are also known (or can be estimated accurately) and given by \( E(W_i) = \xi_i, \ E(X_i) = \mu_i, \ V(W_i) = \omega_{ii} = \omega_i^2 \) and \( V(X_i) = \sigma_{ii} = \sigma_i^2 \), i = 1, 2, ..., n. As before, our objective is to obtain the first two moments of the output \( Y_n \) using the known first origin moments of \( W_i \)'s, \( X_i \)'s, and known covariance structures \( \omega_{ij} \) and \( \sigma_{ij} \).

The first origin moment (or the mean) of \( Y_n \) is easily obtained by applying the linear expected-value operator to \( Y_n \).

\[
E(Y_n) = E(\sum_{i=1}^{n} W_iX_i) = \sum_{i=1}^{n} E(W_iX_i) = \sum_{i=1}^{n} E(W_i)E(X_i) = \sum_{i=1}^{n} (\xi_i \times \mu_i) \quad (3)
\]

The second central moment of \( Y_n \) can be obtained by applying the nonlinear variance-operator \( V \) to \( Y_n \).

\[
V(Y_n) = V(\sum_{i=1}^{n} W_iX_i) = \sum_{i=1}^{n} V(W_iX_i) + 2 \sum_{i=1}^{n-1} \sum_{j>i}^{n} \text{COV}(W_iX_i, W_jX_j) \quad (4)
\]

However, by definition

\[
\text{COV}(W_iX_i, W_jX_j) = E(W_iX_iW_jX_j) - \xi_i\mu_i\xi_j\mu_j = E(W_iW_j) \times E(X_iX_j) - \xi_i\xi_j\mu_i\mu_j
\]
\[ = (\omega_{ij} + \zeta_i \zeta_j) \times (\sigma_{ij} + \mu_i \mu_j) - \zeta_i \zeta_j \mu_i \mu_j = \omega_{ij} \sigma_{ij} + \mu_i \mu_j \omega_{ij} + \zeta_i \zeta_j \sigma_{ij} \]  

(5)

Substituting Eq. (5) into (4) and in light of (2b) results in

\[
V(Y_n) = \sum_{i=1}^{n} (\omega_{ii} \sigma_{ii} + \zeta_{i}^{2} \sigma_{ii}^{2} + \mu_{i}^{2} \omega_{ii}) + 2 \sum_{i=1}^{n-1} \sum_{j>i}^{n} (\omega_{ij} \sigma_{ij} + \mu_{i} \mu_{j} \omega_{ij} + \zeta_{i} \zeta_{j} \sigma_{ij}) 
\]

(6)

One special case of Eq. (6) that occurs frequently in the production control of asphalt paving mixtures is when aggregate characteristics from \( n \) stockpiles (i.e., \( X_i \)'s) are independent (that is \( \sigma_{ij} = 0 \) for all \( i \neq j \)), but the feed rates from each stockpile into a mixing-drum are constrained such that the variable proportions from the \( n \) stockpiles that enter the mixing-drum add to 1, i.e.,

the constraint \( \sum_{i=1}^{n} W_i = 1 \) must be satisfied. Thus, in this special case, Eq. (3) still holds true except for the fact that \( \xi_n = E(W_n) = 1 - \xi_1 - \xi_2 - \ldots - \xi_{n-1} \), while Eq. (6) reduces to

\[
V(Y_n) = \sum_{i=1}^{n} (\omega_{ii} \sigma_{ii} + \zeta_{i}^{2} \sigma_{ii}^{2} + \mu_{i}^{2} \omega_{ii}) + 2 \sum_{i=1}^{n-1} \sum_{j>i}^{n} \mu_{i} \mu_{j} \omega_{ij} 
\]

(7)

Due to the constraint \( \sum_{i=1}^{n} W_i = 1 \), Eq. (7) further reduces to a special form, which is proven below by starting with \( Y_2 = W_1X_1 + W_2X_2 \), bearing in mind that \( W_1 + W_2 = 1 \).

For \( n = 2 \) stockpiles, \( \omega_{22} = V(W_2) = V(1-W_1) = V(W_1) = \omega_{11} \), and \( \omega_{12} = \text{COV}(W_1, W_2) = E[(W_1-\xi_1)\times(W_2-\xi_2)] = E[(W_1-\xi_1)\times(1-W_1-1 + \xi_1)] = -E[(W_1-\xi_1)\times( W_1-\xi_1)] = -\omega_{11} \).

Thus, when \( n = 2 \), Eq. (7) reduces to

\[
V(Y_2) = 2 \sum_{i=1}^{2} (\omega_{ii} \sigma_{ii} + \zeta_{i}^{2} \sigma_{ii}^{2} + \mu_{i}^{2} \omega_{ii}) + 2 \mu_{1} \mu_{2} \omega_{12} = 
\]

\[
\sum_{i=1}^{2} (\omega_{ii} \sigma_{ii} + \zeta_{i}^{2} \sigma_{ii}^{2} + \mu_{i}^{2} \omega_{ii}) - 2 \mu_{1} \mu_{2} \omega_{11} \]. \]  

Substituting \( \omega_{22} = V(W_2) = \omega_{11} \) and \( \xi_2 = E(W_2) = 1 - \xi_1 \) into this last expression and combining common terms yields

\[
V(Y_2) = \sum_{i=1}^{2} \zeta_{i}^{2} \sigma_{ii}^{2} + [(\mu_{1}-\mu_{2})^{2} + (\sigma_{11}+\sigma_{22})] \times \omega_{11} 
\]

(8)

Eq. (8) shows that if an asphalt mixture in a plant-operation is blended from two stockpiles with variable feed rates such that \( W_1 + W_2 = 1 \), then the exact variance of any characteristic of the mixture is given by Eq. (8). Bonaquist and Christensen (2008) report the following equation for the variance of a two-stockpile mixture characteristic, \( m = \alpha a + (1-\alpha)b \), as

\[
V(m) = \alpha^{2} \sigma_{a}^{2} + (1 - \alpha)^{2} \sigma_{b}^{2} + (\bar{X}_{a} - \bar{X}_{b})^{2} \sigma_{a}^{2} 
\]
In our notation, \( n = 2, \ m = Y_2, \ \alpha = W_1, \ 1-\alpha = W_2, \ a = X_1, \ b = X_2, \ \bar{X}_a = E(X_1) = \mu_1, \ \text{and} \ \bar{X}_b = E(X_2) = \mu_2. \) Note that the Bonaquist and Christensen formula for \( V(m) \) is an approximation to our Eq. (8) because the last two terms, \((\sigma_{11} + \sigma_{22})\times\omega_{11},\) are left out of \( V(m) \). However, the last two terms of Eq. (8), \((\sigma_{11} + \sigma_{22})\times\omega_{11},\) are small relative to the other 3 terms unless the \( CV(X_1) = \sigma_1/\mu_1, \ CV(X_2) = \sigma_2/\mu_2, \ \text{and} \ CV(W_1) = \omega_1/\xi_1 \) all exceed 30%. It can be shown, see Appendix 1, that the \( V(Y_2) \) given by Eq. (8), where \( n = 2 \) stockpiles, generalizes to our main result

\[
V(Y_n) = \sum_{i=1}^{n} (\xi_i^2 + \omega_{ii})\sigma_{ii} + \sum_{i=1}^{n-1} (\mu_i - \mu_n)^2\omega_{ii},
\]  

(9a)

for \( n > 2 \) stockpiles. Further, if we use the approximation recommended by Bonaquist and Christensen (2008) for \( n = 2 \) stockpiles, then Eq. (9a) further reduces to

\[
V(Y_n) \approx \sum_{i=1}^{n} \xi_i^2\sigma_{ii} + \sum_{i=1}^{n-1} (\mu_i - \mu_n)^2\omega_{ii}.
\]  

(9b)

Eqs. (9 a & b) assume that only \((n-1)\) out of the random proportions \( W_1, W_2, \ldots, W_n \) are independent due to the constraint \( \sum_{i=1}^{n} W_i = 1, \) i.e., the there are only \((n-1)\) degrees of freedom among the variates \( W_1, W_2, \ldots, W_n. \) Clearly, the stockpile designated as \( n \) impacts the variance given in Eqs. (9a &b). If the user denotes either the stockpile with maximum (or minimum) characteristic as \( n, \) then \( V(Y_n) \) of Eqs. (9) attains its near maximum (or conservative) value. Example 2 below provides an application of the special case of \( V(Y_n) \) as provided by equations (9 a & b).

**Example 2.** Suppose an asphalt plant produces a paving mixture from \( n = 3 \) stockpiles with the same parameter values as in Example 1, i.e., \( E(X_1) = \mu_1 = 35.1, \mu_2 = E(X_2) = 46.4, \mu_3 = 62.8\%, \) \( V(X_1) = \sigma_{11} = 8.60, \) \( V(X_2) = \sigma_{22} = 16.40, \) and \( V(X_3) = \sigma_{33} = 12.96\%^2. \) However, the feed rates cannot be exactly controlled such that \( CV(W_i) = \omega_i/\xi_i = 15\% \) for \( i = 1, \) \( 2, \) \( 3, \) but \( W_1 + W_2 + W_3 = 1 \) at any point in time during the process with \( \xi_1 = E(W_1) = 0.40, \xi_2 = 0.35 \) and \( \xi_3 = E(W_3) = 0.25. \) As noted above, \( E(Y_3) \) remains unaffected and remains as \( E(Y_3) = 45.98\% \) passing the 2.36mm- sieve, but the variance now must be computed from Eq. (9a).

Because the coefficient of variation of each \( W_i \) is assumed to be 15%, then \( \xi_1 = 0.40, \xi_2 = 0.35, \xi_3 = E(W_3) = 0.25 \) imply that \( \omega_{11} = V(W_1) = (0.15\times0.40)^2 = 0.0036, \omega_{22} = (0.15\times0.35)^2 = 0.0028, \text{and} \omega_{33} = (0.15\times0.25)^2 = 0.0014. \) Substituting \( \omega_{ii}'s \) \((i = 1, 2, 3)\) and \( \mu_1 = 35.1, \mu_2 = 46.4, \mu_3 = 62.8\%, \sigma_{11} = 8.60, \sigma_{22} = 16.40, \sigma_{33} = 12.96 \) into Eq. (9a) results in \( V(Y_3) = V(W_1X_1 + \)
W_2X_2 + W_3X_3) = 7.805356, \sigma(Y_3) = 2.79381, and CV(Y_3) = 6.08\%. As expected, if W_i’s are random variables, then \sigma(Y_3) of Example 1 increases from 2.050 \% to 2.794\%. This 36.283\% increase clearly depends on the CV(W_i); e.g., at CV(W_i) = 10\%, \sigma(Y_3) = 2.4071\%, which is an increase of 17.42\%. Further, if we use the approximation of Eq. (9b), used by Bonaquist and Christensen (2008), then at CV(W_i) = 15\%, \sigma(Y_3) \approx 2.77675\% (compared to the exact 2.794\%), showing a very good approximations to our exact value. It should be highlighted that in general the in-plant variability for X_i’s are larger than those in the lab. In Example 2 we assumed the same variability for X_i’s as those of Example 1 for the mere purpose of accessing the increase in the V(Y_3) when W_i’s are random variables.

4. The Mean and Variance of Product of Two Correlated Random Variables

Consider the product Y_1 = W_1X_1, where \nu_{11} = COV(W_1, X_1) = \omega_1 \sigma_{W_1} \sigma_{X_1} \neq 0 is known and the objective is to compute the mean and variance of the product Y_1 = W_1X_1. By definition, COV(W_1, X_1) = E(W_1X_1) - \xi_1 \mu_1, and thus

\[ E(Y_1) = E(W_1X_1) = \xi_1 \mu_1 + \nu_{11} \quad (10) \]

Because W_1 and X_1 are not independent, then the V(Y_1) is no longer given by Eq. (2a) as illustrated below.

\[ V(W_1X_1) = E[(W_1X_1)^2] - E[(W_1X_1)]^2 = E(W_1^2X_1^2) - (\xi_1 \mu_1 + \nu_{11})^2 \quad (11) \]

The 1st term on the RHS of Eq. (11) cannot be exactly computed unless the COV(W_1^2, X_1^2) is known. Therefore, we resort to a Taylor’s expansion of any function f(W_1, X_1) about \xi_1 and \mu_1:

\[ f(W_1, X_1) = f(\xi_1, \mu_1) + \frac{\partial f}{\partial W_1}(\xi_1, \mu_1)(W_1 - \xi_1) + \frac{\partial f}{\partial X_1}(\xi_1, \mu_1)(X_1 - \mu_1) + \]

\[ \frac{1}{2} \frac{\partial^2 f}{\partial W_1^2}(\xi_1, \mu_1)(W_1 - \xi_1)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial X_1^2}(\xi_1, \mu_1)(X_1 - \mu_1)^2 + \]

\[ \frac{\partial^2 f}{\partial W_1 \partial X_1}(\xi_1, \mu_1)(W_1 - \xi_1)(X_1 - \mu_1) + R(W_1, X_1) \quad (12) \]
where $R(W_1, X_1)$ is of order 3 or higher. Because in our special case $f(W_1, X_1) = W_1X_1$, then its Taylor’s expansion from Eq. (12) reduces to

$$Y_1 = W_1X_1 = \xi_1\mu_1 + \mu_1(W_1 - \xi_1) + \xi_1(X_1 - \mu_1) + (W_1 - \xi_1)(X_1 - \mu_1).$$  \hfill (13)

Note that in the special case of $f(W_1, X_1) = W_1X_1$ the Taylor expansion in (12) is an exact identity. In order to obtain the mean of $W_1X_1$, we apply the expected-value operator to both sides of Eq. (13).

$$\text{E}(W_1X_1) \approx \xi_1\mu_1 + 0 + 0 + \text{E}[ (W_1 - \xi_1)(X_1 - \mu_1) ] = \xi_1\mu_1 + \text{COV}(W_1, X_1) = \xi_1\mu_1 + \nu_{11}$$ \hfill (14)

The mean of $W_1X_1$ given in Eq. (14) is identical to that of Eq. (10), as expected.

In order to approximate the variance of $Y_1 = W_1X_1$, we apply the variance operator to Eq. (13) and ignore the last order-2 term. Thus,

$$\text{V}(W_1X_1) \approx \text{V}[ \mu_1(W_1 - \xi_1) + \xi_1(X_1 - \mu_1) ]$$

$$\approx \mu_1^2 \text{V}(W_1 - \xi_1) + \xi_1^2 \text{V}(X_1 - \mu_1) + 2\text{COV}[\mu_1(W_1 - \xi_1), \xi_1(X_1 - \mu_1)]$$

$$\approx \mu_1^2 \omega_{11} + \xi_1^2 \sigma_{11} + 2\xi_1\mu_1\text{COV}(W_1, X_1) = \mu_1^2 \omega_{11} + \xi_1^2 \sigma_{11} + 2\xi_1\mu_1\nu_{11}$$ \hfill (15)

The approximate $\text{V}(W_1X_1)$ in Eq. (15) is fairly close to the exact $\text{V}(Y_1) = \xi_1^2 \sigma_1^2 + \mu_1^2 \omega_1^2 + \omega_1^2 \sigma_1^2$ given in Eq. (2b) for the case when $W_1$ and $X_1$ are independent. Unfortunately, the approximation in Eq. (15) does not reduce to the exact result of $\xi_1^2 \sigma_1^2 + \mu_1^2 \omega_1^2 + \omega_1^2 \sigma_1^2$ when $W_1$ and $X_1$ are independent for which $\nu_{11} = 0$ (because the Taylor expansion was truncated).

However, when both $\omega_1/\xi_1$ and $\sigma_1/\mu_1$ are less than 30%, the product $\omega_1^2 \sigma_1^2 < (0.30 \xi_1)^2 (0.30 \mu_1)^2 = 0.0081 \xi_1^2 \mu_1^2$ so that $\omega_1^2 \sigma_1^2$ is much smaller than either $\mu_1^2 \omega_1^2$ or $\xi_1^2 \sigma_1^2$, and thus, the approximation in (15) is fair agreement with Eq. (2b). For the worst-case scenario of $\text{CV} \geq 30\%$, Eq. (15) further shows that the $\text{V}(Y_1) = \text{V}(W_1X_1)$ is an increasing function of the process correlation coefficient, $\rho_{W_1,X_1}$, between $W_1$ and $X_1$.

Now consider the most general output $Y_n = \sum_{i=1}^n W_iX_i$, where all the $2 \times n$ random variables are correlated with covariance structure
\[
\Sigma_W = \begin{bmatrix}
\omega_{11} & \omega_{12} & \omega_{13} & \cdots & \omega_{1n} \\
\omega_{21} & \omega_{22} & \omega_{23} & \cdots & \omega_{2n} \\
\omega_{31} & \omega_{32} & \omega_{33} & \cdots & \omega_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\omega_{n1} & \omega_{n2} & \omega_{n3} & \cdots & \omega_{nn}
\end{bmatrix}, \quad \text{where } \omega_{ij} = E[(W_i - \xi_i) \times (W_j - \xi_j)] \text{ represents the covariance}
\]

between \(W_i\) and \(W_j\). Similarly, \(X_i\)'s are correlated random variables with means \(E(X_i) = \mu_i\) and covariance structure

\[
\Sigma_X = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1n} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} & \cdots & \sigma_{2n} \\
\sigma_{31} & \sigma_{32} & \sigma_{33} & \cdots & \sigma_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \cdots & \sigma_{nn}
\end{bmatrix}, \quad \text{and } W_i \text{ and } X_i \text{ also are correlated with covariance structure } \Sigma_{WX}
\]

\[
= \begin{bmatrix}
\nu_{11} & \nu_{12} & \cdots & \nu_{1n} \\
\nu_{21} & \nu_{22} & \cdots & \nu_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\nu_{n1} & \nu_{n2} & \cdots & \nu_{nn}
\end{bmatrix}, \quad \text{then we have the following } 2^{\text{nd}} \text{-order approximation for the mean of } Y_n:
\]

\[
E(Y_n) \approx \sum_{i=1}^{n} \xi_i \mu_i + \sum_{i=1}^{n} \nu_{ii}
\]  \hspace{1cm} (16a)

And rough 1\textsuperscript{st}-order approximation for the variance of \(Y_n\) is given by

\[
V(Y_n) \approx \sum_{i=1}^{n} (\mu_i^2 \omega_{ii} + \xi_i^2 \sigma_{ii}) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\mu_i \mu_j \omega_{ij} + \xi_i \xi_j \sigma_{ij}) + 2 \sum_{i=1}^{n} \sum_{j=i}^{n} \mu_i \xi_j \nu_{ij}. \hspace{1cm} (16b)
\]

5. Concluding Remarks

This article has first generalized the known approximate result for the variance of a mixture characteristic having two ingredients to the case of more than \(n = 2\) ingredients. Our Eq. (9a) is an exact formula and (9b) is the corresponding approximation for the practitioner. Secondly, Eqs. (16a\&b) give the approximate formulas for the mean and variance, respectively, of an output \(Y_n = \sum_{i=1}^{n} W_i X_i\) under the most general case that the \(2n\) random variables \(W_i\) and \(X_i\) are correlated.
APPENDIX 1

The Derivation of $V(\sum_{i=1}^{n} W_i X_i)$, for $n > 2$, under the constraint $\sum_{i=1}^{n} W_i = 1$ and the Assumption of Independent $X_i$'s

The constraint $\sum_{i=1}^{n} W_i = 1$ implies that $E(W_n) = \xi_n = 1 - \sum_{i=1}^{n-1} \xi_i$, and $V(W_n) = \omega_{nn} =$

$$V(1-W_1-W_2-W_3-\ldots-W_{n-1}) = \sum_{i=1}^{n-1} \omega_n.$$ Further, $W_1, W_2, \ldots, W_{n-1}$ are jointly independent but $W_1, W_2, \ldots, W_{n-1}$ are correlated with $W_n$, i.e., $\omega_{in} = COV(W_i, W_n) = COV(W_i, 1-W_1-W_2-\ldots-W_{n-1}) = -COV(W_1, W_1+W_2+\ldots+W_{n-1}) = -V(W_1) = -\omega_{111}$. Similarly, $\omega_{in} = -\omega_{ii}$ for all $i = 1, 2, \ldots, n-1$. The use of Eq. (9) leads to

$$V(Y_n) = V(\sum_{i=1}^{n} W_i X_i) = \sum_{i=1}^{n} (\omega_{ii} \sigma_{ii}^2 + \xi_i^2 \sigma_{ii}^2 + \mu_i^2 \omega_{ii}) + 2 \sum_{i=1}^{n-1} \sum_{j>i}^{n} \mu_i \mu_j \omega_{ij} = \sum_{i=1}^{n} (\omega_{ii} \sigma_{ii}^2 + \xi_i^2 \sigma_{ii}^2) +$$

$$\sum_{i=1}^{n} \mu_i^2 \omega_{ii} + 2 \sum_{i=1}^{n-1} \sum_{j>i}^{n} \mu_i \mu_j \omega_{ij}.$$ Substituting $\omega_{nn} = \sum_{i=1}^{n-1} \omega_i$ and $\omega_{ij} = COV(W_i, W_j) = 0$ for all $i \neq j$ into the last formula, we obtain

$$V(Y_n) = \sum_{i=1}^{n} (\omega_{ii} \sigma_{ii}^2 + \xi_i^2 \sigma_{ii}^2) + \sum_{i=1}^{n-1} \mu_i^2 \omega_{ii} + \mu_n^2 \omega_{nn} + 2 \sum_{i=1}^{n-1} \mu_i \mu_n \omega_{ii}$$

$$= \sum_{i=1}^{n} (\omega_{ii} \sigma_{ii}^2 + \xi_i^2 \sigma_{ii}^2) + \sum_{i=1}^{n-1} \mu_i^2 \omega_{ii} + \mu_n^2 \sum_{i=1}^{n-1} \omega_{ii} - 2 \sum_{i=1}^{n-1} \mu_i \mu_n \omega_{ii}$$

$$= \sum_{i=1}^{n} (\omega_{ii} \sigma_{ii}^2 + \xi_i^2 \sigma_{ii}^2) + \sum_{i=1}^{n-1} (\mu_i^2 + \mu_n^2 - 2 \mu_i \mu_n) \omega_{ii}$$

$$= \sum_{i=1}^{n} (\omega_{ii} + \xi_i^2) \sigma_{ii}^2 + \sum_{i=1}^{n-1} \mu_i^2 \omega_{ii},$$

which completes the proof.
References