

The Logic Behind the ANOVA for Exercise 37 on page 395 of Devore's 7th Edition **Maghsoodloo**

In Exercise 37 on page 395 of Devore's 7th edition) there are 5 treatments (or 5 independent population Brands), whose sample means are to be compared simultaneously in order to test $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu$, where μ represents the grand population average of all 5 processes (or treatments). For example, the data layout shows that $y_{24} = 14.9$ microns (μ) while $y_{43} = 14.4$, etc; further, there are $n = 6$ random observations from each level (or each population). Hence, $N = 5 \times 6 = 30$, and there are a total of 29 degrees of freedom in the entire experiment. In the following development, I will first break down the Total SS (Sum of Squares) into two orthogonal (i.e., additive) components : (1) due to differences within the five samples (or within subgroups), (2) due to differences between the 5 sample means. The Total SS, denoted SS_T , will be obtained from the deviations of individual observations from the grand sample means $\bar{y}_{..} = \frac{y_{..}}{30} = \frac{426.7}{30} = 14.2233\bar{3}$, while within SS will be obtained from the 6 observations in the i^{th} treatment (or brand) from the corresponding means $\bar{y}_{i.}$, $i = 1, 2, 3, 4, 5$.

$$\begin{aligned}
 SS(\text{Total}) &= \sum_{i=1}^{5 \text{ brands}} \sum_{j=1}^{n=6} (y_{ij} - 14.2233\bar{3})^2 \equiv \sum_{i=1}^5 \sum_{j=1}^{n=6} [(y_{ij} - \bar{y}_{i.}) + (\bar{y}_{i.} - \bar{y}_{..})]^2 = \dots \\
 &\equiv \sum_{i=1}^5 \sum_{j=1}^{n=6} [(y_{ij} - \bar{y}_{i.})^2] + \sum_{i=1}^5 \sum_{j=1}^{n=6} [(\bar{y}_{i.} - \bar{y}_{..})^2] = \\
 &\equiv \sum_{i=1}^5 \sum_{j=1}^{n=6} [(y_{ij} - \bar{y}_{i.})^2] + \sum_{i=1}^5 6[(\bar{y}_{i.} - \bar{y}_{..})^2] \equiv \sum_{i=1}^5 \sum_{j=1}^{n=6} e_{ij}^2 + 6 \sum_{i=1}^5 \hat{\tau}_i^2 = \\
 &\equiv SS(\text{Within the 5 levels}) + SS(\text{Between Brand Means}) \\
 SS_T &\equiv SS(\text{Residuals}) + SS(\text{Treatments}) = SS_{\text{RES}} + SS_{\text{Tr}}
 \end{aligned}$$

Or:

53.69366667 (with 29 df) = 22.83833333 (with 5×5 df) + 30.85533333 (with 4 df)

Further, $e_{ij} = y_{ij} - \bar{y}_{i.}$ is called the ij^{th} residual while $\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..}$ is called the effect of the i^{th} treatment (or Brand).

There are two main assumptions in Fixed-Effects ANOVA: (1) y_{ij} 's \sim NID(μ_i, σ_i^2),

(2) $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \dots = \sigma_a^2 = \sigma_\epsilon^2 = \sigma^2$, where "a" represents the number of treatments.

The assumption 2 above implies that we should pool our sample variances

given by $S_i^2 = \frac{1}{5} \sum_{j=1}^{n=6} [(y_{ij} - \bar{y}_{i.})^2] = \frac{\text{CSS}_i(\text{Within})}{5} = 1.425667, 1.363000,$

0.666667, 0.882667, 0.229667 ($i = 1, 2, 3, 4, 5$) from the 5 independent

populations to obtain one overall estimate of error variance $\sigma_\epsilon^2 = \sigma^2 = \sigma_y^2$.

That is, from the within treatments we may estimate σ_y^2 from $\hat{\sigma}_y^2 =$

$$\frac{\sum_{i=1}^5 \text{CSS}_i(W)}{5 \times 5} = \frac{7.12833\bar{3} + 6.8150 + 3.3333\bar{3} + 4.41333\bar{3} + 1.148333\bar{3}}{25} = \frac{22.838333\bar{3}}{25} = 0.91353333 \text{ with } 25 \text{ df.}$$

Another estimate of error variance

may be obtained by first estimating the variance between the five

population means from $\hat{\sigma}_{\bar{y}}^2 = \frac{1}{4} \sum_{i=1}^5 [(\bar{y}_{i.} - \bar{y}_{..})^2] = \frac{\sum_{i=1}^5 \hat{\tau}_i^2}{4} = \frac{5.14255\bar{5}}{4} =$

1.2856388889 with 4 df. However, forming the F statistic as the ratio of

$\frac{\hat{\sigma}_{\bar{y}}^2}{\hat{\sigma}_y^2}$ would be exactly like comparing "Apples and Oranges" because the

numerator estimates the variance between means while the denominator

estimates the variance amongst the individual measurements within the same treatment! Therefore, we convert the variance of means in the numerator of $\hat{\sigma}_{\bar{y}}^2 / \hat{\sigma}_y^2$ by multiplying it by the size of each sample $n = 6$ in order to convert $\hat{V}(\bar{y})$ to $\hat{V}(y)$. This implies that another estimate of σ_y^2 may be obtained from $n \times \hat{\sigma}_{\bar{y}}^2 = 6 \times 1.2856388889 = 7.713833333$. Hence, the

correct value of the F statistic is $F_0 = \frac{7.713833333}{0.91353333} =$

$\frac{MS(\text{Between Samples})}{MS(\text{Within Samples})} = 8.443954$, and the corresponding Pr level of

testing $H_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu$ is given by $P\text{-value} = P(F_{4, 25} \geq 8.443954) = 0.00018715$. Since this $P\text{-value}$ is much smaller than 5%, we can strongly reject $H_0: \tau_i = 0$ ($i = 1, 2, \dots, a$) and conclude that at least two of the 5 population means differ significantly. This implies that the 5 treatments (or Brands) have a statistically significant impact on the dependent variable “Motor Vibration”. Then, Tukey’s Post-ANOVA must be used next to ascertain where the exact differences are.

Tukey’s Post-ANOVA For Fixed-Effects only if H_0 is Rejected

Step 1: Arrange the $a = 5$ brand means in ascending order

$$\bar{y}_{5.} = 13.08 \quad \bar{y}_{3.} = 13.67 \quad \bar{y}_{1.} = 13.68 \quad \bar{y}_{4.} = 14.73 \quad \bar{y}_{2.} = 15.95$$

Step 2: Obtain the critical value of Tukey’s (Studentized Range) from Table A10 (p. 682), $Q_{0.05,5,25}$, and use it to compute the 95% half-confidence interval band (or width) as shown below

$$\begin{aligned} w_{ij} &= Q_{0.05,5,25} \times \sqrt{\frac{MS_{\text{Error}}(1/n_i + 1/n_j)}{2}}, \quad (i \neq j, i, j = 1, 2, \dots, a) \\ &= 4.15 \times \sqrt{\frac{0.9135333(1/6 + 1/6)}{2}} = 4.15 \times \sqrt{\frac{0.9135333}{6}} = 4.15 \times se(\bar{y}) \\ &= 1.6193. \end{aligned}$$

Step 3: Underline all the means of step 1 which do not differ by as much as w_{ij} .

$$\bar{y}_{5.} = 13.08 \quad \bar{y}_{3.} = 13.67 \quad \bar{y}_{1.} = 13.68 \quad \bar{y}_{4.} = 14.73 \quad \bar{y}_{2.} = 15.95$$

Step 4: Use the underlined means to draw the conclusion that they statistically come from the same population from the standpoint of treatment means. Thus, Brands 5, 3 and 1 means (or averages) are statistically the same; Brand 4 mean is statistically different from 5, and brand 2 mean differs significantly from those of 5, 3, and 1.

Additional Information: $USS = 6122.789917$; $CF = 6069.0962465$;
 $SS_T = SS(\text{Total}) = USS - CF = 53.693666667$; $USS_1 = 1130.53000786$;
 $USS_2 = 1533.23$; $USS_3 = 1123.99997368$; $USS_4 = 1306.83996159$;
 $USS_5 = 1028.18997437$; $y_i = 82.10, 95.7, 82.0, 88.4, 78.50 \longrightarrow y_{..} = 426.700$.

$$SS(\text{Model}) = \frac{82.10^2 + 95.7^2 + 82.0^2 + 88.4^2 + 78.50^2}{6} - CF = 30.8553333$$

$$SS(\text{Error or Residuals}) = SS(\text{Total}) - SS(\text{Model or Brands}) = 22.8383333.$$

The ANOVA Table for the Exercise 37 on page 395 of Devore

SOURCE OF VARIATION	DF	SS	MS = SS/df	$F_0 = MS_{BT}/MS_E$	PR LEVEL = P-value
Total	29	53.693666667			$\hat{\alpha}$
Model (OR Brands)	4	30.8553333	7.713833333	8.443954	
ERROR	25	22.8383333	0.91353333		0.00018715

The 5% critical (or threshold) value of $F_{4,25}$ is $F_{0.05,4,25} = 2.758710$. Thus, the critical (or rejection) region of the test statistic F_0 is $[2.758710, \infty)$.

In order to test $H_0: \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \dots = \sigma_a^2 = \sigma_\epsilon^2 = \sigma^2$, use the Bartlett

statistic $\chi_0^2 = 2.30258509q/c$, where $q = (N - a) \times \log_{10}(S_p^2) -$

$\sum_{i=1}^a (n_i - 1) \log_{10}(S_i^2)$, $c = 1 + [\sum_{i=1}^a (n_i - 1)^{-1} - (N - a)^{-1}] / [3(a - 1)]$, and reject H_0

only if $\chi_0^2 > \chi_{0.05, a-1}^2$. $c = 1 + (1 - 1/25)/12 = 1.08$; $q = 25 \log_{10}(0.9135333) -$

$5 \sum_{i=1}^5 \log_{10}(S_i^2) = 1.9216$; $\chi_0^2 = 4.0967$; $P\text{-value} = \Pr(\chi_4^2 \geq 4.0967) = 0.3931 \gg 0.05$

→ Do not reject $H_0: \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = \sigma_5^2 = \sigma_\epsilon^2 = \sigma^2$.