

## Statistical Inference About Parameters of Two Normal Populations

### 1. CONFIDENCE INTERVALS FOR DIFFERENCE IN MEANS OF TWO NORMAL PROCESSES WHEN BOTH VARIANCES ARE KNOWN

Reference: CHAPTER 9 OF Devore's 8<sup>th</sup> Edition

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**Example 41.** A consumer group tested 2 major brands of radial tires, X and Y, to determine if there were significant differences in expected (or mean) tread life measured in 1000 miles. The data (in 1000 miles) are given below:

X: 51.5, 53, 52, 47, 63, 51, 51, 51, 46, 56, 52.5, 42.5, 58, 52, 47, 49, 51.5

Y: 57, 62, 48, 51.5, 54.5, 58, 57, 54, 63, 58, 62, 59, 56, 62, 58

From past experience it is known that the tread life X is  $N(\mu_x, 22)$  and  $Y \sim N(\mu_y, 18)$ . Our objective is to develop a 95% (2-sided) CI for the mean difference  $\mu_x - \mu_y$ . Since a point estimator of  $\mu_x - \mu_y$  is  $(\hat{\mu}_x - \hat{\mu}_y) = \bar{x} - \bar{y} = 51.4118 - 57.3333 = -5.92157$ , we have to make use of the sampling distribution of  $\bar{x} - \bar{y}$  in order to obtain the requisite CI. Since  $\bar{x}$  is  $N(\mu_x, 22/17)$  and is independent of  $\bar{y} \sim N(\mu_y, 18/15)$ , then  $\bar{x} - \bar{y}$  is  $N(\mu_x - \mu_y, 22/17 + 18/15 = 2.49412)$  as depicted below in Figure 18. Figure 18 clearly illustrates that

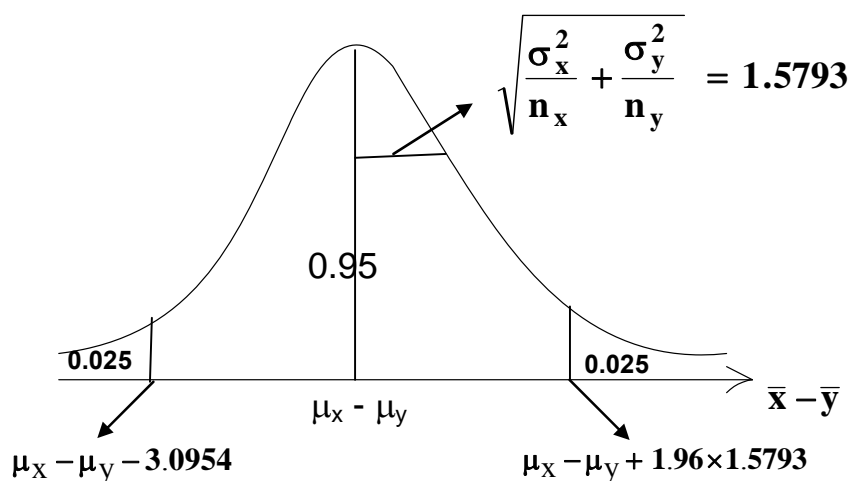


Figure 18. The Sampling Distribution of  $\bar{x} - \bar{y}$

$$\Pr(\mu_x - \mu_y - 3.09533 \leq \bar{x} - \bar{y} \leq \mu_x - \mu_y + 3.09533) = 0.95$$

or

$$\Pr(-\bar{x} + \bar{y} - 3.09533 \leq -\mu_x + \mu_y \leq -\bar{x} + \bar{y} + 3.09533) = 0.95$$

$$\Pr(\bar{x} - \bar{y} + 3.09533 \geq \mu_x - \mu_y \geq \bar{x} - \bar{y} - 3.09533) = 0.95$$

Rearranging this last Pr statement, we obtain the desired result:

$$\Pr(\bar{x} - \bar{y} - 3.09533 \leq \mu_x - \mu_y \leq \bar{x} - \bar{y} + 3.09533) = 0.95.$$

Substituting for  $\bar{x} - \bar{y} = -5.92157$  into the above confidence Pr statement, the 95%

CI for  $\mu_x - \mu_y$  is given by

$$-9.01690 \leq \mu_x - \mu_y \leq -2.82624.$$

Since the above interval excludes 0, there is apparently a significant difference between the two brand means, i.e., the null hypothesis  $H_0 : \mu_x - \mu_y = 0$  must be rejected at the 5% LOS. Note that the 95% CI for  $\mu_y - \mu_x$  is given by  $2.82624 \leq \mu_y - \mu_x \leq 9.01690$ , which also excludes zero.

**Exercise 71.** A study was conducted to determine the impact of viscosity on the coating thickness produced by a paint operation. Up to a certain paint viscosity, higher viscosities cause thicker coatings. The data are provided below:

X = Low Viscosity : 1.09, 1.12, 0.83, 0.88, 1.62, 1.49, 1.59, 0.83, 1.04, 1.34, 1.83, 1.65, 1.71, 1.76

Y = High Viscosity : 1.46, 1.51, 1.59, 1.40, 0.94, 0.98, 0.89, 1.03, 2.05, 2.17, 2.06, 2.02, 1.51, 1.46,  
1.42, 1.40, 1.53, 2.07

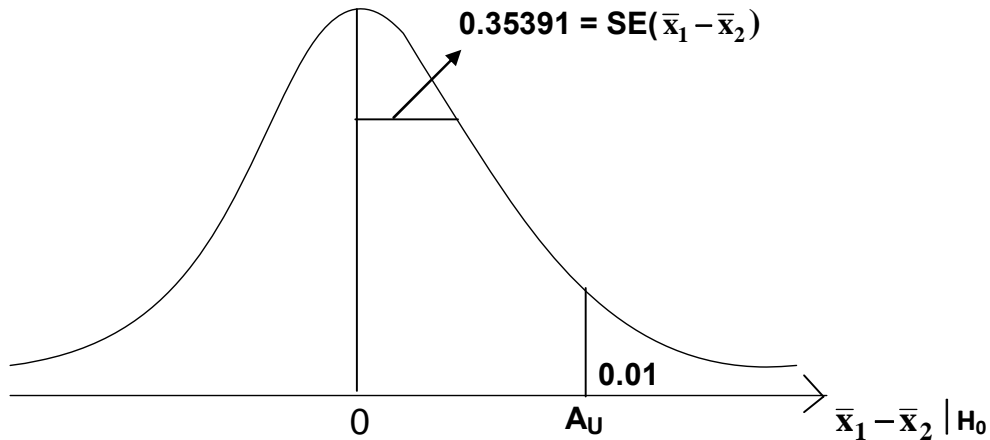
Assuming that X is  $N(\mu_x, 0.130)$  and Y is also  $N(\mu_y, 0.170)$  but independent of X, obtain the 95% lower one sided CI for  $\mu_y - \mu_x$ . ANS:  $-0.03934 \leq \mu_y - \mu_x < \infty$ . Note that this interval does include 0, and therefore the null hypothesis  $H_0 : \mu_y - \mu_x = 0$  cannot be rejected at the 5% level! (b) Work Exercise 9 on page 356 of Devore, using the large-sample approximation to Student's t.

## 2. TEST OF HYPOTHESIS ON EQUALITY OF MEANS OF TWO NORMAL POPULATIONS WITH KNOWN VARIANCES

For the sake of illustration, consider the Exercise 6, page 355 of Devore (8e) where  $n_x = 40$  with  $\sigma_x = \sigma_1 = 1.60 \text{ kgf/cm}^2$  for the Modified Mortar (the treatment group), and  $n_y = 32$ ,  $\sigma_2 = \sigma_y = 1.4$  for the unmodified mortar (the control group). Our objective is to test if the modified mortar

(the treatment group) has a greater average bond strength than that of unmodified, i.e., we

wish to test  $H_0: \mu_1 = \mu_2$  vs  $H_1: \mu_1 > \mu_2$  at  $\alpha = 0.01$ . Since  $\bar{X}_1 - \bar{X}_2$  is  $N(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}) = N(\mu_1 - \mu_2, 0.12525)$ , then under the null hypothesis  $H_0: \mu_1 - \mu_2 = 0$ , the SMD of our test statistic  $\bar{X}_1 - \bar{X}_2$  follows the normal distribution depicted in Figure 19.

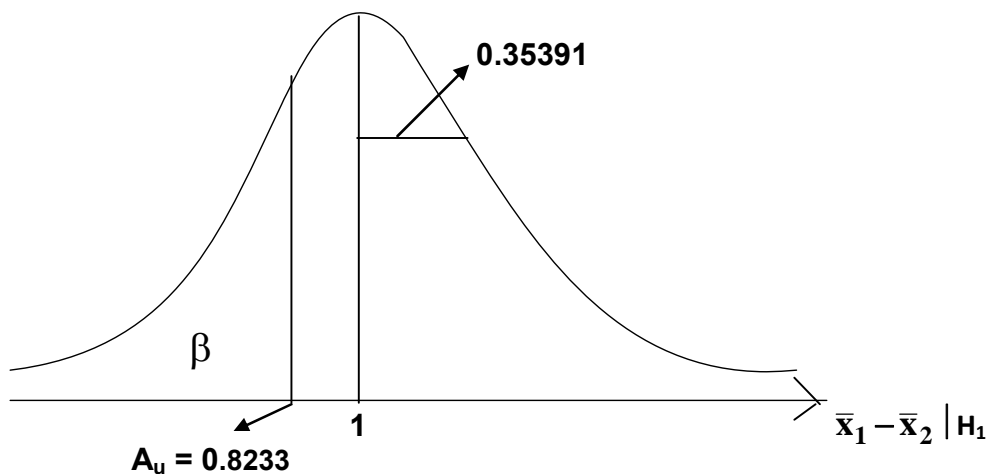


**Figure 19.** The Null Sampling Distribution of  $\bar{X}_1 - \bar{X}_2$  Given that  $H_0$  is True

The upper limit for the acceptance interval  $AI = (-\infty, A_u)$  is  $A_u = 0 + Z_{0.01} \times SE(\bar{X}_1 - \bar{X}_2) = 2.3263 \times 0.35391 = 0.82331$ . Therefore, the critical (or rejection) region for the test statistic  $\bar{X}_1 - \bar{X}_2$  is  $[0.8233, \infty)$ , i.e., we must reject  $H_0$  at the 1% level iff  $\bar{X}_1 - \bar{X}_2 > 0.8233$ . Because  $\bar{X}_1 - \bar{X}_2 = 18.12 - 16.87 = 1.25 \text{ kgf/cm}^2$  is well inside the critical region  $\bar{AI} = AI' = 0.8233 < \bar{X}_1 - \bar{X}_2$  then we do have sufficient evidence, at the 1% level, to reject  $H_0$ . Once  $H_0$  is rejected at  $\alpha = 0.01$  with the given data, then the probability level (or *P-value*) of the test must be smaller than 0.01 as computed next. Since the above test is right-sided, the *P-value* =  $P(\bar{X}_1 - \bar{X}_2 \geq 1.25) = P(Z_0 \geq 3.532) = 0.0002062$ , and hence  $\hat{\alpha} = 0.0002062 < \alpha = 0.01$ , as expected because the test statistic  $\bar{X}_1 - \bar{X}_2$  rejected  $H_0$  at the 1% LOS. Further, the value of  $Z_0 = 3.5320 > Z_{0.01} = 2.3263$ , again implying sufficient evidence to reject  $H_0$  at the LOS of 1%.

To compute the type II error  $Pr$ , recall that 1st we have to assume  $H_0$  false. Consequently,

suppose  $\mu_1 - \mu_2 = 1 \neq 0$  (i.e.,  $H_0$  is assumed false) as illustrated in Figure 20. Then how do we compute the prior Pr of accepting  $H_0: \mu_1 - \mu_2 = 0$  given that  $H_0$  is now assumed false. Figure 20



**Figure 20.** The (alternate) Sampling Distribution of  $\bar{X}_1 - \bar{X}_2$  assuming  $H_0$  is False

clearly shows that  $\beta(\text{at } \mu_1 - \mu_2 = 1) = \Pr(\bar{X}_1 - \bar{X}_2 \leq A_u \mid \mu_1 - \mu_2 = 1) = \Pr(-\infty < \bar{X}_1 - \bar{X}_2 \leq 0.8233 \mid \mu_1 - \mu_2 = 1) = \Pr(Z \leq -0.4993) = \Phi(-0.4993) = 0.3088$ .

**Exercise 72.** (a) Work Exercise 6 on page 355 of Devore. For part (c) use the pertinent version of the formula page 124 of my notes. Note that Devore uses  $m$  for  $n_1 = n_x$  and  $n$  for  $n_2 = n_y$ . (b) Draw the 5%-level OC curve for the Exercise 6 on page 355 of Devore by tabulating the values of  $\beta$  for  $\mu_1 - \mu_2 = -0.50, 0, 0.50, 0.5821, 1, \text{ and } 1.25$ . Note that you already know three points on the OC curve.

### 3. SAMPLE SIZE DETERMINATION GIVEN TWO SPECIFIC POINTS ON THE OC CURVE

To illustrate the procedure, suppose we wish to test  $H_0: \mu_1 - \mu_2 = 0$  VS the alternative  $H_1: \mu_1 - \mu_2 \neq 0$ , where  $X \sim N(\mu_1, 0.0004)$  and  $Y \sim N(\mu_2, 0.000625)$ . Our objective now is to design a sampling procedure such that our OC curve goes thru the three points  $(0, 0.95)$  and  $(\pm 0.02, 0.10)$ , i.e., at  $\delta = 0, \beta = 1 - \alpha = 0.95$ , but  $\beta = 0.10$  when  $\delta = \mu_1 - \mu_2 = \pm 0.02$ . Note that Devore uses  $\Delta$  for

the population mean difference  $\mu_1 - \mu_2$ , while I am using  $\delta = \mu_1 - \mu_2$ . We need the information only about 2 points; however, because of symmetry, the OC curve will also automatically go thru the 3<sup>rd</sup> point, and this in turn will determine the three unknowns  $n$ ,  $A_L$  and  $A_U$ .

We will work with the two points  $(\mu_1 - \mu_2 = 0, \beta = 0.95)$  and  $(-0.02, 0.10)$ . From the point

$(0, 0.95)$ , we obtain  $A_L = 0 - 1.96 \sqrt{\frac{0.02^2}{n_1} + \frac{0.025^2}{n_2}}$ . Secondly, Figure 21 shows that  $A_L = -0.02 +$

$1.28155 \times SE(\bar{x}_1 - \bar{x}_2)$ . Equating these last 2 expressions, we obtain  $A_L = 0 - 1.96 \times$

$$\sqrt{\frac{0.02^2}{n_1} + \frac{0.025^2}{n_2}} = -0.02 + 1.2816 \times SE(\bar{x}_1 - \bar{x}_2), \text{ or}$$

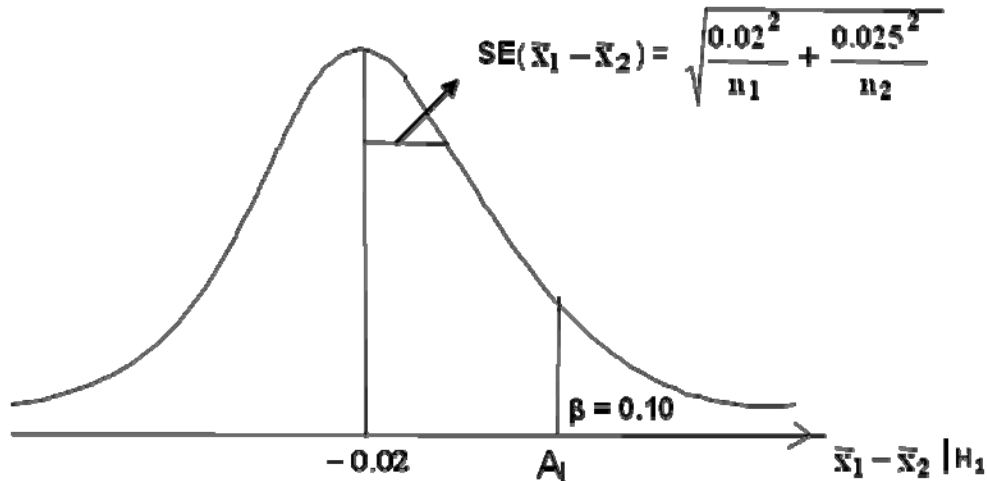


Figure 21. The Sampling Distribution of  $\bar{x}_1 - \bar{x}_2$  assuming that  $H_0$  is false.

$$\begin{aligned}
 -1.96 \sqrt{\frac{0.02^2}{n_1} + \frac{0.025^2}{n_2}} &= -0.02 + 1.2816 \times \sqrt{\frac{0.02^2}{n_1} + \frac{0.025^2}{n_2}} \\
 \longrightarrow 0.02 &= 3.2415 \times \sqrt{\frac{0.000400}{n_1} + \frac{0.000625}{n_2}}. \quad (36)
 \end{aligned}$$

Squaring both sides of equation (36) and simplifying gives rise to

$$0.0004 = \frac{0.00420297}{n_1} + \frac{0.00656714}{n_2}.$$

(37) Equation (37) has two unknowns, and therefore, there are infinite number of solutions. For example, one possible solution is to take  $n_1 = 15$  and  $n_2 = 55$ ; another solution set is ( $n_1 = n_x = m = 20$ ,  $n_2 = n_y = n = 35$ ). In other words, if you specify the value of  $n_1$  (or  $n_2$ ), then I will in turn

determine the other sample size such that  $\frac{0.00420297}{n_1} + \frac{0.00656714}{n_2} \leq 0.0004$ . The question

arises how should we allocate our total resources  $N = n_1 + n_2$  between the two populations in order to minimize the  $V(\bar{x}_1 - \bar{x}_2)$ , which also automatically minimizes the  $SE(\bar{x}_1 - \bar{x}_2)$ . It can be proven that our total resources  $N$  must be allocated proportional to the standard deviation of the two populations, i.e., the larger  $\sigma_i$ ,  $i = 1$  and  $2$  is, the larger the corresponding  $n_i$  should be.

Partially differentiating  $V(\bar{X}_1 - \bar{X}_2) = \sigma_1^2/n_1 + \sigma_2^2/n_2$  with respect to  $n_1$  and  $n_2$  and requiring that

both partial derivatives be zero leads to the allocations  $n_1 = \left(\frac{\sigma_1}{\sigma_1 + \sigma_2}\right) \times N$  and  $n_2 = N - n_1$ . Thus,

our optimum solution for the above example is  $n_1 = 0.444\bar{4}N$  and  $n_2 = 0.555\bar{5}N$ . Inserting these into equation (36) yields  $N \geq 53.19383$  or  $N = 54$ . Hence  $n_1 = 0.444\bar{4}N = 24$  and  $n_2 = 30$ .

It can be shown that in general the total sample size  $N = n_1 + n_2$  for a 2-sided test, that includes the constraint  $n_1 = N\sigma_1 / (\sigma_1 + \sigma_2)$ , must be selected from the equation  $N =$

$(Z_{\alpha/2} + Z_{\beta})^2 (\sigma_1 + \sigma_2)^2 / (\delta_1 - \delta_0)^2$ , where  $\delta_0$  is the hypothesized value of  $\mu_1 - \mu_2$  under  $H_0$ , and  $\delta_1$  is a value of  $\mu_1 - \mu_2$  under  $H_1$  for which the type II error  $Pr$  is required to be  $\beta$ . Once  $N$  is determined, then  $n_1 = N\sigma_1 / (\sigma_1 + \sigma_2)$ . If the alternative is one-sided, then replace  $Z_{\alpha/2}$  with  $Z_{\alpha}$ .

#### **4. CONFIDENCE INTERVAL ON THE MEAN DIFFERENCE OF TWO INDEPENDENT NORMAL POPULATIONS WHEN $\sigma_1^2 = \sigma_2^2 = \sigma^2$ IS UNKNOWN (THE COMPLETELY RANDOMIZED DESIGN)**

For the sake of illustration consider the Experiment reported in the Journal of Waste and Hazardous Materials (Vol. 6, 1989), where  $X_1$  = weight of calcium in standard cement (the control

group), while  $X_2$  = weight of calcium in cement doped with lead (the treatment group).

Reduced levels of calcium would cause the hydration mechanism to get blocked and allow water to attack various locations of cement structure. Ten samples of standard cement gave  $\bar{x}_1 = 90.0$  with  $S_1 = 5.0$  while 15 samples of lead-doped cement resulted in  $\bar{x}_2 = 87.0$  with  $S_2 = 4.0$ . Assuming that

$X_1 \sim N(\mu_1, \text{unknown } \sigma^2)$  and  $X_2 \sim N(\mu_2, \sigma^2)$ , then  $\bar{x}_1 - \bar{x}_2$  is  $N(\mu_1 - \mu_2, \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2})$ , where it is

assumed that  $\sigma^2$  is the common value of the unknown  $\sigma_1^2 = \sigma_2^2$ . It was shown in STAT 3600 that if

$\hat{\theta}$  is an unbiased estimator of a parameter  $\theta$  with standard error  $se(\hat{\theta}) = S_{\hat{\theta}}$ , then the sampling distribution of  $(\hat{\theta} - \theta) / S_{\hat{\theta}}$  follows that of (Gosset's) Student's-t with  $df$  (degrees of freedom) equal

to that of  $S_{\hat{\theta}}$ . Accordingly, let  $\hat{\theta} = \bar{x}_1 - \bar{x}_2$ ; then  $\bar{x}_1 - \bar{x}_2$  is an unbiased estimator of  $\mu_1 - \mu_2$  with

$V(\bar{x}_1 - \bar{x}_2) = \sigma^2 \left( \frac{1}{n_1} + \frac{1}{n_2} \right)$ . Since the common value of the process variances  $\sigma^2$  is unknown, we

must pool both unbiased estimators  $S_1^2$  and  $S_2^2$  of  $\sigma^2$  to obtain one unbiased estimator of  $\sigma^2$ ,

which is given by their weighted average of the two sample variances based on their  $df$  (or DOF).

$$S_p^2 = \frac{v_1 S_1^2}{v_1 + v_2} + \frac{v_2 S_2^2}{v_1 + v_2} = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{(n_1 - 1) + (n_2 - 1)} = \frac{CSS_1 + CSS_2}{n_1 + n_2 - 2}. \quad (38)$$

Note that  $E(S_p^2) = \sigma^2$ . Therefore, the  $se(\bar{x}_1 - \bar{x}_2) = S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$ , and as a result the rv  $[(\bar{x}_1 - \bar{x}_2)$

$- (\mu_1 - \mu_2)] / (S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}})$  has a t-sampling distribution with  $v = n_1 + n_2 - 2$   $df$ . To obtain the

two-sided 95% CI for  $\mu_1 - \mu_2$ , we make use of the confidence Pr statement ( $n_1 = 10, n_2 = 15, t_{0.025,23} = 2.069$ )

$$\Pr(-2.069 \leq T_{23} \leq 2.069) = 0.95.$$

Substituting  $[(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)] / (S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}})$  for  $T_{23}$  in this last Pr statement

and solving for  $(\mu_1 - \mu_2)$  leads to the desired 95% CI given below.

$$\bar{x}_1 - \bar{x}_2 - 2.069 \times S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + 2.069 \times S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} .$$

To obtain the lower and upper confidence limits, we need to compute  $S_p^2$  using  $n_1 = 10$ ,  $S_1^2 = 25$ ,  $n_2 = 15$  and  $S_2^2 = 16$ . From equation (38),  $S_p^2 = \frac{9(25)}{23} + \frac{14(16)}{23} = 19.52174$ , and hence

$$se(\bar{x}_1 - \bar{x}_2) = 4.41834 \sqrt{\frac{1}{10} + \frac{1}{15}} = 1.8038, \text{ with } \bar{x}_1 = 90.0 \text{ and } \bar{x}_2 = 87.0. \text{ Thus, } L(\mu_1 - \mu_2) = 3 -$$

$2.069 \times 1.8038 = -0.732$  and  $U(\mu_1 - \mu_2) = 3 + 3.732 = 6.732$ . Since this 95% CI,  $-0.732 \leq \mu_1 - \mu_2 \leq 6.732$ , includes zero, then there does not exist a significant difference between the two population parameters  $\mu_1$  and  $\mu_2$  at the 5% level. This implies that doping cement with lead does not significantly alter water hydration mechanism from a statistical standpoint at  $\alpha = 0.05$ . However, the null hypothesis  $H_0 : \mu_1 - \mu_2 = -1$  must be rejected at the 5% level because  $-1$  lies outside the 95% CI.

Similar developments as above lead to the upper one-sided CI for the parameter difference  $\mu_1 - \mu_2$  given below:

$$-\infty < \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + t_{\alpha, n_1 + n_2 - 2} \times S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} .$$

The lower one-sided CI for  $\mu_1 - \mu_2$  with confidence coefficient  $1 - \alpha$  is given by

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha, n_1 + n_2 - 2} \times S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \leq \mu_1 - \mu_2 < +\infty.$$

When the variances of the two independent normal populations,  $\sigma_1^2$  and  $\sigma_2^2$ , are unequal and unknown, then the CI for  $\mu_1 - \mu_2$  must be obtained using the equation (9.2), on page 357 of Devore's 8<sup>th</sup> edition, but the  $df$   $v$ , where  $\text{Min}(n_1 - 1, n_2 - 1) < v \leq n_1 + n_2 - 2$ , is given by the

equation near the bottom of Devore's page 357. As a general rule of thumb, I would recommend against using the pooled t procedure outlined on pp. 124-126 of my notes if  $S_1^2 > 2S_2^2$ , or vice versa. When the sample variances are significantly different, then the two-sample t procedure (outlined in section 9.2. pp. 357-360 of Devore's 8<sup>th</sup> edition) must be applied for both obtaining a CI for  $\mu_1 - \mu_2$  and testing the null hypothesis  $H_0: \mu_1 - \mu_2 = \delta$ .

To illustrate the two-independent-sample t-test, we work exercise 21 on page 362 of Devore (8e). Since  $S_1^2 = (0.53)^2 = 0.2809$  and  $S_2^2 = (0.87)^2 = 0.7569$ , showing that  $S_2^2 \gg 2S_1^2$ , then the assumption of  $\sigma_1^2 = \sigma_2^2 = \sigma^2$  is very tenuous at best! Therefore, we use the two-sample t procedure, outlined in section 9.2 of Devore's 8<sup>th</sup> edition page 357, which does not require the assumption of  $\sigma_1^2 = \sigma_2^2 = \sigma^2$ .

The sample statistics for the Exercise 21, p. 362, are for the case of W/O probing  $n_1 = m = 8$ ,  $\bar{x}_1 = 1.71$  mm,  $S_1 = 0.53$ , and for the CTS (Carpal Tunnel Syndrome) subjects  $n_2 = n = 10$ ,  $\bar{x}_2 = 2.53$  mm,  $S_2 = 0.87$ . We wish to test  $H_0: \mu_2 - \mu_1 = 0$  against the alternative  $H_1: \mu_2 - \mu_1 > 0$  at  $\alpha = 0.01$ . Our test statistic from equation 9.2 of Devore (p. 357) is

$$t_0 = \frac{2.53 - 1.71 - 0}{\sqrt{\frac{0.87^2}{10} + \frac{0.53^2}{8}}} = 2.4634$$

with  $df = 15.11$  (computed from the equation below Devore's Eq. (9.2) on page 357). Since  $t_{0.01,15} = 2.602$  and  $t_{0.01,16} = 2.583$ , then our critical threshold value is  $t_{0.01,15.11} \cong 2.6003$ . Since our test statistic  $t_0 = 2.4634$  does not exceed this threshold value, then the two samples provide insufficient evidence, at the 1% level, to conclude that the true average gap detection for the CTS population exceeds that of normal subjects. The *P-value* of the test is computed from  $\hat{\alpha} \cong \Pr(T_{15.11} \geq 2.4634) = 0.01312$ , which exceeds  $\alpha = 0.01$  as expected. However, our test statistic  $t_0 = 2.4634$  does provide sufficient evidence, at the 5% level, to conclude that  $\mu_2 > \mu_1$ .

Unfortunately, the SMD of the two-independent-sample t statistic given in equation (9.2) of Devore when  $H_0$  is false, as far as I know, is intractable for small and moderate sample sizes, say  $2 \leq n < 50$ , and therefore, the power of the t-test cannot be estimated directly. For the pooled t-

test, the OC curves of Table A.17 on page A-28 of Devore's 8<sup>th</sup> edition may be used only when  $n_1 = n_2 = n$  with the abscissa  $d = (\delta - \delta_0)/(2S_p)$  and the sample size for the OC curve as  $n_{OC} = 2n - 1$ .

**Exercise 73.** Work Exercises 18, 23(c) and 25 on pages 362-3 of Devore (8e).

## 5. CONFIDENCE INTERVAL ON $\mu_1 - \mu_2$ FOR PAIRED

### OBSERVATIONS (I.E., THE RANDOMIZED COMPLETE BLOCKS)

As an example, consider the Example 9.9, on pp. 367-8 of Devore's 8<sup>th</sup> edition, where 16 subjects (or blocks) were selected at random and two responses ( $X_{1j}, X_{2j}$ ) were obtained from each subject (or block). When  $j = 5$ , the paired responses from block number 5 are ( $x_{15} = 90, x_{25} = 84$ ). It

is assumed that the random vector  $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$  has a bivariate normal distribution with unknown  $V(X_1) =$

$\sigma_1^2$ ,  $V(X_2) = \sigma_2^2$ , and unknown correlation coefficient  $\rho$  (note that  $X_{1j}$  and  $X_{2j}$  cannot possibly be independent because they originate from the same  $j^{\text{th}}$  block or  $j^{\text{th}}$  subject). Letting  $D_j = X_{1j} - X_{2j}$ , the rv  $D$  is also normally distributed with

$$E(D) = E(X_1 - X_2) = \mu_1 - \mu_2 = \mu_D, \quad (39a)$$

and  $V(D) = \sigma_1^2 + \sigma_2^2 - 2\text{COV}(X_1, X_2)$ , or

$$\sigma_D^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_{12}. \quad (39b)$$

Clearly, all the parameters in equations (39) are unknown and have to be estimated by sample statistics. To obtain a CI for  $\mu_D = \mu_1 - \mu_2$ , we use the fact that the sampling distribution of  $(\bar{d} - \mu_D)/se(\bar{d})$  is that of W. S. Gosset's T rv with  $v = (n - 1)$  *df*. For the Example 9.9 on pp. 367-368 of Devore's 8<sup>th</sup> edition, the number of blocks  $n = 16$  and hence the *df* of  $(\bar{d} - \mu_D)\sqrt{n}/S_d$  is  $v = (n - 1) = 15$ . The 16 paired differences for our example are provided in the Table in the middle of

page 367 of Devore. The  $n = 16$  differences give rise to  $\bar{d} = \frac{1}{16} \sum_{j=1}^n d_j = 6.75$  and

$$S_d^2 = \frac{1}{15} \sum_{j=1}^{16} (d_j - \bar{d})^2 = 67.8 \rightarrow S_d = 8.2341 \rightarrow se(\bar{d}) = 2.05852. \text{ Table A.5, p. A-9 of}$$

Devore's 8<sup>th</sup> ed., shows that the  $\Pr(-2.131 \leq T_{15} \leq 2.131) = 0.95$ , or

$$\Pr(-2.131 \leq \frac{(\bar{d} - \mu_D)\sqrt{n}}{S_d} \leq 2.131) = 0.95$$

Rearranging the above 2-sided inequality inside the parentheses yields

$$\Pr[\bar{d} - t_{0.025;15} \times S_d / \sqrt{n} \leq \mu_D \leq \bar{d} + 2.131 \times S_d / \sqrt{n}] = 0.95$$

Therefore,  $(\mu_D)_L = LB(\mu_1 - \mu_2) = 6.75 - 2.131(2.05852) = 2.3633$  and  $UB(\mu_1 - \mu_2) = 11.1367$ , where LB stands for the lower bound and UB for upper bound. Since this 95% CI interval  $2.3633 \leq \mu_1 - \mu_2 \leq 11.1367$  excludes zero, it provides conclusive evidence that the 2 population averages are significantly different, i.e., the null hypothesis  $H_0 : \mu_1 - \mu_2 = 0$  must be rejected at the 5% level.

The testing procedure for the above Example 9.9 is well outlined by Devore at the top of page 368. The critical level of the test is given by  $\hat{\alpha} \cong 2 \times \Pr(T_{15} \geq 3.27906) = 2(0.00253602) = 0.0050721$  (From Microsoft Excel function: `tdist(3.27906,15,2)`, which is consistent with the rejection of  $H_0$  since the 95% CI excluded the hypothesized value of the population mean difference equaling zero.

The power of the paired t test can be estimated using the OC curves of Table A.17 on page A-28 of Devore's 8<sup>th</sup> edition with abscissa  $d \cong |\mu_D - \mu_0| / S_d$  and the sample size  $n$  equal to the number of pairs (or blocks).

**Exercise 74.** (a) For the data of Example 9.9, pages 367-368 of Devore, obtain the point unbiased estimates of  $\mu_1 - \mu_2$  and  $\sigma_D^2$ . For  $\sigma_D^2$  use equations (39b) and the unbiased estimator

$$\hat{\sigma}_D^2 = \frac{1}{n-1} \sum_{j=1}^n (d_j - \bar{d})^2 = \frac{1}{n-1} \sum_{j=1}^n [(x_{1j} - x_{2j}) - (\bar{x}_1 - \bar{x}_2)]^2$$

$$\hat{\sigma}_D^2 = \frac{1}{n-1} \sum_{j=1}^n [(x_{1j} - \bar{x}_1) - (x_{2j} - \bar{x}_2)]^2$$

$$\begin{aligned}
\hat{\sigma}_D^2 &= \frac{1}{n-1} \sum_{j=1}^n [(x_{1j} - \bar{x}_1)^2 + (x_{2j} - \bar{x}_2)^2 - 2(x_{1j} - \bar{x}_1)(x_{2j} - \bar{x}_2)] \\
&= S_1^2 + S_2^2 - 2 \frac{1}{n-1} \sum_{j=1}^n (x_{1j} - \bar{x}_1)(x_{2j} - \bar{x}_2) \\
&= S_1^2 + S_2^2 - 2\hat{\sigma}_{12} = S_1^2 + S_2^2 - 2S_{12}/(n-1)
\end{aligned}$$

where

$$\hat{\sigma}_{12} = \frac{1}{n-1} \sum_{j=1}^n (x_{1j} - \bar{x}_1)(x_{2j} - \bar{x}_2) = \frac{S_{12}}{n-1}$$

and

$$S_{12} = \sum_{j=1}^n (x_{1j} - \bar{x}_1)(x_{2j} - \bar{x}_2) = \text{Sum of Cross-Products}$$

$$S_{12} = \sum_{j=1}^n x_{1j} x_{2j} - \left( \sum_{j=1}^n x_{1j} \sum_{j=1}^n x_{2j} \right) / n = \sum_{j=1}^n (x_{1j} x_{2j}) - n\bar{x}_1\bar{x}_2 = \sum_{j=1}^n (x_{1j} x_{2j} - \bar{x}_1\bar{x}_2).$$

(b) Work Exercises 36, 37, 40, and 46 on pages 371-374 of Devore (8e).

## 6. (SIR R. A.) FISHER'S F DISTRIBUTION

A continuous rv,  $X$ , has the (Sir Ronald A. Fisher's) F distribution iff its pdf is given by

$$f(x) = C x^{(v_1-2)/2} (v_2 + v_1 x)^{-(v_1+v_2)/2}, \quad 0 \leq x < \infty.$$

By now, surely you know the constraint on the normalizing constant  $C$ , that makes the above  $f(x)$  a

density function, and which leads to the specific value of  $C = \frac{v_1^{v_1/2} v_2^{v_2/2} \Gamma(\frac{v_1+v_2}{2})}{\Gamma(v_1/2)\Gamma(v_2/2)}$ , where  $v_1$

is called the *df* (or DOF) of the numerator and  $v_2$  that of the denominator, for reasons that will be explained in the following paragraphs. It can be shown that the modal point of the F distribution occurs at  $MO = v_2(v_1 - 2)/[v_1(v_2 + 2)]$ ,  $v_1 \geq 2$ . It can also be verified that  $0 \leq MO < 1$ , and for very large  $v_1$  and  $v_2$  the value of  $MO$  is close to 1 but always less than 1. Further, the 1<sup>st</sup> two moments of F are given by  $E(F) = v_2/(v_2 - 2)$  for  $v_2 > 2$ , and

$$V(F) = \frac{2v_2^2(v_2 + v_1 - 2)}{v_1(v_2 - 2)^2(v_2 - 4)}, \quad v_2 > 4.$$

These last two expressions imply that the mean of  $F$  does not exist for  $v_2 \leq 2$  and the variance of  $F$  does not exist for  $v_2 \leq 4$ . The skewness of the  $F$  rv is given by

$$\alpha_3 = \frac{2v_1 + v_2 - 2}{v_2 - 6} \sqrt{\frac{8(v_2 - 4)}{v_1(v_1 + v_2 - 2)}}, \quad v_2 > 6$$

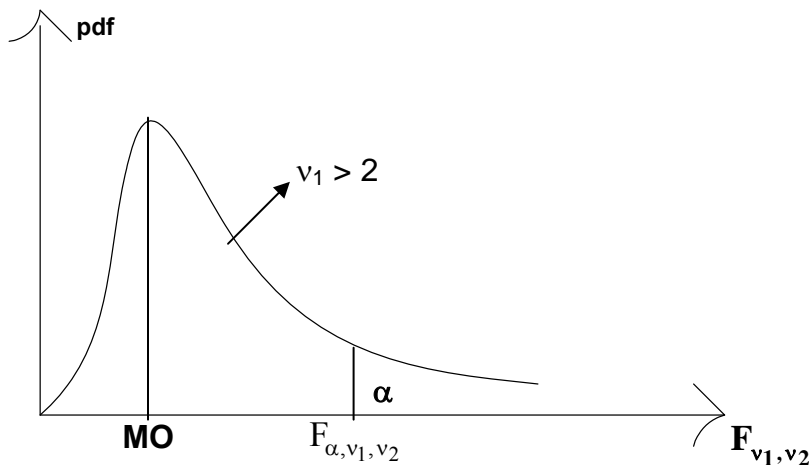
so that the  $F$  distribution is positively skewed ( $MO < x_{0.50} < \mu$ ). Its kurtosis,  $\beta_4$ , is given by

$$\beta_4 = \alpha_4 - 3 = \frac{12[(v_2 - 2)^2(v_2 - 4) + v_1(v_1 + v_2 - 2)(5v_2 - 22)]}{v_1(v_2 - 6)(v_2 - 8)(v_1 + v_2 - 2)}, \quad v_2 > 8$$

where  $\alpha_4 = E[(X - \mu)^4 / \sigma^4] = E[(X - \mu)^4] / \sigma^4 = E(Z^4)$ .

**Exercise 75.** Graph the density functions of the Fisher's  $F$  rv for  $(v_1 = 1, v_2 = 2)$ ,  $(v_1 = 2, v_2 = 3)$ , and for  $(v_1 = 3, v_2 = 4)$ . Note that the graph of the  $F$  distribution resembles Figure 22 only for  $v_1 > 2$ . When  $v_1 = 2$ , the mode occurs at the origin and when  $v_1 = 1$ ,  $f(F_{1,v_2})$  has an asymptote at the origin.

The graph of  $f(F)$  for  $v_1 > 2$  is given in Figure 22. Figure 22 clearly shows that the  $\Pr(F_{v_1, v_2})$



**Figure 22.** The Graph of the  $F$  Distribution for  $v_1 > 2$

$> F_{\alpha, v_1, v_2} ) = \alpha$ , i.e.,  $F_{\alpha, v_1, v_2}$  represents the  $100 \times \alpha$  percentage point of the F distribution with  $v_1$  *df* for the numerator and  $v_2$  *df* for the denominator. The percentage points of the F distribution are tabulated in A.9 on pp. A-14 through A-19 of Devore for  $\alpha = 0.10, 0.05, 0.01$  and  $0.001$ , and I have provided more percentage points on my website under Finverse. The easiest way to obtain any percentage points is to use the Microsoft excel function  $\text{Finv}(\alpha, v_1, v_2)$ . For example, Table A.9 on pages A-14 shows that  $F_{0.05, 4, 10} = 3.48$ , i.e., the  $\Pr(F_{4, 10} > 3.48) = 0.05$ , and  $F_{0.01, 4, 10} = 5.99$ , i.e., the  $\Pr(F_{4, 10} \leq 5.99) = 0.99$ , while MS Excel gives  $\text{Finv}(0.05, 4, 10) = 3.47805$ .

## THE APPLICATION OF THE F DISTRIBUTION TO SAMPLING OF NORMAL POPULATIONS

Statistical theory shows that an F rv can be generated through the ratio of two independent scaled (with respect to their *df*)  $\chi^2$  rvs as depicted in equation (40) below.

$$F_{v_1, v_2} = \frac{\chi_{v_1}^2 / v_1}{\chi_{v_2}^2 / v_2} \quad (40)$$

Equation (40) clearly shows that  $v_1$  is the *df* of the  $\chi^2$  rv in the numerator and  $v_2$  is the *df* of the  $\chi^2$  in the denominator, and as stated above both the numerator and denominator are scaled wrt (with respect to) their *df*.

As an application of equation (40), consider 2 independent normal (machining) processes  $N(\mu_x, \sigma_x^2)$  and  $N(\mu_y, \sigma_y^2)$ . In order to compare the variances of two machines, we sample the two normal processes of sizes  $n_x$  and  $n_y$ , respectively. As you well know by now that the sampling distributions of both  $(n_x - 1)S_x^2 / \sigma_x^2$  and  $(n_y - 1)S_y^2 / \sigma_y^2$  follow  $\chi^2$  with  $(n_x - 1)$  and  $(n_y - 1)$  *df*, respectively. Substitution into equation (40) yields:

$$F_{n_x - 1, n_y - 1} = \frac{[(n_x - 1)S_x^2 / \sigma_x^2] / (n_x - 1)}{[(n_y - 1)S_y^2 / \sigma_y^2] / (n_y - 1)} = \frac{S_x^2 / \sigma_x^2}{S_y^2 / \sigma_y^2} \quad (41)$$

This above result shows that the ratio of two normal sample variances,  $S_x^2 / S_y^2$ , possesses a

sampling distribution which is the Fisher's F when the ratio  $S_x^2 / S_y^2$  is multiplied by the scaling factor  $\sigma_y^2 / \sigma_x^2$ . If it can be hypothesized that  $\sigma_y^2 = \sigma_x^2$ , then the sampling distribution of  $S_x^2 / S_y^2 \sim F_{n_x-1, n_y-1}$ . Put differently, under the null hypothesis  $H_0 : \sigma_y^2 = \sigma_x^2$ , the null SMD of the statistic  $S_x^2 / S_y^2$  follows the Fisher's F with  $v_1 = n_y - 1$  *df* for the numerator and  $v_2 = n_x - 1$  *df* for the denominator.

**Example 42.** A random sample of size  $n_x = 26$  is drawn from a  $N(\mu_x, 25)$  process and one of size  $n_y = 21$  is selected from another  $N(\mu_y, 49)$  process. Prior to the drawing of the two samples, compute the  $\Pr(S_x^2 / S_y^2 \leq 3.3712)$ .

Solution.  $\Pr(S_y^2 / S_x^2 \leq 3.3712) = \Pr\left[\frac{S_y^2 / \sigma_y^2}{S_x^2 / \sigma_x^2} \leq \frac{3.3712 \times (25)}{49}\right] = \Pr(F_{20,25} \leq 1.72) = 0.90,$

which shows that the 90<sup>th</sup> percentile of the rv  $F_{20,25}$  is equal to 1.72, or  $F_{0.10,20,25} = 1.72$ .

## THE PERCENTAGE POINTS OF THE F DISTRIBUTION FOR $\alpha > 0.50$ (I.E., THE LEFT TAIL of $F(v_1, v_2)$ )

Table A.9 of Devore provides some percentage points of  $F(v_1, v_2)$  only for  $\alpha < 0.50$ , i.e., pages A-14 through A-19 of Devore tabulate only the right tail of the F distribution, and only for  $\alpha = 0.10, 0.05, 0.01$ , and  $0.001$ . The question arises as to why the left tail such as  $F_{0.99}(5, 10)$  is not tabulated while the corresponding right tail  $F_{0.01}(5, 10)$  is given to be 5.64 on page A-14?

Apparently there must exist a relationship between  $F_\alpha$  and  $F_{1-\alpha}$  as illustrated below for  $v_1 = 5$  and

$$v_2 = 10. \quad \Pr(F_{5,10} \geq F_{0.99,5,10}) = 0.99 \rightarrow \Pr\left[\frac{\chi_5^2 / 5}{\chi_{10}^2 / 10} \geq F_{0.99}(5,10)\right] = 0.99$$

$$\rightarrow \Pr\left[\frac{\chi_{10}^2 / 10}{\chi_5^2 / 5} \leq \frac{1}{F_{0.99,5,10}}\right] = 0.99 \quad \rightarrow \Pr(F_{10,5} \leq \frac{1}{F_{0.99,5,10}}) = 0.99$$

$$\longrightarrow \Pr(F_{10,5} \geq \frac{1}{F_{0.99,5,10}}) = 0.01 \longrightarrow F_{0.01; 10, 5} = \frac{1}{F_{0.99,5,10}}.$$

Thus, in general we have the result:

$$F_{\alpha, v_1, v_2} = \frac{1}{F_{1-\alpha, v_2, v_1}}, \text{ for all } 0 < \alpha < 1, v_1 \text{ and } v_2. \quad (42)$$

**Exercise 76.** A random sample of size  $n_x = 13$  and one of size  $n_y = 21$  are selected from 2 independent normal processes with  $\sigma_x^2 = 6$  and  $\sigma_y^2 = 10$ . Compute the  $\Pr(S_x^2 / S_y^2 < 0.621)$ . ANS: approximately 0.025. (b) Use Microsoft Excel to obtain  $F_{0.995}(15, 20)$  and  $F_{0.999,15,20}$ . (c) Obtain the value of the constant C such that the  $\Pr(C \leq F_{5,10} \leq 8.316C) = 0.80$ .

## 7. CONFIDENCE INTERVALS FOR THE RATIO $\sigma_x^2 / \sigma_y^2$ FROM

### TWO INDEPENDENT NORMAL POPULATIONS

All three types of CIs for  $\sigma_x^2 / \sigma_y^2$  are of interest, i.e., a 2-sided CI:  $L \leq \sigma_x^2 / \sigma_y^2 \leq U$ , a lower 1-sided:  $L \leq \sigma_x^2 / \sigma_y^2 < \infty$ , and an upper 1-sided:  $0 < \sigma_x^2 / \sigma_y^2 \leq U$ . Henceforth,  $\sigma_x^2$  will represent the variance of process X (such as machine X) and  $\sigma_y^2$  that of process Y.

**Example 43.** The diameter of steel rods manufactured on machines 1 and 2 are assumed  $N(\mu_1, \sigma_1^2)$  and  $N(\mu_2, \sigma_2^2)$ , respectively. Two random samples of sizes  $n_1 = 15$  and  $n_2 = 18$  are selected yielding the statistics  $\bar{x}_1 = 8.73$  inches,  $S_1^2 = 0.35$ ,  $\bar{x}_2 = 8.68$  and  $S_2^2 = 0.40$ . The objective is to obtain a 95% 2-sided CI for  $\sigma_1^2 / \sigma_2^2$ . We use the fact that the rv  $(S_1^2 / \sigma_1^2) / (S_2^2 / \sigma_2^2)$  has the Fisher's  $F_{14,17}$  sampling distribution depicted by Figure 23 given next page. Since the 2.5 percentage point of the rv  $F_{14,17}$  is  $F_{0.025,14,17} = 2.75$  and its 97.5 percentage point from Eq. (42) is  $F_{0.975,14,17} = 1 / F_{0.025,17,14} = 1/2.90 = 0.3448$ , Figure 23 below clearly shows that

$$\Pr(0.3448 \leq F_{14,17} \leq 2.7526) = 0.95.$$

Or  $\Pr(0.3448 \leq \frac{S_1^2 / \sigma_1^2}{S_2^2 / \sigma_2^2} \leq 2.7526) = 0.95$

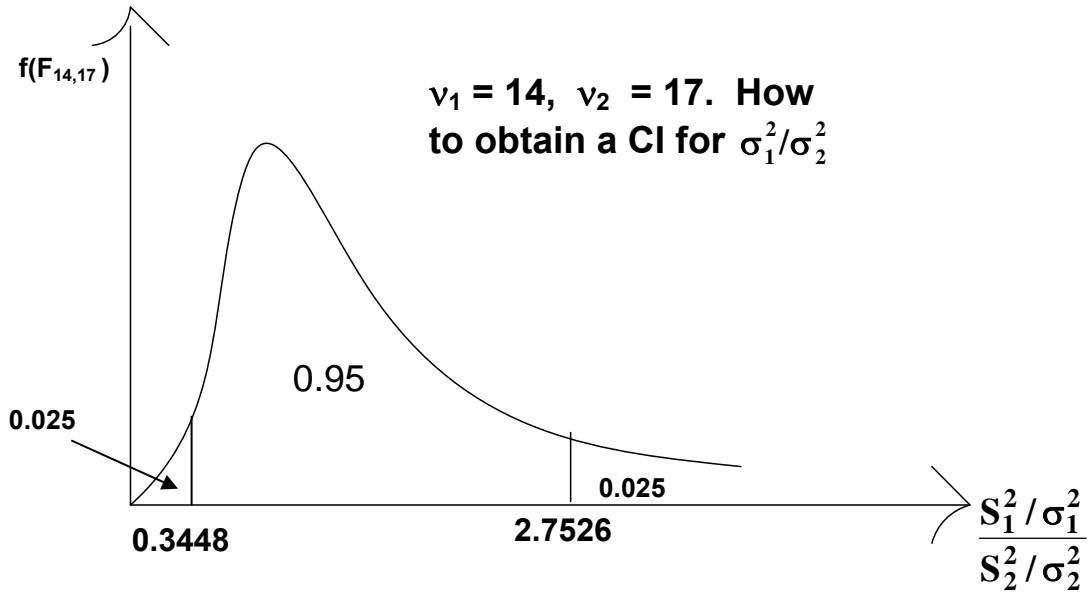


Figure 23. The Sampling Distribution of the rv  $\frac{S_1^2 / \sigma_1^2}{S_2^2 / \sigma_2^2} \sim F_{14, 17}$

$$\Pr(0.3448 \leq \frac{S_1^2 \sigma_2^2}{S_2^2 \sigma_1^2} \leq 2.7526) = 0.95 \longrightarrow \Pr(0.3448 S_2^2 / S_1^2 \leq \frac{\sigma_2^2}{\sigma_1^2} \leq 2.7526 S_2^2 / S_1^2) = 0.95$$

$$\longrightarrow \Pr\left(\frac{S_1^2}{0.3448 S_2^2} \geq \frac{\sigma_1^2}{\sigma_2^2} \geq \frac{S_1^2}{2.7526 S_2^2}\right) = 0.95, \Pr\left(\frac{S_1^2}{2.7526 S_2^2} \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{S_1^2}{0.3448 S_2^2}\right) =$$

$$0.95 \rightarrow L(\sigma_1^2 / \sigma_2^2) = \frac{S_1^2}{2.7526 S_2^2}, \text{ and } U(\sigma_1^2 / \sigma_2^2) = \frac{S_1^2}{(0.3448 S_2^2)}.$$

The use of the sample statistics  $S_1^2 = 0.35$  and  $S_2^2 = 0.40$  leads to  $L(\sigma_1^2 / \sigma_2^2) = 0.3179$  and

$U(\sigma_1^2 / \sigma_2^2) = 2.5377 \rightarrow 0.3179 \leq \sigma_1^2 / \sigma_2^2 \leq 2.5377$  at the 95 confidence level. Note that this CI

implies that the null hypothesis  $H_0 : \frac{\sigma_1^2}{\sigma_2^2} = 1$  cannot be rejected at the 5% level because the

hypothesized value of  $(\frac{\sigma_1^2}{\sigma_2^2})_0 \equiv 1$  is inside this CI:  $0.3179 \leq \sigma_1^2/\sigma_2^2 \leq 2.5377$

**Exercise 77.** (a) Repeat the above Example, i.e., obtain the same CI :  $0.3179 \leq$

$\sigma_1^2/\sigma_2^2 \leq 2.5377$ , using the pdf of  $\frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2}$ . (b) Obtain a 95% lower 1-sided CI for  $\sigma_y/\sigma_x$  for the

following data, where the rv  $X$  = Drying Time of White Paint and rv  $Y$  = Drying Time of Yellow Paint. State the corresponding hypotheses  $H_0$  and  $H_1$ . (c) Obtain a 95% lower 1-sided CI for  $\mu_y - \mu_x$  also for the following data and draw appropriate conclusions from your CIs.

$X$  : 120, 112, 116, 122, 115, 110, 120, 107 minutes

$Y$  : 126, 124, 116, 125, 109, 130, 125, 117, 129, 120 minutes.

ANS: (b)  $0.6421 \leq \sigma_y/\sigma_x < \infty$ , (c)  $1.8536 \leq \mu_y - \mu_x < \infty$ .

**Exercise 78.** Work Exercises 61, 62, 63, and 64 on pp. 385-6 of Devore (8e).

## 8. TEST OF HYPOTHESIS ABOUT THE VARIANCES OF TWO INDEPENDENT NORMAL POPULATIONS

Suppose we are interested in comparing the variability of machine  $X$  with that of machine  $Y$ , i.e., we wish to test the null hypothesis  $H_0 : \sigma_x^2 = \sigma_y^2$  versus one of the following three

alternatives:  $H_1 : \sigma_x^2 \neq \sigma_y^2$ ,  $H_1 : \sigma_x^2 < \sigma_y^2$ , or  $H_1 : \sigma_x^2 > \sigma_y^2$ , at the nominal pre-

assigned LOS  $\alpha = 5\%$ . Note that, in general, the most prevalent alternative is  $H_1 : \sigma_x^2 \neq \sigma_y^2$ .

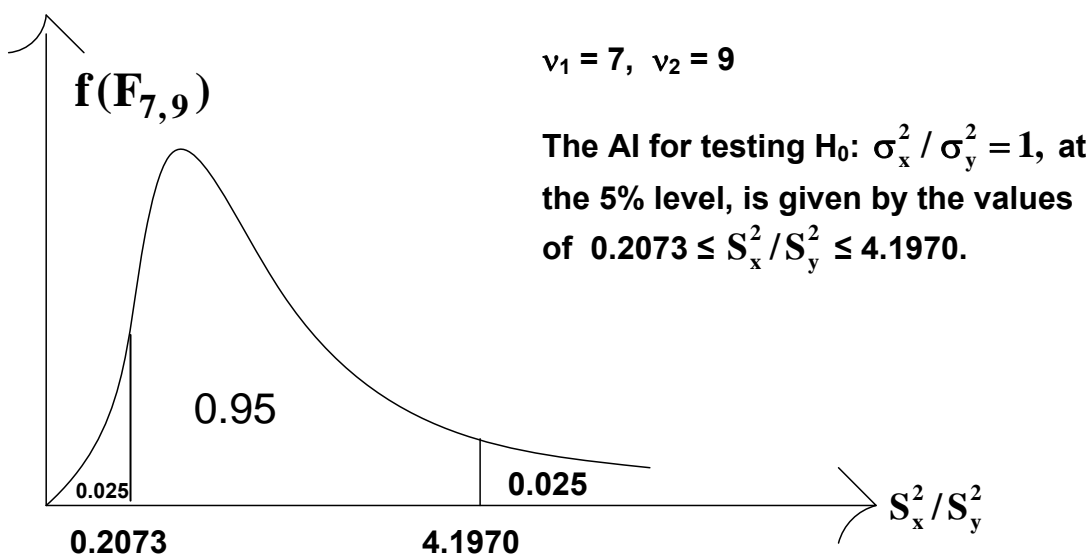
Assuming that the operations on the two machines are completely independent, then from

equation (41), the rv  $F_{n_x-1, n_y-1} = \frac{S_x^2/\sigma_x^2}{S_y^2/\sigma_y^2}$  has the Fisher's F distribution with  $df$  of the numerator

$v_1 = n_x - 1$  and that of the denominator  $v_2 = n_y - 1$ . Equation (41) implies that under the null

hypothesis  $H_0 : \sigma_x^2 = \sigma_y^2$ , the sampling distribution of the statistic  $S_x^2/S_y^2 \sim F_{n_x-1, n_y-1}$ .

As an example consider the data in Exercise 77 on page 136 of my notes, where  $X =$  Drying Time of White Paint, and  $Y =$  Drying Time of Yellow Paint with  $n_x = 8$  and  $n_y = 10$ . Our objective is to test  $H_0 : \sigma_x = \sigma_y$  VS the alternative  $H_1 : \sigma_x \neq \sigma_y$  at the 5% LOS. Since the 2.5 percentage point of  $F_{7,9}$  is  $F_{0.025,7,9} = 4.20$  and the 97.5 percentage point is  $F_{0.975,7,9} = 1/F_{0.025,9,7} = 1/4.8232 = 0.2073$ , then our AI for the test statistic  $S_x^2/S_y^2$  is  $AI = [0.2073, 4.1970] = 0.2073 \leq S_x^2/S_y^2 \leq 4.1970$  as depicted in Figure 24. Although Devore does not provide the 2&1/2 percentage points of the F



**Figure 24.** The Sampling Distribution of  $S_x^2/S_y^2$  Assuming  $H_0$  is True

distribution, you may easily use Microsoft Excel to obtain  $\text{FINV}(0.025,7,9) = 4.197047$  and  $\text{FINV}(0.025,9,7) = 4.823221$ . My website also lists the inverse function values of F. Because the value of our test statistic  $F_0 = S_x^2/S_y^2 = 28.2143/42.7666 = 0.65973$  falls inside the decision interval  $AI = [0.2073, 4.1970]$ , then the two data sets do not provide sufficient evidence to reject  $H_0$ , and therefore, we cannot deduce that the variances of the two machines are significantly different at the 5% level. By now you should be well cognizant of the fact that the *P-value* of the above test will be larger than 5%. To compute the critical level  $\hat{\alpha}$ , we have to make use of our observed test statistic  $F_0 = 0.65973$ . That is, even if the F distribution is skewed, the approximate *P-value* is  $\hat{\alpha} =$

$2 \times \Pr(F_{7,9} \leq 0.65973) = 2 \times 0.298484 \cong 0.597$ . This *P-value* implies that if we decide to reject  $H_0$ , then the Pr of committing a Type I error for such a decision is approximately 0.597.

Secondly, how do we approximate the Type II error Pr for the above example if  $H_0$  is false, say  $\sigma_y = 1.5\sigma_x$ ? Unlike many Statistics texts, Devore does not provide OC curves for testing  $H_0: \sigma_x^2 = \sigma_y^2$ , because the type II error Pr can be computed analytically. The abscissa of such OC curves, only for the case  $n_x = n_y = n$ , is defined as  $\lambda = \sigma_y/\sigma_x$  (or  $\lambda = \sigma_x/\sigma_y$ ). For the example under consideration,  $\lambda = 1.50$  so that  $\beta \cong 0.83$  (using  $n = n_x = n_y \cong 8$  in Chart VI (o) on page 669 of Montgomery and Runger (2003), third edition). Fortunately, the Pr of committing a type II error for the F test can be computed analytically, as illustrated below, and most likely this is the reason Devore does not provide the OC curves at the end of his text.

$$\begin{aligned}
 \beta(\text{at } \lambda = 1.5) &= \Pr(0.2073 < F_0 < 4.1970 \mid \sigma_y = 1.5 \sigma_x) \\
 &= \Pr\left(0.2073 < \frac{S_x^2}{S_y^2} < 4.1970 \mid \sigma_y/\sigma_x = 1.5\right) \\
 &= \Pr\left(0.2073 \times \frac{\sigma_y^2}{\sigma_x^2} < \frac{S_x^2/\sigma_x^2}{S_y^2/\sigma_y^2} < 4.1970 \times \frac{\sigma_y^2}{\sigma_x^2} \mid \lambda = 1.5\right) \\
 &= \Pr\left(0.2073 \times 2.25 < \frac{S_x^2/\sigma_x^2}{S_y^2/\sigma_y^2} < 4.1970 \times 2.25\right) \\
 &= \Pr(0.4665 < F_{7,9} < 9.4434) = \text{cdf}(9.4434) - \text{cdf}(0.4665); \text{ using MS. Excel} \\
 &= [1 - \text{FDIST}(9.4434, 7, 9)] - [1 - \text{FDIST}(0.4665, 7, 9)] \\
 &= \text{FDIST}(0.4665, 7, 9) - \text{FDIST}(9.4434, 7, 9) = 0.83634 - 0.00160 \\
 &\beta(\text{at } \lambda = 1.5) = 0.8348 \text{ (From MS Excel).}
 \end{aligned}$$

**Exercise 79.** (a) Repeat the above example using the sampling distribution of  $S_x^2 / S_y^2$ . (b) Redo the above example using the alternative  $H_1: \sigma_y > \sigma_x$  at the LOS  $\alpha = 5\%$ . (c) Compute the value of  $\beta$  for the above 2-sided test of hypothesis  $H_0: \sigma_x = \sigma_y$  if  $\sigma_y = 2\sigma_x$ . (d) For Exercises 72 on page 387 of Devore (8e), also test for variance equality of the two types of joints.

## 9. STATISTICAL INFERENCE ABOUT DIFFERENCES BETWEEN TWO POPULATION PROPORTIONS

Devore covers this topic in section 9.4 (pp. 375-380). To illustrate the procedure, I will go thru the solution of Exercise 86 on page 389 of Devore in detail. Here our null hypothesis is  $H_0 : p_1 = p_2$  VS the alternative  $H_1 : p_1 \neq p_2$ , or  $H_0 : p_1 - p_2 = 0$  vs  $H_1 : p_1 - p_2 \neq 0$ , where  $p_1$  is the proportion of eggs surviving at 11 °C and  $p_2$  is the proportion of eggs surviving at 30°C. Therefore, points unbiased estimates of  $p_1$  and  $p_2$  are, respectively,  $\hat{p}_1 = 73/91 = 0.8022$  and  $\hat{p}_2 = 102/110 = 0.9273$ . We wish to ascertain if the sample difference of  $\hat{p}_2 - \hat{p}_1 = 0.1251$  is significantly different from zero to warrant the rejection of  $H_0 : p_2 - p_1 = 0$ .

Recall that in Chapter 7 we used the fact that the SMD of  $\hat{p} = X/n$  is approximately normal as long as  $n > 50$  and  $0.10 < p < 0.90$ , with mean  $E(\hat{p}) = p$  and  $SE(\hat{p}) = \sqrt{pq/n}$ . Since the two populations are independent, then the null SMD of  $\hat{p}_2 - \hat{p}_1$  is also approximately Gaussian with  $E(\hat{p}_2 - \hat{p}_1) = p_2 - p_1 = 0$  and

$$V(\hat{p}_2 - \hat{p}_1) = V(\hat{p}_2) + V(\hat{p}_1) = \frac{p_2 q_2}{n_2} + \frac{p_1 q_1}{n_1} \quad (43)$$

as depicted in Figure 25. Since  $p_2$  is hypothesized to be equal to  $p_1$ , then in Fig. 25 the  $E(\hat{p}_2 - \hat{p}_1)$

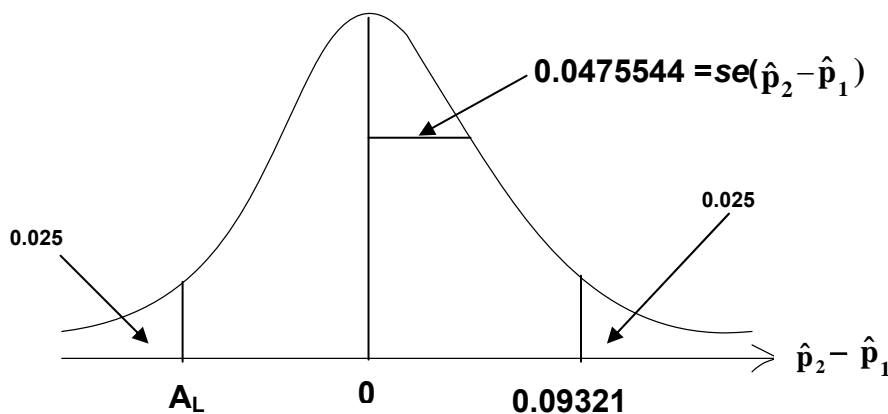


Figure 25. The Approximate Null SMD of  $\hat{p}_2 - \hat{p}_1$

= 0 under  $H_0$ , and further, the estimate of the  $SE(\hat{p}_2 - \hat{p}_1)$  will be obtained assuming that  $p_2 = p_1 = p$ . Thus, the pooled estimate of  $p$  from the combined samples is  $\hat{p} = (73 + 102) / (91 + 110) = 0.87065$ . Therefore, under  $H_0$   $se(\hat{p}_2 - \hat{p}_1) = \sqrt{\frac{\hat{p}\hat{q}}{n_1} + \frac{\hat{p}\hat{q}}{n_2}} = \sqrt{\frac{0.112621}{91} + \frac{0.112621}{110}} = 0.0475544$ .

The 5%-level AI =  $(-0.09321, 0.09321)$  implies that we cannot reject  $H_0$  iff  $-0.09321 \leq \hat{p}_2 - \hat{p}_1 \leq 0.09321$ . Since the value of our test statistic  $\hat{p}_2 - \hat{p}_1 = 0.1251$  falls outside this last AI, the data provide sufficient evidence at the 5% level to reject  $H_0$  and to conclude that the two survival rates are significantly different. The  $P$ -value of the test is given by  $\hat{\alpha} = 2 \times \Pr(\hat{p}_2 - \hat{p}_1 \geq 0.1251) = 2 \times \Pr(Z \geq 2.63014) = 2 \times \Phi(-2.63014) = 0.008535$ , which as expected is much less than  $\alpha = 0.05$ .

**Exercise 80.** (a) Obtain the 95% CI for  $p_2 - p_1$  of Exercise 86 on page 389 of Devore and determine if your CI is consistent with my test above. Note that in deriving the 95% CI, you may not assume  $p_2 = p_1 = p$ , and therefore the  $SE(\hat{p}_2 - \hat{p}_1)$  will have to be estimated from (see equation (43))  $se(\hat{p}_2 - \hat{p}_1) = \sqrt{\hat{p}_1\hat{q}_1/n_1 + \hat{p}_2\hat{q}_2/n_2}$ . I have not obtained this CI, but Minitab gives (0.02993, 0.22023) as the answer. (b) Work Exercises 53 and 55 on pp. 381 of Devore's 8<sup>th</sup> edition.

## Summary of Chapters 7, 8 and 9

We have finally come to the end of SI on parameters of one or two normal populations, and therefore, we will provide a summary of statistics and their sampling distributions in conducting statistical inference. In all cases except in the case of SI on proportion(s), the tacit assumption was made that the underlying distribution (or the parental variable  $X$ ) was normal (or Laplace-Gaussian).

(i) The sampling distributions of the rvs  $(\bar{x} - \mu)\sqrt{n} / \sigma$ ,  $(\bar{d} - \mu_d)\sqrt{n} / \sigma_d$ , and  $[(\bar{x} - \bar{y}) - (\mu_x - \mu_y)] / \sqrt{\sigma_x^2/n_x + \sigma_y^2/n_y}$ , are  $N(0, 1)$ , while those of  $(\bar{x} - \mu_x)\sqrt{n} / S_x$ ,  $(\bar{d} - \mu_d)\sqrt{n} / S_d$  (for paired-samples), and  $[(\bar{x} - \bar{y}) - (\mu_x - \mu_y)] / [S_p \sqrt{1/n_x + 1/n_y}]$  follow the Student's-t with  $(n-1)$ ,  $(n-1)$ , and  $(n_x + n_y - 2)$   $df$ , respectively. This last t-statistic also requires

the assumption that  $\sigma_x = \sigma_y$ . Further, you must be cognizant of the fact that the above six sampling distributions are used only in conducting statistical inference(s) on one or two population means. If the assumption  $\sigma_x^2 = \sigma_y^2$  is not tenable (or rejected at the 20% level), then the two-independent-sample t-statistic for testing  $H_0: \mu_x - \mu_y = \delta_0$  is given by  $t_0 = [(\bar{x} - \bar{y}) - \delta_0] /$

$$\sqrt{\frac{S_x^2}{n_x} + \frac{S_y^2}{n_y}}, \text{ which has an approximate Student's t-distribution with } df, v, \text{ given near the bottom}$$

of page 357 of Devore and repeated below:

$$v = \frac{(S_x^2 / n_x + S_y^2 / n_y)^2}{(S_x^2 / n_x)^2 / (n_x - 1) + (S_y^2 / n_y)^2 / (n_y - 1)} = \frac{v_y v_x (k^2 + 1)^2}{v_y k^4 + v_x},$$

where  $k = (S_x \sqrt{n_y}) / (S_y \sqrt{n_x})$  is called the *se* ratio.

(ii) The sampling distribution of  $(n - 1)S^2/\sigma^2$  follows a  $\chi_{n-1}^2$  and was used to conduct statistical inference on one process variance  $\sigma^2$ . However, the Chi-square distribution has numerous other applications, a few of which will be discussed in Chapter 14.

(iii) The sampling distribution of  $\frac{S_x^2 / \sigma_x^2}{S_y^2 / \sigma_y^2}$  follows the Fisher's F distribution with *df* of the

numerator  $v_1 = n_x - 1$  and that of the denominator  $v_2 = n_y - 1$ . Thus far, the F distribution was used only to conduct statistical inference on  $\sigma_x^2 / \sigma_y^2$ . Under  $H_0: \sigma_x^2 = \sigma_y^2$ , the SMD of  $S_y^2 / S_x^2 \sim F_{n_y-1, n_x-1}$ .

In Chapters 10, 12, and 13 you will, however, learn that the Fisher's F distribution has extensive applications in almost all facets of sciences and engineering. For example, you will study that the F-statistic is used to test the null hypothesis  $H_0: \mu_1 = \mu_2 = \mu_3 = \dots = \mu_a$  VS the alternative that at least two treatment means differ significantly. However, the F-test in this case will always be right-tailed even if the alternative is not one-sided.

(iv) The sampling distribution of sample proportion,  $\hat{p} = X/n$  where  $X \sim B(n, p)$ , is approximately normal with population mean  $p$  and its SE given by  $SE(\hat{p}) = \sqrt{pq/n}$ . The

normal approximation requires that  $n > 50$ ,  $0.10 < p < 0.90$ , and  $n \times p > 10$ . To be on the conservative side, it will be best to have  $np > 15$ . When  $np < 10$ , the Poisson approximation to the binomial is appropriate; the region  $10 < np < 15$  is gray, and for which it is not clear-cut as to which approximation (Poisson or normal) to the  $B(n, p)$  is more accurate.

Further, the SMD of  $\hat{p}_1 - \hat{p}_2$  is also approximately normal with  $E(\hat{p}_1 - \hat{p}_2) =$

$p_1 - p_2$  and  $SE(\hat{p}_1 - \hat{p}_2) = \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}$ . Only under  $H_0: p_1 = p_2 = p$  this last SE reduces to

$$SE(\hat{p}_1 - \hat{p}_2 | H_0) = \sqrt{\frac{pq}{n_1} + \frac{pq}{n_2}}.$$

Finally, you should always bear in mind that all CIs are simply tests of hypotheses in disguise! Therefore, to be consistent, the lower one-sided CI:  $\theta_L \leq \theta < \infty$  always corresponds to a right-tailed test on the parameter  $\theta$ , and vice versa for an upper one-sided CI. Further, the student must not confuse a decision interval (DI) for a test-statistic with the corresponding CI for the parameter under the null hypothesis. The former are values of the test-statistic that lead to a decision regarding  $H_0$ , while the latter consists of parameter values at a given confidence level. Note that I have also referred to decision intervals as acceptance intervals.