Computing the Probabilities of all Outcomes when 5 Cards Are Drawn at random W/O Replacement (and then also with replacement) from a Well-Shuffled Deck of 52 Cards

Let \( A_i \) represent the event that exactly \( i \) cards that are same, \( i = 0, 2, 3, 4 \). The index \( i \) can equal 5 only in the case of with replacement.

1(Drawing W/O Replacement). Then, the event \( A_0 \) represents an outcome such as \( \{5J96A\} \), where none of the cards are the same. Thus,

\[
P_0 = \Pr(A_0) = \frac{52 \times 48 \times 44 \times 40 \times 36}{52 \times 51 \times 50 \times 49 \times 48} = 0.507082833133,
\]

i.e., roughly 51% of the times all the 5 cards will be distinct. Note that the above hands include straights, flushes and straight-flushes.

1(With Replacement). It is best to think of this situation as drawing 5 cards at random from 5 distinct deck of well-shuffled of 52 cards, one card from each deck of 52 cards.

\[
P_0 = \Pr(A_0) = \frac{52 \times 48 \times 44 \times 40 \times 36}{52 \times 52 \times 52 \times 52 \times 52} = 0.41595182242
\]

2(W/O Replacement). The event \( A_2 \) represents an outcome such as \( \{5J9JA\} \), where exactly 2 cards are the same and the other 3 are distinctly different, i.e., \( A_2 \) represents exactly one-pair.

\[
P_2 = \Pr(A_2) = \frac{4 \times 3 \times 48 \times 44 \times 40}{52 \times 51 \times 50 \times 49 \times 48} \times \binom{5}{2} = 0.42256902761.
\]

The multiplier \( \binom{5}{2} \) is needed because of \( (JJA59), (JAJ59), (JA5J9), \ldots \).
(A59JJ), of which there are 10 combinations, i.e., this is how many ways we can place the 2 jacks in the 5 drawings.

2(With Replacement). (5 decks of card; exactly one card drawn at random at random from each)

\[ \Pr(A_2) = \left(\frac{4}{52}\right)^2\left(\frac{48}{52}\right)\left(\frac{44}{52}\right)\left(\frac{40}{52}\right) \times \binom{5}{2} \times 13 = 0.462168692 \]

3(W/O Replacement). The event \(A_3\) represents an outcome such as \{5J9JJ\}, where exactly 3 cards are the same and the other 2 are distinctly different, i.e., \(A_3\) represents exactly 3-of-a-kind.

\[ P_3 = \Pr(A_3) = \frac{4 \times 3 \times 2 \times 48 \times 44}{52 \times 51 \times 50 \times 49 \times 48} \times \binom{5}{3} \times 13 = 0.021128451380 \]

3(With Replacement).

\[ \Pr(A_3) = \left(\frac{4}{52}\right)^3\left(\frac{48}{52}\right)\left(\frac{44}{52}\right) \times \binom{5}{3} \times 13 = 0.04621686916 \]

4(W/O Replacement). The event \(A_4\) represents an outcome such as \{JJ9JJ\}, where exactly 4 cards are the same and the other per force is different, i.e., \(A_4\) represents exactly 4-of-a-kind.

\[ P_4 = \Pr(A_4) = \frac{13 \times \binom{4}{4} \times \binom{48}{1}}{\binom{52}{5}} = 0.0002400960384154 \]

Or:

\[ P_4 = \Pr(A_4) = \frac{4 \times 3 \times 2 \times 1 \times \binom{5}{4} \times 48 \times 13}{52 \times 51 \times 50 \times 49 \times 48} \]

4(With Replacement).

\[ \Pr(A_4) = \left(\frac{4}{52}\right)^4\left(\frac{48}{52}\right) \times \binom{5}{4} \times 13 = 0.00210076677987 \]

5(Only With Replacement).

\[ \Pr(A_5) = \left(\frac{4}{52}\right)^5 \times 13 = 0.00003501277966 \]
6(W/O Replacement). Let the event $A_{22}$ represent an outcome such as \{5J9J5\}, where there are exactly 2 different pairs but the 5th card is distinctly different from each of the 2 pairs, i.e., $A_{22}$ represents exactly Two-pairs.

$$P_{22} = \Pr(A_{22}) = \frac{\binom{13}{2} \times (\binom{4}{2})^2 \times (\binom{4}{2}) \times 44}{\binom{52}{5}} = 0.04753901561.$$  

6(With Replacement).

$$P_{22} = \Pr(A_{22}) = \frac{4}{52}^2 \left(\frac{4}{52}\right)^2 \left(\frac{48}{52}\right) \times (\binom{13}{2}) \times (\binom{5}{2}) \times (\binom{3}{2}) = 0.06932530374$$

7(W/O Replacement). Let the event $A_{23}$ represent an outcome such as \{5J5J5\}, which is called a full-house.

$$P_{23} = \Pr(A_{23}) = \frac{\binom{13}{2} \times (\binom{4}{3}) \times (\binom{4}{2}) \times 2}{\binom{52}{5}} = 0.001440576.$$  

7(With Replacement).

$$P_{23} = \Pr(A_{23}) = (\binom{13}{2}) \times 2 \times (\binom{5}{2}) \times \left(\frac{4}{52}\right)^2 \left(\frac{4}{52}\right)^3 = 0.00420153356.$$