CONTINUOUS RANDOM VARIABLES

Let X be a continuous rv with range space R_x (a subset of real numbers). Since the Pr that the rv, X, takes on a single specific value in R_x is zero, then Prs over a continuum must be defined only over intervals within R_x. The function, f(x), is said to be a Pr density function (pdf) over R_x iff

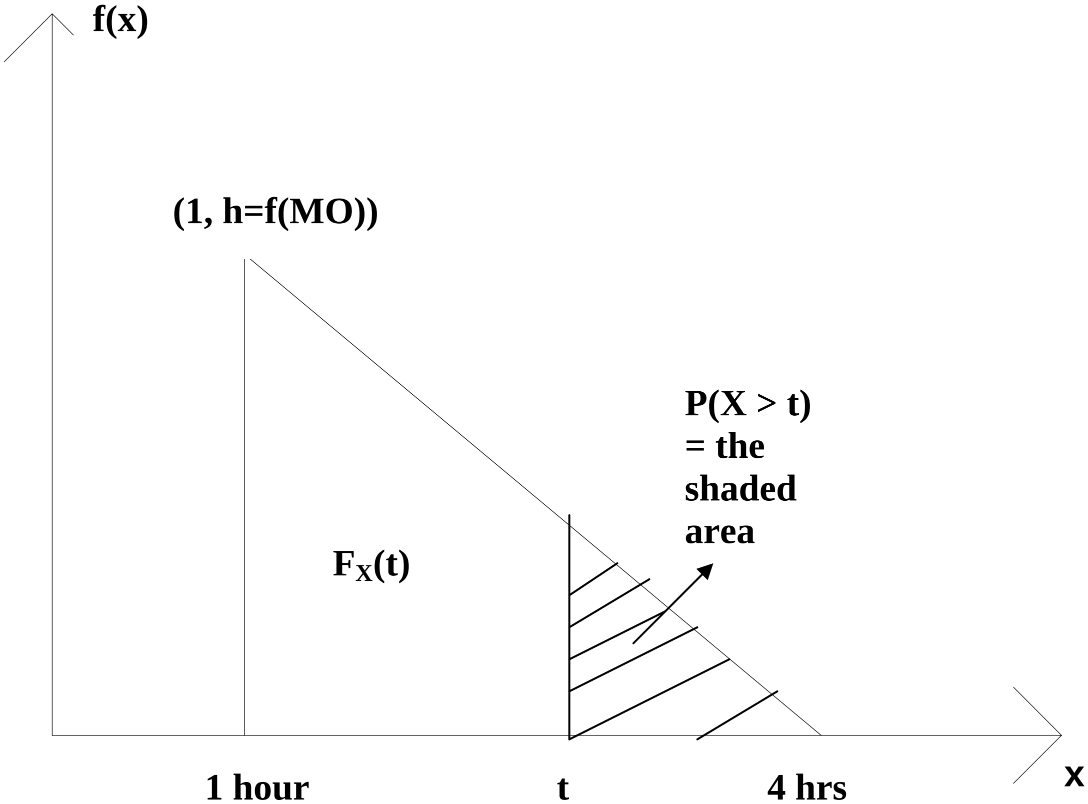
(1) $f(x) \geq 0$ for all x in R_x, (2) $\int_{R_X} f(x)\,dx = 1$, (3) f(x) must be (piecewise) continuous, (4) f(x) = 0 for all $x \notin R_x$, and (5) $P(a \leq X \leq b) = P(a \leq X < b) = P(a < X < b) = P(a < X \leq b) = \int_a^b f(x)dx$. Note that all these last 4 Prs are equal because the pr for a single point (or any set of measure 0) for any continuous rv is identically zero.

<u>Example .</u> The repair time of certain engines, X in hours, of an automobile is given by the pdf $f(x) = a + bx$, $1 \leq x \leq 4$ hours and f(x) = 0 elsewhere. The interval [1, 4 hrs] is called the range space of X.

(a) Determine the constants a and b such that f(x) is a Pr density function (pdf).

The graph of the above function is given below. Because the area under a triangle is equal to (1/2)base×height, then to

make the total area equal to 1 we require that (1/2)3×h = 1 → h = 2/3.



Note that the modal point of the above pdf is MO = 1 and f(MO) = h.

To obtain the Pdf itself, we insert the points (1, 2/3) and (4, 0) into f(x) = a + bx.

2/3 = a + b and 0 = a + 4b → – 2/3 = 3b → b = – 2/9, and a = +8/9 →

f(x) = = (– 2/9)x + 8/9, R_x = [1, 4 hours].

$$\int_{R_X} f(x)\,dx = \int_1^4 \left(\frac{-2x+8}{9}\right) dx = \frac{1}{9}\left[-x^2+8x\right]_1^4 = \frac{1}{9}\,[16 - (-1+8)]$$

= (16 – 7)/9 = 1 → Thus, f(x) is a pdf.

(b) Compute the Pr that the next repair time lies in the interval [1.5, 2 hours].

P(1.5 < X ≤ 2 hours) = $\int_{1.5}^{2}(-2x/9+8/9)\,dx = \left[\frac{-x^2}{9}+\frac{8x}{9}\right]_{1.5}^{2}$ =

$\frac{12}{9} - \frac{9.75}{9}$ = $\frac{2.25}{9}$ = 1/4 = 0.25 .

(c) **Obtain the cdf (cumulative distribution function) of repair time, X, at time t.**

F_X (t) = P(X ≤ t) = $\frac{1}{9}\int_{1}^{t}(-2x+8)\,dx = \frac{1}{9}\left[-x^2+8x\right]_1^t$ = $\frac{1}{9}$ (– t^2 + 8t -7)

Thus, F_X (x) = $\begin{cases} 0, & x \le 1, \\ (-x^2+8x-7)/9, & 1 \le x \le 4, \\ 1, & x \ge 4 \text{ hours} \end{cases}$

Because the 2nd derivative of the above F(x) < 0, then F(x) is strictly concave downward. Further, a function is strictly convex (upward) over R_x iff its 2nd derivative exceed zero for all x in R_x.

Note that once the cdf of a rv is obtained, then all Prs can be computed using the cdf. For example, we may use the cdf to recompute the probability of part (b) above.

P(1.5 < X ≤ 2 hours) = P(X ≤ 2) – P(X ≤ 1.5) = F_X(2)– F_X(1.5) = (– 4 + 16 – 7)/9 – (– 2.25 +12 – 7)/9 = 5/9 – (2.75/9) = 2.25/9 = 1/4 = 0.25, as before.

(d) Use the cdf to compute the conditional Pr that a repair job will last more than 3 hours given that it has already lasted 2

hours.

P(X > 3 hours $|$ **X ≥ 2) =** $\frac{P(X>3)}{P(X>2)} = \frac{1-P(X<3)}{1-P(X\leq 2)} = \frac{1-F_X(3)}{1-F_X(2)} =$

$\frac{1-8/9}{1-5/9} = \frac{9-8}{9-5}$ **= 1/4 = 0.25.**

THE EXPECTED VALUE OF A CONTINUOUS RV

Definition: **Recall that if X is a discrete rv over the range R_x with pmf p(x), then the population mean of the rv, X, is defined as its expected value and is given by**

$$\mu = E(X) = \sum x\, p(x),$$ **where the sum extends over R_x.**

If X is continuous with pdf f(x), then

$$\mu = E(X) = \int_{R_X} x\, f(x)\, dx\ .$$

Since an integral is simply summing over a continuum and f(x)dx represents the pr element at x (actually over an interval of length dx), then the form of E(X) for a continuous rv is identical to that of the discrete case.

Example (continued). **Compute the average repair time per job (or mean time to complete a maintenance task).**

$$\mu = E(X) = \int_1^4 x(-2x/9 + 8/9)\,dx = \left[-\frac{2x^3}{27} + \frac{4x^2}{9}\right]_1^4 = 2.00.$$

THE VARIANCE OF A CONTINUOUS RV

As before, the population variance is the long-run (or weighted) average of deviations from the population mean (μ) squared, i.e.,

$$V(X) = \sigma^2 = E[(X - \mu)^2] = \int_{R_X} (x-\mu)^2 f(x)dx = \int_{R_x} (x^2 - 2\mu x + \mu^2) f(x)dx$$

$$=\ldots = \int_{R_X} x^2 f(x)dx - \mu^2. \quad E(X^2) = \int_1^4 x^2(-2x/9 + 8/9)dx = \left[-\frac{x^4}{18} + \frac{8x^3}{27} \right]_1^4$$

$$= 4.50 \rightarrow \sigma_x^2 = V(X) = 0.50 \rightarrow CV_x = \frac{\sqrt{0.50}}{2} = 35.35534\%$$

THE MOMENTS OF A RANDOM VARIABLE

Definition: Let X be a rv with the range space R_x and let c be any known constant. Then the k^{th} moment of X about the constant c is defined as

$$M_k(X) = E[(X - c)^k]. \qquad (12)$$

In the field of statistics only 2 values of c are of interest: c = 0 and c = μ. Moments about c = 0 are called origin moments and are denoted by μ_k', i.e., $\mu_k' = E(X^k)$, where c = 0 has been inserted into equation (12). Moments about the population mean, μ, are called central moments and are denoted by μ_k, i.e,

$\mu_k = E[(X-\mu)^k]$, where $c = \mu$ has been inserted into (12).

STATISTICAL INTERPRETATION OF MOMENTS

By definition of the k^{th} origin moment, we have:

$$\mu_k' = \begin{cases} E(X^k) = \sum_{R_X} x^k p(x), & \text{if } X \text{ is a discrete rv} \\ E(X^k) = \int_{R_X} x^k f(x)dx, & \text{if } X \text{ is continuous} \end{cases}$$

(1) Whether X is discrete or continuous, $\mu_1' = E(X) = \mu$, i.e., the 1st origin moment is simply the population mean (i.e., μ_1' measures central tendency).

(2) Since the population variance, σ^2, is the weighted average of deviations from the mean squared over all elements of R_X, then $\mu_2 = E[(X-\mu)^2] = \sigma^2$. Therefore, the 2nd central moment, $\mu_2 = \sigma^2$, is a measure of dispersion (or variation, or spread) of the population. Further, the 2^{nd} central moment can be expressed in terms of origin moments using the binomial expansion of $(X-\mu)^2$, as shown below.

$$\mu_2 = E[(X-\mu)^2] = E[(X^2 - 2\mu X + \mu^2)] = E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - \mu^2 = \mu_2' - (\mu_1')^2 = \sigma^2 . \qquad (13)$$

<u>Example (continued)</u>. $\mu_1' = E(X) = 2.00$, $\mu_2' = E(X^2) = 4.5$

(3) The 3rd central moment, μ_3, is a measure of skewness (bear in mind that $\mu_3 \equiv 0$ for all symmetrical distributions). If X is continuous, then

$\mu_3 = E[(X-\mu)^3] = \int_{R_X} (x^3 - 3\mu x^2 + 3x\mu^2 - \mu^3) f(x)\,dx$

$$= \mu_3' - 3\mu_2'\mu + 2\mu^3 \qquad (14)$$

For our example, we have shown that $\mu = \mu_1' = 2$, $\mu_2' = 4.5$.

$$\mu_3' = E(X^3) = \int_1^4 x^3(-2x/9 + 8/9)\,dx = \left[-\frac{2x^5}{45} + \frac{2x^4}{9}\right]_1^4 = 11.20$$

$$\rightarrow \mu_3 = \mu_3' - 3\mu_2'\mu + 2\mu^3 = 11.20 - 3(4.5)(2) + 2(2^3) = 0.20$$

If X is measured in hours, the units of μ_3 are expressed in terms of hours^3. To obtain a unit-less measure of asymmetry (for comparative purposes), we standardize μ_3 to obtain the coefficient of skewness (most authors refer to this coefficient simply as skewness) given below:

$$\alpha_3 = \mu_3 / \sigma^3 .$$

$\alpha_3 = 0.20/(0.5)^{1.5} = 0.565685425 > 0$ which is unit-less and shows that our pdf is positively skewed (long tail on the RHS), and thus $\mu > x_{0.50} > MO$

It is not common for the value of $\alpha_3 = \mu_3/\sigma^3$ to lie outside the interval $[-2, 2]$.

(4) The 4th central moment is a measure of Kurtosis (peakedness in the middle and heavy Prs at the tails) and is given by (for a continuous rv)

$$\mu_4 = \int_{R_X} (x-\mu)^4 f(x)\,dx = \int_{R_x} (x^4 - 4x^3\mu + 6x^2\mu^2 - 4x\mu^3 + \mu^4) f(x)\,dx$$

This last expression after simplifying reduces to

$$\mu_4 = \mu'_4 - 4\,\mu'_3\,\mu + 6\,\mu'_2\,\mu^2 - 3\,\mu^4 \qquad (15)$$

For our example, $\mu'_4 = \int_1^4 x^4(-2x/9 + 8/9)\,dx = \left[-\frac{x^6}{27} + \frac{8x^5}{45}\right]_1^4 = 30.20$

$\rightarrow \mu_4 = \mu'_4 - 4\,\mu'_3\,\mu + 6\,\mu'_2\,\mu^2 - 3\,\mu^4 = 30.20 - 4(11.2)(2) + 6(4.5)(2^2) - 3(2^4) = 0.60.$

Again in order to obtain a unit-less measure of kurtosis, we 1st standardize μ_4 to obtain a (unitless) measure of kurtosis defined as

$$\alpha_4 = \beta_2 = \mu_4 / \sigma^4 .$$

we then normalize the value of $\alpha_4 = \beta_2 = \mu_4/\sigma^4$ by 3 and use the terminology kurtosis = $(\mu_4 / \sigma^4) - 3 = (0.6)/0.5^2 - 3 = 2.40 - 3 = - 0.60 = \beta_4$.

All Triangular distributions in the universe have a Kurtosis $\beta_4 = \alpha_4 - 3 = \mu_4 / \sigma^4 - 3 = - 0.60000$ (Platykurtic).

The percentiles of Continuous rv X (i.e., inverting the cdf)

The median is a point on the abscissa such that $F_X(x_{0.5}) = 0.50$. For our example,

$\frac{1}{9}(-x^2_{0.50} + 8x_{0.5} - 7) = 0.50 \rightarrow -x^2_{0.50} + 8x_{0.5} - 7 = 4.5 \rightarrow$

$x^2_{0.50} - 8x_{0.5} + 11.5 = 0 \rightarrow x_{0.50} = (8 \pm \sqrt{64 - 4\times 11.5})/2 =$

$4 \pm 2.121320 \rightarrow x_{0.50} = 4 - 2.121320 = 1.87868 \rightarrow$ MO = 1 < $x_{0.50}$= 1.87868 < μ = 2 because $\alpha_3 > 0$.

Recall from chapter 1 that the IQR = Q3 − Q1 = $x_{0.75} - x_{0.25}$ is another measure of variability. To obtain the 25th percentile (or the 0.25 quantile) of the above triangular density, we set its cdf equal to 0.25 and solve for the corresponding value of x : $\frac{1}{9}(-x^2_{0.25} + 8x_{0.25} - 7) = 0.25 \rightarrow$ Q1 = $x_{0.25}$ = 1.401924.

Similarly Q3 = $x_{0.75}$ = 2.50. Hence, the value of the IQR for the triangular density of our example is IQR = 1.098076.

In general to obtain any percentile, x_p, of the rv X, we set the cdf equal to p, where 0 < p < 1. Thus, for our example the percentile (or quantile) function, x_p, is obtained from

$F(x_p) = p \rightarrow \frac{1}{9}(-x_p^2 + 8x_p - 7) = p \rightarrow -x_p^2 + 8x_p - 7 = 9p$

$\rightarrow x_p^2 - 8x_p + (7 + 9p) = 0 \rightarrow x_p = (8 \pm \sqrt{64 - 4(7+9p)})/2 \rightarrow$

$x_p = (8 \pm \sqrt{36 - 36p})/2 \rightarrow \quad x_p = (8 \pm 6\sqrt{1-p})/2 \rightarrow$

$$x_p = 4 - 3\sqrt{1-p}$$

The percentile function $x_p = 4 - 3\sqrt{1-p}$, $0 \le p \le 1$, is also called the quantile function. Therefore, the inverse function of F(x) is given by

$$F^{-1}(x) = 4 - 3\sqrt{1-x}, \quad 0 \le x \le 1$$

because $F[F^{-1}(x)] = \frac{1}{9}[-(4 - 3\sqrt{1-x})^2 + 8(4 - 3\sqrt{1-x}) - 7] =$ x. Similarly, $F^{-1}[F(x)] = 4 - 3\sqrt{1-F(x)} =$

$4 - 3\sqrt{1-(-x^2+8x-7)/9} = 4 - \sqrt{9-(-x^2+8x-7)} =$

$4 - \sqrt{x^2 - 8x + 16} = 4 - \sqrt{(4-x)^2} = x.$

Examples of Inverse Functions

(a) $f(x) = x^2,\ x \ge 0 \rightarrow f^{-1}(x) = \sqrt{x}$

(b) $f(x) = x^{1/3} \rightarrow f^{-1}(x) = x^3, \quad -\infty < x < \infty$

(c) $f(x) = e^x \rightarrow f^{-1}(x) = \ln(x)$

(d) $f(x) = \sin(x) \rightarrow f^{-1}(x) = \arcsin(x)$

(e) $f(x) = \frac{1}{2}\ln(\frac{1-x}{1+x}) \rightarrow f^{-1}(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(f) $f(x) = 10^x \rightarrow f^{-1}(x) = \log_{10}(x)$

The Gamma Function

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx = \int_0^\infty u\, dv = u v\Big]_0^\infty - \int_0^\infty v\, du \qquad (20)$$

$u = x^{n-1}$, $dv = e^{-x}\,dx$ One integration by parts yields

$$\Gamma(n) = \left[x^{n-1}(-e^{-x}) \right]_0^{\infty} + (n-1)\int_0^{\infty} x^{(n-1)-1} e^{-x}\,dx = (n-1)\,\Gamma(n-1)$$

$$\text{or } \Gamma(n+1) = n\,\Gamma(n) \qquad (21)$$

Thus, if we integrate (20) a total of (n – 1) times by parts, we obtain

$$\Gamma(n) = (n-1)(n-2)(n-3)\ldots1.\Gamma(1)$$

But $\Gamma(1) = \int_0^{\infty} x^0 e^{-x}\,dx = \left[-e^{-x} \right]_0^{\infty} = 1$, and hence

$\Gamma(n) = (n-1)(n-2)(n-3)\ldots1 = (n-1)! \rightarrow \Gamma(1) = (1-1)! = 0! = 1$

For example, $\Gamma(5) = 4! = 24$, while $\Gamma(7) = 6! = 720$, etc.

Further, it can be proven that $\Gamma(1/2) = \sqrt{\pi}$.

If n = 3/2, from equation (21), $\Gamma(n+1) = n\,\Gamma(n)$ so that $\Gamma(3/2) = \Gamma(\frac{1}{2} + 1) = \frac{1}{2}\,\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$. Similarly, $\Gamma(5/2) = \Gamma(\frac{3}{2} + 1) = \frac{3}{2}\,\Gamma(3/2) = 3\sqrt{\pi}\,/\,4$. You should verify for yourself that $\Gamma(7/2) = 15\sqrt{\pi}\,/\,8$, etc.

In the expression for the gamma function $\Gamma(n)$, let $x = \lambda t \rightarrow dx = \lambda\,dt$. Substituting the transformation $x = \lambda t$ into equation (20) yields

$$\Gamma(n) = \int_0^{\infty} (\lambda t)^{n-1} e^{-\lambda t} \lambda\,dt \quad \rightarrow \quad 1 = \int_0^{\infty} \frac{\lambda}{\Gamma(n)} (\lambda t)^{n-1} e^{-\lambda t}\,dt \,, \qquad (23)$$

which implies that the integrand must be a density function over the range $R_x = [0, \infty)$. However, the integrand in (23), namely $\frac{\lambda}{\Gamma(n)}(\lambda t)^{n-1} e^{-\lambda t}$, is simply the Gamma pdf, denoted by $f(x; n, \lambda)$, i.e., $f(x; n, \lambda) = \frac{\lambda}{\Gamma(n)}(\lambda t)^{n-1} e^{-\lambda t}$, $0 \le t < \infty$, where λ is called the scale and n is called the shape parameter because $\alpha_3 = 2 / \sqrt{n}$.

Specific cases of the Gamma density

(1) $n = 1 \rightarrow f(x) = \frac{\lambda}{\Gamma(1)}(\lambda t)^{1-1} e^{-\lambda t} = \frac{\lambda}{0!} e^{-\lambda t} = \lambda e^{-\lambda t} \rightarrow$ The exponential pdf at the Poisson occurrence rate of λ per time unit.

<u>Application Example</u>

The time to failure, X in hours, of a computer that guides a 4-engine aircraft has the pdf $f(x) = c\, e^{-0.0004\, x}$, $x \ge 0$ and $f(x) = 0$ for $x < 0$. (a) Determine the constant c such that f(x) is a Pr density function.

$$\int_{R_X} f(x)\, dx = \int_0^{\infty} c e^{-0.0004x}\, dx = c \left[\frac{e^{-0.0004x}}{-0.0004}\right]_0^{\infty} = c\,[-(0-1)/0.0004]$$

$$= c / 0.0004 = 1 \quad \rightarrow \quad c = 0.0004.$$

Therefore, the only unique value of the constant c that makes the function $c\, e^{-0.0004\, x}$, $x \ge 0$ a pdf (Pr density function) is c =

0.0004.

$f(x) = 0.0004\, e^{-0.0004\,x}$ is called the exponential density function at a rate of $\lambda = 0.0004$ failures (or Poisson events) per hour.

(b) Compute the Pr that the computer will survive a 10-hour flight.

$$P(X > 10 \text{ hours}) = \int_{10}^{\infty} 0.0004 e^{-0.0004x}\, dx = e^{-0.0004 \times 10} = e^{-0.004} =$$

$0.996007989344 \cong$ roughly 4 failures in 1000 flights.

The above Pr is called the reliability (or survival Pr) of the PC at 10 hours.

This implies that the failure Pr of the PC during the 10-hour flight is $P(X \le 10) = 1 - 0.996007989344 = 0.003992010656$, while the reliability at 10 hours is equal to R(at 10 hours) = 0.996008

(c) Obtain the cdf (cumulative distribution function) of TTF, X, at time t.

$$F_X(t) = P(X \le t) = \int_0^t \lambda e^{-\lambda x}\, dx = \left[\frac{e^{-\lambda x}}{-1}\right]_0^t = 1 - e^{-\lambda t} = 1 - e^{-0.0004t}$$

Note that once the cdf of a rv is obtained, then all Prs can be computed using the cdf. For example, we may use the cdf to recompute the reliability of the PC in part (b) above.

$R(10 \text{ hours}) = P(TTF > 10) = 1 - P(X \le 10) = 1 - F_X(10) =$

$1 - (1 - e^{-\lambda t}) = e^{-\lambda t} = e^{-10\lambda} = e^{-0.004} = 0.996008 =$

Pr[X(10 hours) = 0]

(d) Use the cdf to compute the Pr that the PC will fail within the interval (10, 20 hours).

$P(10 \le X \le 20 \text{ hours}) = P(X \le 20) - P(X \le 10) = F(20) - F(10) =$

$$= (1 - e^{-20\lambda}) - (1 - e^{-10\lambda}) = e^{-10\lambda} - e^{-20\lambda} =$$

$$= R(10) - R(20) = e^{-0.004} - e^{-0.008} = 0.0039761.$$

(e) Suppose it is known that on a certain flight the PC has already lasted 7 hours. What is the Pr that its lifetime will go beyond 10 hours?

$$P(X > 10 \mid X > 7) = \frac{P(X > 10 \cap X > 7)}{P(X > 7)} = \frac{P(X > 10)}{P(X > 7)} = \frac{e^{-10\lambda}}{e^{-7\lambda}} = e^{-3\lambda}$$

$= P(X > 3 \text{ hours} \mid X > 0) = P(X > 3 \text{ hours}) = R(3) = e^{-0.0012} =$ **0.99880072**

The above developments show that the exponential density function is memory-less because the conditional pr that the PC's lifetime exceeds 10 hours given that it has lasted 7 hours is the same as the unconditional pr that the PC lasts 3 hours from time zero. Thus, in general the exponential pdf has the following memory-less property

$$P(X > a+b \mid X > a) = P(X > b) = e^{-\lambda b}.$$

The discrete analogue of the exponential density is the Geometric distribution g(x; p). To my knowledge, these are the only two memory-less statistical distributions.

The percentile (or quantile) function of the exponential at the rate λ is given by $x_p = \frac{1}{\lambda}\ln(\frac{1}{1-p})$, $0 \le p \le 1$, i.e., the inverse function of the exponential density is given by $F^{-1}(x) = \frac{1}{\lambda}\ln(\frac{1}{1-x})$. For example, the median is given by $x_{0.50} = F^{-1}(0.50) = \frac{1}{\lambda}\ln(\frac{1}{1-0.5}) = \frac{1}{\lambda}\ln(2) = 0.6931472/\lambda < 1/\lambda = \mu$ because the exponential is positively skewed with $\alpha_3 = 2$ and its kurtosis is equal to $\beta_4 = 6$.

Consider a Poisson event that occurs at a rate of λ per unit of time. Then the average number of occurrences during an interval of length t is λt. Let X(t) represent the number of Poisson events (such as failures) occurring during an interval of length t, i.e., $R_{X(t)} = \{0, 1, 2, 3, ...\}$. Because the Poisson pmf is given by $P[X(t) = x] = \frac{(\lambda t)^x}{x!} e^{-\lambda t}$, x = 0, 1, 2, 3, 4, ….., then

$$P[X(t) = 0] = \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t} .$$

Next, define a continuous rv, T, as follows:

T = the time between the occurrences of 2 successive Poisson events (or the interoccurrence or intervening time of 2 successive Poisson events). Then the following two events are equivalent.

[T > t] = [No. of failures during t = X(t) = 0]

Thus,

$$R(t) = P(T > t) = P[X(t) = 0] = e^{-\lambda t} = 1 - P(T \leq t) = 1 - F(t) \rightarrow$$

$$F(t) = 1 - e^{-\lambda t} \quad \rightarrow f(t) = dF(t)/dt = \lambda e^{-\lambda t}$$

The above developments show that if the number of occurrences of an event is Poisson distributed, then the intervening time, T, of the next Poisson event measured from the last occurrence is exponentially distributed.

2. n > 1

Example (continued)

Consider a 2-PC standby system that guides an aircraft each PC with a failure rate of $\lambda = 0.0004$/hour. Compute the system reliability for a mission of 10-hour flight.

R(10) = P(Time to the 2nd failure measured form zero $\geq$ 10 hours) = $P(T_2 \geq 10$ hours) = P[X(10 hours) $\leq$ 1 failure] =

$$= \sum_{x=0}^{1} \frac{(0.004)^x}{x!} e^{-0.004} = 0.99999202 \cong 8 \text{ failures in a}$$

million.

Note that X(10 hours) denotes the number of failures occurring during 10 hours, which has a Poisson pmf with mean $\mu = \lambda t = 0.0004 \times 10 = 0.004$ failures per 10 hours.

The system MTTF = $E(T_2) = n/\lambda = 2/0.0004 = 5000$ hours, and

$V(T_2) = \sigma^2 = n/\lambda^2 = 2/(0.0004^2) = 12500000$ hours^2 $\rightarrow \sigma = 3535.53391 \rightarrow$ $CV(T_2) = 70.711\%$.

Now consider a 3-PC (or 3-unit) standby system that guides an aircraft each PC with a failure rate of $\lambda = 0.0004$/hour.

Compute the system reliability for a mission of 10-hour flight.

$R(10) = P(T_3 \geq 10 \text{ hours}) = P[X(10 \text{ hours}) \leq 2 \text{ failures}] =$

$$= \sum_{x=0}^{2} \frac{(0.004)^x}{x!} e^{-0.004} = 0.9^7 8936528 = 10.635 \text{ failures in a billion.}$$

The system MTTF = $E(T_3) = n/\lambda = 3/0.0004 = 7500$ hours, and $V(T_3) = \sigma^2 = n/\lambda^2 = 18.75 \times 10^6$ hours^2 $\rightarrow \sigma = 4330.127 \rightarrow CV(T_3) = 57.735\%$.

We can also compute the above reliability at 10 hours directly from the gamma pdf as follows:

$$R(10) = P(T_3 \geq 10 \text{ hours}) = \int_{10}^{\infty} \frac{\lambda}{\Gamma(3)} (\lambda x)^{3-1} e^{-\lambda x}\, dx \;;\; \text{Let } y = \lambda x$$

$$\rightarrow dy = \lambda(dx) \rightarrow \quad R(10) = \int_{0.004}^{\infty} \frac{1}{2!} (y)^2 e^{-y}\, dy = \frac{1}{2} \{ -y^2 e^{-y} \Big]_{0.004}^{\infty} + \int_{0.004}^{\infty} 2(y) e^{-y}\, dy \} = \frac{1}{2} \{ (0.004)^2 e^{-0.004} + [-2ye^{-y} \Big]_{0.004}^{\infty} + 2 \int_{0.004}^{\infty} e^{-y}\, dy] \} = \frac{1}{2} \{ (0.004)^2 e^{-0.004} + 2(0.004)e^{-0.004} + 2e^{-0.004} \} =$$

$$e^{-0.004} + (0.004)e^{-0.004} + \frac{1}{2} (0.004)^2 e^{-0.004} = 0.9^7 8936528$$

The application of the gamma pdf, when n is a positive integer, occurs often in reliability engineering, where $T_n = X_1 + X_2 + ... + X_n$ represents the lifetime (or time to failure = TTF) of an n-unit standby system (i.e., units 2 thru n are cold spares). Unit 1 is energized at time 0 and its TTF is X_1. ASA unit 1 fails, unit 2 is energized (by means of a perfect switch) and its TTF measured from last failure is X_2; when unit 2 fails, unit 3 is put on-line with lifetime X_3, etc. When the value of n in the Gamma pdf is a positive integer, then the gamma pdf is called Erlang. The Erlang pdf has also widespread applications in queuing where T_n represents the arrival time (measured from zero) of the nth customer to a queuing process.

It can be proven that if X_i's (i = 1, 2,..., n) are exponentially and independently distributed each with the same identical rate λ, then the system lifetime, $T_n = \sum_{i=1}^{n} X_i$, is a rv having a gamma pdf with parameters n (a positive integer) and rate λ. As a result, $E(T_n) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} (1/\lambda) = n/\lambda$, and similarly $V(T_n)$ =

n/λ^2. It can be shown that the 3rd and the 4th standardized moments of the Gamma pdf are given by $\alpha_3 = 2/\sqrt{n}$ and $\alpha_4 = 3 + \frac{6}{n}$ so that the kurtosis of the Gamma density is equal to 6/n.

These last 2 moments show that as n $\to \infty$, the distribution of T_n approaches normality because the 3rd and 4th standardized moments of a normal distribution are $\alpha_3 = 0$ and $\alpha_4 = E(Z^4) = 3$. Further, because λ determines the standard deviation, it is called the scale parameter while n is called the shape parameter because n determines the skewness and kurtosis.

Exercise 33 p.69 (a) $\lambda = 0.20$ failures/Week

$F_T(2)$ = P(Time to the 1st downtime < 2 weeks) = (T < 2 weeks)

$$\int_0^2 0.20e^{-0.20t}\,dt = -e^{-0.20t}\Big]_0^2 = 1 - e^{-0.20(2)} = 1 - e^{-0.40} = 0.32968.$$

Let X(2 weeks) = No. of failures during 2 weeks.

$F_T(2) = P(T < 2) = P[X(2\text{ weeks}) \ge 1] = 1 - P[X(2) = 0] = 1 -$ Poisson pmf(at x = 0 and $\lambda t = 0.40$)$= 1 - p(0; \lambda t = 0.40) = 1$

$-e^{-0.40} = 0.32968.$

(b) $P[X(6) = 4] = \dfrac{(\lambda t)^x}{x!}e^{-\lambda t} = \dfrac{(0.20\times 6)^4}{4!}e^{-1.20} = 0.02602318$

$$P[X(6) \le 6] = \sum_{x=0}^{6}\frac{(1.20)^x}{x!}e^{-1.20} = 0.99974888754$$

(c) $P(T < 4) = F_T(4) = \int_0^4 0.20e^{-0.20t}\,dt = 1 - e^{-0.80} = 0.550671 =$

$P[X(4\text{ weeks}) \ge 1] = 1 - P[X(4) = 0] = 1 - p(0; \lambda t = 0.80)$

(e) Compute the Pr that the time to the 4th failure (measured from the last failure, i.e., T_4) exceeds 9 weeks.

$P(T_4 > 9 \text{ weeks}) = P[X(9 \text{ weeks}) \leq 3] = F(3; \lambda t = 1.80) =$ 0.8912916

Compute the Pr that the time to the 6th failure is less than 15 weeks.

$P(T_6 \leq 15 \text{ weeks}) = P[X(15 \text{ weeks}) \geq 6] = 1 - P[X(15 \text{ weeks}) \leq 5]$

$= 1 - F(5; \lambda t = 3) = 1 - \sum_{x=0}^{5} \frac{3^x}{x!} e^{-3} = 1 - 0.916082058 = 0.083918.$