

Sample (or Outcome) Space: This is the universe or the set of all possible outcomes of an experiment (denoted by U). An experiment is a test, an action or a process that generates observation(s). Experiments can be deterministic or random. Random experiments are also called nondeterministic or stochastic. All experiments in this course are stochastic (or random) in nature.

Example 3. Two fair dice, red and white, are tossed and the "up" faces are observed. The sample space is given by the set:
 $U = \{(1, 1), 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$, where $32 = (3 \text{ on red, } 2 \text{ on white})$ for brevity. Or $U = \{(i, j) \mid i, j = 1, 2, 3, 4, 5, 6\}$.

Note that the combination "12" = (1, 2) means that the red die comes up 1 and the white die shows 2, while the ordered pair "21" = (2, 1) implies just the opposite. Further, the universe U contains $N(U) = 36$ outcomes because $6 \times 6 = 36$ and all outcomes are equally likely because both dice are fair and balanced.

Example 4. A lot of 10 items has 2 nonconforming units (NCUs) and 8 good units. One unit is drawn after another at random (and w/o replacement) until the last defective unit is identified. The drawing is stopped ASA the 2nd defective is removed from the lot. Let C denote a conforming unit and D denote a NCU (or defective). Then, the set of all possible outcomes is $U = \{DD, (DCD, CDD), (DCCD, CDCD, CCDD), (DCCCD, CDCCD, CCDCD, CCCDD), \dots, CCCCCCDD, CCCCCCDD\}$, where $N(U) = \sum_{i=2}^{10} (i - 1) = \sum_{i=1}^9 i = 9(10) / 2 = 45$ outcomes.

Example 5. Three coins (nickel, dime and quarter) are tossed and the 3 faces are observed. The sample space for this experiment is $U = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$, where $N(U) = 8$.

Definition: An event, A , is a set of possible outcomes or simply a subset of the universe U . If an event contains no outcome, it is said to be null or empty,

denoted by ϕ ; if an event has exactly one outcome, it is called simple (or elementary); if it has more than one outcome, then it is said to be a compound event.

Example 6. Refer to Example 3 above: Define the event $A = \{\text{the 2 faces of the dice sum to a total of 7}\}$. Then $A = \{16, 25, 34, 43, 52, 61\}$ is a compound event for which $N(A) = 6$.

Example 7. Refer to Example 4. Define $B = \{\text{At most 5 units are drawn at random in order to remove both NCUs from the lot}\} = \{DD, DCD, CDD, DCCD, CDCD, CCDD, DCCCD, CDCCD, CCDCD, CCCDD\}$, where $N(B) = 10$.

Example 8. Refer to Example 5: Define the event $C = \{\text{Exactly 2 heads come up}\} = \{HHT, HTH, THH\}$, where $N(C) = 3$.

Example 9. Consider the Exercise 4 on page 50 of Devore. (a) $U = \{FFFF, FFFV, FFVF, FVFF, VFFF, FFVV, FVfV, FVfV, VFFV, VFfV, VVfV, FVVV, VFVV, VVFV, VVfV, VVVV\}$, where $N(U) = 16$. (b) The event $B = \{FFFF, FFFV, FFVF, VFFF\}$, where $N(B) = 4$. (c) $C = \{FFFF, VVVV\}$. (d) The event $D = \{FFFF, FFFV, FFVF, FVFF, VFFF\}$, where $N(D) = 5$. (e) $C \cup D = \{FFFF, FFFV, FFVF, FVFF, VFFF, VVVV\}$ and $C \cap D = \{FFFF\}$.

Since events are sets, then we define and conclude that : (i) A' = The compliment of A = The event that A does not occur, is also denoted by \bar{A} . (ii) $A \cup B$ = At least one of the 2 events occur = either A or B occurs (or both A and B occur). (iii) $A \cap B$ occurs iff Both A and B occur simultaneously. Events A and B are mutually exclusive (MUEX) iff $A \cap B = \phi$ (the null set).

Definition: The occurrence probability (Pr) of an event A is defined as the function

$$P(A) = \frac{N(A)}{N(U)}, \text{ iff all the elementary outcomes in } U \text{ are equally likely.}$$

Example 10. Refer to the Example 6 above: $P(A) = 6/36 = 1/6$. In Example 7, $P(B) = 10/45$ because all the outcomes in U are equally likely. For the Example 8 above, $P(C) = 3/8$.

The function $P(A) = \text{Pr}(A)$ must satisfy the following axioms:

- (1) $0 \leq P(A) \leq 1$ because $0 \leq N(A) \leq N(U)$.
- (2) $P(U) \equiv 1$, and $P(\phi) \equiv 0$, i.e., the Pr of a null set is identically zero.
- (3) If A and B are MUEX, then $P(A \cap B) = P(\phi) = 0$, and

$P(A \cup B) = P(A) + P(B)$; further, if A_1, A_2, A_3, \dots is a countably infinite

jointly MUEX events, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Since A and $A' = \bar{A}$ are MUEX, then $A \cup A' = U$ and as a result $P(A \cup A') = 1 = P(A) + P(A') \rightarrow P(A') = 1 - P(A)$, or $P(\bar{A}) = 1 - P(A)$.

How do we assign probabilities to an event A?

- (1) By experimentation, (2) Estimates from prior data, (3) Analytical consideration of experimental conditions, (4) Subjective assumptions.

For example If the odds in favor of an event A are 4 to 3, then $P(A) = 4/(4+3) = 4/7$, i.e., we are willing to bet \$4.00 in order to win \$3.00. In general, if the odds in favor of an event A are n_1 to n_2 , then we are making the subjective assumption that its

occurrence Pr is $P(A) = \frac{n_1}{n_1 + n_2} \rightarrow (n_1 + n_2)P(A) = n_1 \rightarrow n_2 P(A) = n_1 [1 - P(A)] \rightarrow \frac{n_1}{n_2} =$

$\frac{P(A)}{1 - P(A)} = \frac{P(A)}{P(\bar{A})}$. Thus, the odds that A will occur is stated as the ratio of $P(A)$ to

$1 - P(A) = P(\bar{A})$, i.e., odds in favor of A are $P(A)$ to $1 - P(A)$, while the odds against A

are stated as $P(A')$ to $P(A)$. For Example, If $P(A) = 0.80 = 4/5$, then the odds in favor of A are 4 to 1; if $P(A) = 0.60 = 3/5$, then the odds in favor of the event A are 3 to 2.

On the other hand, if $P(A) = 0.05 = 1/20$, then the odds against its occurrence are 19 to 1, i.e., we are willing to bet one dollar against \$19.00 (in order to net \$19).

The common notation $O(A)$ used in statistics is counterintuitive because it always

describes the odds against the event A, i.e., $O(A) = n_2/n_1 = \frac{n_2 / (n_1 + n_2)}{n_1 / (n_1 + n_2)} =$

$P(\bar{A})/P(A)$ and is read as odds against A are n_2 to n_1 . So, I will make the

counterintuitive notation that the odds for A is given by $\bar{O}(A) = P(A) / P(A')$. For

Example, if a bookie gives you a 2 to 1 odds that Auburn beats Bama this year, then you must bet \$1.00 (i.e., pay in upfront) and if Auburn wins you will receive a total of \$3.00. On the other hand, if the same bookie gives a Bama fan a 1 to 3 odds, then a Bama fan must bet \$3.00 and if Bama wins, s/he will receive \$4.00. Note that bookies must give different odds in order to guarantee a net profit after the game is played. Thus, in this scenario, if 12000 Auburn fans bet the minimum of \$1.00, and 10000 Bama fans bet the minimum of \$3.00, the bookie takes in $\$12000.00 + 10000 \times 3 = \$42,000.00$. If Auburn wins, the book pays off \$36000.00, and if Bama wins, the bookie pays off \$40,000.00. If we define the event $A = \{\text{Auburn beats Bama in 2009}\}$, then in the above scenario the bookie's odds against A is $O(A) = 2/1 = 2 = P(\bar{A}) / P(A) \rightarrow P(A) = 1/3$, while if we define the event $B = \{\text{Bama beats Auburn in 2009}\}$, then the bookie's odds against B is $O(B) = 1/3 \rightarrow$ the bookie's Pr that Bama wins is equal to $P(B) = 3/4$.

A Result that will be shown in class using a Venn Diagram. Let A and B belong to the same sample space U. Then in general $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for all events A and B in a universe U.

Example 11. A repairman claims that the Pr is $P(A) = 0.85$ that the air conditioning compressor is all right, and $P(B) = 0.64$ that the fan motor is all right, and 0.45 that both are all right. Can his claim be true?
 $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.85 + 0.64 - 0.45 = 1.04$. Clearly, his claim is false! If he revises his claim to $P(B) = 0.55$, would his assignment of Prs be correct then? Had he claimed that $P(A \cap B) = 0.60$, could his claim be true? Answers: Yes!

Exercise 1. In a group of 220 ISE UG students, 105 are enrolled in statistics (event A), 75 in OR (Event B) and 45 in both. An ISE student is selected at random from a frame that lists all ISE students in alphabetical order. What is the Pr that s/he is not enrolled in either course?

Exercise 2. A plant has 400 employees, 300 of whom got a raise in

the year 2008, 250 got an increase in pension benefits and 40 got neither.

An employee is selected at random from a frame that lists all plant employees in an alphabetical order. Compute the Pr that s/he got both. ANS: 0.475.

Exercise 3. Among 200 persons (selected at random) interviewed as part of an urban mass transportation study, some live more than 5 miles from the city center (event A), some regularly drive their own car (event B), and some would gladly switch to a public mass transportation if it were made available (event C). Further, it was determined that $N(A) = 113$, $N(B) = 130$, $N(C) = 117$, $N(A \cap B) = 100$, $N(A \cap C) = 84$, $N(B \cap C) = 86$ and $N(\bar{A} \cap \bar{B} \cap \bar{C}) = 35$. One person is selected at random from a frame that has listed the names of the 200 persons. (a) Draw a Venn diagram and use it to compute the Pr that the randomly selected person fits all 3 profiles. (b) Compute the Pr that at least one of the 3 events occur. (c) Compute the $P(B \cup C)$ and $P(\bar{B} \cup C)$. (d) Compute the $P(\bar{A} \cap \bar{B} \cap \bar{C})$, $P[\bar{A} \cup \bar{B} \cup C]$, $P[A \cup (B \cap C)]$, $P[(A \cup B) \cap C]$, and $P[(A \cup C) \cap B]$. ANS: (a) 0.375, (b) 0.825, (c) 0.805, 0.78, (d) 0.095, 0.875, 0.620, 0.475, 0.555.

Multiplication Principle: If an event A_1 can occur in n_1 different ways, event A_2 can occur in n_2 ways, ..., and event A_K in n_K different ways, then the event A_1 , followed by A_2 , ..., followed by A_K can occur one after another in $n_1 \times n_2 \times \dots \times n_K$ different ways.

Example 12. In how many different ways can a local union with 40 members select a president, followed by a VP and then a secretary treasurer? ANS : $40 \times 39 \times 38 = 59280$ different ways.

Example 13. A test consists of 20 T/F questions. If a student flips a coin to select his answers, in how many different ways can s/he answer all 20 questions? ANS: $2 \times 2 \times \dots \times 2 = 2^{20} = 1048576$ different ways. If 60% performance is passing, it will be shown in Chapter 3 the Pr that he/she will pass the test is $\text{Pr} = 0.2517223$.

Exercise 4. A dime is tossed, then a quarter is tossed followed by a die. Determine the total number of outcomes for this compound experiment. (ANS: 24)

Permutations: The total number of different ways that n distinct objects can be permuted is given by ${}_n P_n = n!$. As an example, consider the 3 distinct objects A, B, C. The permutations of these objects are ABC, ACB, BAC, BCA, CAB, CBA, of which there are six. Therefore,

$${}_3 P_3 = 3! = 6.$$

The total number of ways that r ($\leq n$) objects can be selected from n objects and then permuted is given by

$${}_n P_r = \frac{n!}{(n-r)!}.$$

For example, consider 4 objects A, B, C and D. Then, the elements of ${}_4 P_2$ are: AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC, of which there are $12 = 4!/(4-2)! = 24/2$, while the elements of ${}_4 P_3$ are: ABC, ACB, BAC, BCA, CAB, CBA, ABD, ADB, BAD, BDA, DAB, DBA, ..., DCB, of which there are $4!/(4-3)! = 24$.

Combinations: The total number of different ways that r distinct objects can be selected from n ($r \leq n$) is given by

$${}_n C_r = \binom{n}{r} = {}_n P_r / r! = \frac{n!}{(n-r)! r!}.$$

For example, consider the combinations of the 4 objects A, B, C and D, taken 2 at a time:

AB, AC, DA, BC, DB, CD,

of which there are $6 = {}_4 C_2 = 4!/(2! \times 2!)$ of them.

Exercise 5. Five cards are drawn at random from a well-shuffled deck of 52 cards. What is the Pr of getting exactly 3 kings and 2 aces? 0.0^{592345} .

(b) Thirteen cards are drawn at random from a deck of 52 cards. What is the Pr that all 4 aces will be drawn? ANS: 0.0026411

(c) Five cards are drawn at random from a well-shuffled deck of 52 cards W/O replacement. What is the Pr that the 5 cards form a straight? What is this Pr for the case of with replacement? ANS: 0.00392465, 0.0032193241

(d) Compute the Pr of a full-house for a 5-card draw of a poker hand.

(e) A production lot contains 80 units of which 6 are NC. A random sample of

size $n = 10$ is drawn from the lot and the number of NCU's is counted. What is the Pr that the sample contains exactly 2 NCU's? What is the Pr that the sample contains at least one NCUs? ANS: 0.13731, 0.5637

(f) Five cards are drawn at random with replacement from well-shuffled deck of 52 cards. Compute the Pr that there 0, 1, 2, 3, 4, or 5 aces amongst the 5 cards.

CONDITIONAL PROBABILITY

Examples 14. Consider 100 individuals in a ballroom. Sixty of the 100 are college graduates, 55 are married and 35 are married college graduates. The name of the 100 individuals is written on 100 different slips and put in a box. One slip is drawn completely at random. Now define the following events.

Let $A = \{\text{The slip randomly drawn is an individual who is a college graduate}\}$.

Let $B = \{\text{The randomly selected person is married}\}$. Clearly, $O(A) = 40/60$, $P(A) = 60/100 = 0.60$, $O(B) = 45/55$, $P(B) = 55/100 = 0.55$, and $P(A \cap B) = 0.35$.

(a) What is the Pr that the selected individual is an unmarried college graduate?

$$P(A \cap \bar{B}) = P(A) - P(A \cap B) = 0.25$$

Similarly, $P(\bar{A} \cap B) = 0.20$ gives the Pr that the selected person is W/O a college degree but married.

(c) Suppose you are told that the selected person is a college graduate, i.e., we know that the event A has already occurred. Now what is the Pr that the selected person is also married? The Pr of the event B given that A has already occurred is given by

$$P(B|A) = \frac{N(A \cap B)}{N(A)} = \frac{35}{60} = \frac{N(A \cap B)/N(U)}{N(A)/N(U)} = \frac{P(A \cap B)}{P(A)} = 0.5833\bar{3}.$$

The event A in part (c) above is called the reduced universe, RU , to account for the conditional information provided by the slip that was drawn at random. In general, we deduce from the above that $P(A \cap B) = P(A) \times P(B|A)$, or $P(A \cap B) = P(B) \times P(A|B)$ holds true for any 2 events in the same universe U .

Two events A and B in the same universe are stochastically independent if

$$P(A|B) = P(A) \text{ and also } P(B|A) = P(B), \text{ in which case}$$

$$P(A \cap B) = P(A) \times P(B) \text{ iff A and B are stochastically independent.}$$

As an example, suppose a 2nd person is selected at random from the ballroom (W/O replacing the 1st slip back in the box). Let the event C = {the 2nd person is also a college grad}. Then

$$P(A \cap C) = P(A) \times P(C|A) = (60/100) (59/99) = 0.3576,$$

and events C and A would not be independent because the occurrence Pr of C depends on A. Note that $P(C|\bar{A}) = 60/99$.

Had the 2nd drawing been performed with replacement then

$$P(A \cap C) = P(A)P(C) = (60/100) \times (60/100) = 0.36$$

so that events A and C would be independent because P(C) would not depend on whether A occurs or not.

Exercise 6. (a) Compute the Pr that the 1st person selected at random from the ballroom is neither a college grad nor married. (b) Compute the Pr that if 2 persons are selected at random (and W/O replacement from the box with 100 slips), one is a CG and the other is married (or both). (c) Repeat part (b) if the 2nd random selection is done with replacement. ANS: 0.20, 0.539393, 0.5375.

Bayes' Theorem

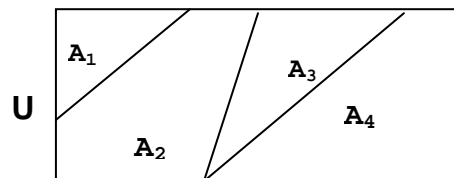
The events A_1, A_2, \dots, A_k form a partition of a sample space U iff (1) $A_i \cap A_j = \phi$ for all $i \neq j$ (i.e., A_i and A_j must be MUEX for all $i \neq j$), and (2)

$$A_1 \cup A_2 \cup \dots \cup A_k = \bigcup_{i=1}^k A_i = U. \text{ Let B denote any event in the universe U. Then}$$

$$P(B) = \sum_{i=1}^k P(B \cap A_i) \text{ because the events } B \cap A_i \text{ (} i = 1, 2, \dots, k \text{) are also jointly MUEX.}$$

Thus

$$P(B) = \sum_{i=1}^k P(B \cap A_i) = \sum_{i=1}^k [P(A_i) \times P(B|A_i)].$$



In the Bayes' theorem, the objective is to compute the conditional Pr, $P(A_j | B)$, for a specified $j = 1, 2, 3, \dots, k$. Then

$$P(A_j | B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(A_j) \times P(B | A_j)}{\sum_{i=1}^k [P(A_i) \times P(B | A_i)]}$$

Example 15. A job shop has 3 machines M_1 , M_2 and M_3 where the FNC from them are 0.02, 0.05, and 0.03, respectively. An item is drawn at random from the combined well-mixed daily outputs of the 3 machines and found to be defective. What is the Pr that the randomly selected item was manufactured by machine 2 if the relative production rates (PRs) of the 3 machines are 0.40, 0.30 and 0.30, respectively.

Let $A_i = \{\text{the randomly selected item came from } M_i\}$, $i = 1, 2, 3$, and event $B = \{\text{the item is defective or NC}\}$. Note that the event B in the Bayes' formula is always the one that has already occurred. (In fact in this case, the event B represents Average Outgoing Quality, denoted by AOQ.)

$$P(B) = \sum_{i=1}^k [P(A_i) \times P(B | A_i)] = 0.40 \times 0.02 + 0.30 \times 0.05 + 0.30 \times 0.03 = \text{AOQ} = 0.032 = \text{AOQ}$$

$$P(A_2 | B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{P(A_2) \times P(B | A_2)}{0.032} = \frac{0.30 \times 0.05}{0.032} = 0.46875.$$

Exercise 7. Repeat the above example 15 for the case when 2 items are drawn from the daily output and both found to be defective. Assume that the proportions 0.40, 0.30 and 0.30 stay roughly the same for the 2nd randomly selected unit. What is the Pr that both units came from machine M_2 ? ANS: 0.21973 (approximately).

Exercise 8. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 6\}$, and $C = \{1, 3, 5, 7\}$. (a) Determine if A , B and C are MUEX. (b) If a number is selected at random, compute the $P(A)$, $P(B)$, $P(C)$ and the Pr that both A and B occur. Determine if any two of the 3 events are stochastically independent.