Reference: Chapter 15 of Ebeling Weibull Underlying Life Distribution

Failure Data With (Types I & II) and Multiple Censoring Maghsoodloo

For Type II censoring we have n units on test and our objective is to test exactly a priori fixed number $N_f = r^* < n$ of them to failure in order to obtain the failure times $t_1, t_2, ..., t_r$ and then use these observed failure instances to obtain the MLEs of θ and β. We should first obtain the proper LF as provided below, where ${}_nP_r$ is the permutations of n units taken r at a time.

$$L(\theta, \beta) = \left[({}_nP_{r^*}) \prod_{i=1}^{r^*} \frac{\beta}{\theta}(t_i/\theta)^{\beta-1} e^{-(t_i/\theta)^\beta} dt_i \right] \times \left[e^{-(t_{r^*}/\theta)^\beta} \right]^{(n-r^*)} \qquad \textbf{(Type II)}$$

$$= {}_nP_r \left(\frac{\beta}{\theta}\right)^{r^*} \left[\prod_{i=1}^{r^*} (t_i/\theta)^{\beta-1} dt_i \right] \times e^{-\sum_{i=1}^{r^*} (t_i/\theta)^\beta} \times \left[e^{-(n-r^*)(t_{r^*}/\theta)^\beta} \right] \qquad \textbf{(137a)}$$

For Type I censoring (where testing time T = t* is censored) the LF modifies to

$$L(\theta,\beta) = \left(\frac{\beta}{\theta}\right)^r \left[\prod_{i=1}^{r} (t_i/\theta)^{\beta-1} \right] \times e^{-\sum_{i=1}^{r} (t_i/\theta)^\beta} \times \left[e^{-(n-r)(t^*/\theta)^\beta} \right] \qquad \textbf{(Type I, 137b)}$$

For multiple censoring the LF becomes

$$L(\theta,\beta) = \prod_{i=1}^{S_F} \frac{\beta}{\theta}(t_i/\theta)^{\beta-1} e^{-(t_i/\theta)^\beta} dt_i \prod_{j=1}^{S_C} e^{-(t_j^\dagger/\theta)^\beta} = \left(\frac{\beta}{\theta}\right)^r e^{-\sum_{i=1}^{r} t_i/\theta} e^{-\sum_{j=1}^{n-r} t_j^\dagger/\theta} \prod_{i=1}^{r} (t_i/\theta)^{\beta-1} dt_i,$$

(Multiple, 137c)

where $S_F = r$ represents the failed units and $S_C = n-r$ represents the censored units.

For Type II censoring, taking the natural logarithm of Eq. (137a) yields

$$\mathcal{L}(\theta, \beta) = \ln({}_nP_{r^*}) + r^*\left(\ln\frac{\beta}{\theta}\right) + (\beta-1)\sum_{i=1}^{r^*}\ln(t_i/\theta) + \sum_{i=1}^{r^*}\ln(dt_i) - \sum_{i=1}^{r^*}(t_i/\theta)^\beta - (n-r)(t_{r^*}/\theta)^\beta$$

Because the derivatives of 1st and 4th terms on the RHS will vanish, the above log-likelihood function reduces to

$$\mathbf{L}(\theta, \beta) = r^*\ln(\beta) - r\ln(\theta) + (\beta-1)\sum_{i=1}^{r^*}[\ln(t_i) - \ln(\theta)] - \sum_{i=1}^{r^*}(t_i/\theta)^{\beta} - (n-r)(t_{r^*}/\theta)^{\beta}$$

$$= r^*\ln(\beta) + (\beta-1)\sum_{i=1}^{r^*}\ln(t_i) - \beta r^*\ln(\theta) - \sum_{i=1}^{r^*}(t_i/\theta)^{\beta} - (n-r^*)(t_{r^*}/\theta)^{\beta} \qquad \textbf{(Type II, 138a)}$$

For Multiple censoring

$$\mathbf{L}(\theta, \beta) = r\left(\ln\frac{\beta}{\theta}\right) + (\beta-1)\sum_{i=1}^{r}\ln(t_i/\theta) + \sum_{i=1}^{r}\ln(dt_i) - \sum_{i=1}^{r}(t_i/\theta)^{\beta} - \sum_{j=1}^{n-r}(t_j^{+}/\theta)^{\beta} \qquad \textbf{(138b)}$$

We 1st take the partial derivative of $\mathbf{L}(\theta, \beta)$ in (138a) wrt θ and will require it to be zero.

$$\frac{\partial \mathbf{L}}{\partial \theta} = -\frac{\beta r^*}{\theta} + \beta\,\theta^{-\beta-1}\sum_{i=1}^{r^*} t_i^{\beta} + \beta(n-r^*)\theta^{-\beta-1}(t_{r^*}^{\beta}) = -\frac{\beta r^*}{\theta} + \beta\theta^{-\beta-1}\left[\sum_{i=1}^{r^*} t_i^{\beta} + (n-r^*)\,t_{r^*}^{\beta}\right] = 0$$

(Type II, 139a)

In order to solve for $\hat{\theta}$ from equation (139a), we multiply throughout by $\theta^{\beta+1}/\beta$, which will reduce (139a) to $-r\,\theta^{\beta} + \sum_{i=1}^{r} t_i^{\beta} + (n-r)(t_r^{\beta}) = 0 \rightarrow$ Thus for

Type II Censoring: $$\hat{\theta} = \left[\left[\sum_{i=1}^{r^*} t_i^{\hat{\beta}} + (n-r^*)t_{r^*}^{\hat{\beta}}\right]/r^*\right]^{1/\hat{\beta}} \qquad \textbf{(Type II, 139b)}$$

For multiple censoring, the partial derivative of $\mathbf{L}(\theta, \beta)$ in (138b) wrt θ is given by

$$\frac{\partial \mathbf{L}}{\partial \theta} = -\frac{r}{\theta} - (\beta-1)\sum_{i=1}^{r}(1/\theta) + \beta\,\theta^{-\beta-1}\sum_{i=1}^{r} t_i^{\beta} + \beta\,\theta^{-\beta-1}\sum_{j=1}^{n-r}(t_j^{+})^{\beta} \xrightarrow{\text{Set to}} 0 \qquad \textbf{(139c)}$$

The solution to Eq. (139c) will lead to the following MLE of $t_c = \theta$ for the case of

Multiple Censoring: $$\hat{\theta} = \left[\left[\sum_{i=1}^{r} t_i^{\hat{\beta}} + \sum_{j=1}^{n-r}(t_j^{+})^{\hat{\beta}}\right]/r\right]^{1/\hat{\beta}} \qquad \textbf{(Multiple, 139d)}$$

Note that the MLEs of θ given in (139b &d) reduce to equation (125a) for the case of no censoring when all n units are tested to failure. Next we partially differentiate $\mathbf{L}(\theta, \beta)$ in equation (138a) wrt β and set it equal to zero in order to obtain the MLE of β for type II censoring.

$$\frac{\partial \mathbf{L}}{\partial \beta} = \frac{r^*}{\beta} + \sum_{i=1}^{r^*} \ln(t_i) - r^*\ln(\theta) - \sum_{i=1}^{r^*}\left[(t_i/\theta)^{\beta} \times \ln(t_i/\theta)\right] - (n - r^*)(t_{r^*}/\theta)^{\beta} \times \ln(t_{r^*}/\theta) \xrightarrow{\text{Set to}} 0$$

This last equation for Type II will simplify to

$$\frac{r^*}{\hat{\beta}} + \sum_{i=1}^{r^*} \ln(t_i) - r^*\ln(\hat{\theta}) - \sum_{i=1}^{r^*} (t_i/\hat{\theta})^{\hat{\beta}} \ln(t_i/\hat{\theta}) - (n - r^*)(t_{r^*}/\hat{\theta})^{\hat{\beta}} \times \ln(t_{r^*}/\hat{\theta}) = 0 \qquad \text{(140a)}$$

Combining Eqs. (140a) and (139b), using the fact that $\hat{\theta}^{-\hat{\beta}} = \dfrac{r^*}{\sum_{i=1}^{r^*} t_i^{\hat{\beta}} + (n - r^*)t_{r^*}^{\hat{\beta}}}$

and simplifying results in (note that r* is not a rv in the case of Type II)

Type II Censoring: $\dfrac{r^*}{\hat{\beta}} + \sum_{i=1}^{r^*} \ln(t_i) - \hat{\theta}^{-\hat{\beta}} \sum_{i=1}^{r^*} t_i^{\hat{\beta}} \ln(t_i) - (n - r^*)\hat{\theta}^{-\hat{\beta}}(t_{r^*}^{\hat{\beta}}) \times \ln(t_{r^*}) \rightarrow$

$$\frac{1}{\hat{\beta}} + \frac{1}{r^*}\sum_{i=1}^{r^*} \ln(t_i) - \frac{\sum_{i=1}^{r^*} t_i^{\hat{\beta}} \ln(t_i) + (n\text{-}r^*)t_{r^*}^{\hat{\beta}} \ln(t_{r^*})}{\sum_{i=1}^{r^*} t_i^{\hat{\beta}} + (n\text{-}r^*)t_{r^*}^{\hat{\beta}}} = 0 \qquad \text{(Type II, 140a)}$$

This last equation (140a) and Eq. (139b) have to be solved simultaneously for $\hat{\beta}$ and $\hat{\theta}$ in order to obtain the ML estimators of type II censoring. However, there will never exist a closed- form solution for the two MLEs $\hat{\beta}$ and $\hat{\theta}$, and hence the approximate solutions have to be found using similar procedure that I have outlined on pages 241-245 of my notes for the case of no censoring. For Multiple Censoring, the log-likelihood function was

$$\mathbf{L}(\theta, \beta) = r(\ln\beta - \ln\theta) + (\beta - 1)\sum_{i=1}^{r} \ln(t_i/\theta) - \sum_{i=1}^{r} (t_i/\theta)^{\beta} - \sum_{j=1}^{n-r} (t_j^{+}/\theta)^{\beta} \qquad \text{(140b)}$$

and its partial derivative wrt β is given by

$$\frac{\partial \mathbf{L}(\theta,\beta)}{\partial\beta} = \frac{r}{\beta} + \sum_{i=1}^{r}\ln(t_i/\theta) - \sum_{i=1}^{r}(t_i/\theta)^{\beta}\ln(t_i/\theta) - \sum_{j=1}^{n-r}(t_j^{+}/\theta)^{\beta}\ln(t_j^{+}/\theta) \xrightarrow{\text{Set to}} 0$$

Multiple Censoring: $$\sum_{i=1}^{r}[\frac{1}{\hat{\beta}} + \ln(t_i/\hat{\theta}) - (t_i/\hat{\theta})^{\hat{\beta}}\ln(t_i/\hat{\theta})] - \sum_{j=1}^{n-r}(t_j^{+}/\hat{\theta})^{\hat{\beta}}\ln(t_j^{+}/\hat{\theta}) = 0$$

` (141)

For multiple censoring, the two Eqs. (139d) and (141) have to be solved numerically in order to obtain their simultaneous solutions for $\hat{\beta}$ and $\hat{\theta}$. For type I censoring, I have summarized the equations to be solved simultaneously to obtain the MLE of β and θ at the end of this Chapter.

To illustrate the procedure, I will go through the Example 5.19 on pages 299-303 of E. A. Elsayed that involves type II censoring.

<u>The Example 5.19 on pages 299-303 of Elsayed.</u> In this experiment n = 30 identical units are put on test at time zero W/O replacement (U-type) and testing is stopped at the instant of t_{22} = 33 time units. The instances of failure are t_i: 18.5, 20, 20.5, 21.5, 22, 22.5, 23.5, 24, 24.3, 24.6, 25, 25.3, 25.6, 26, 26.3, 26.7, 27, 28, 29, 30, 32, 33; thus r* = 22. From the problem statement I could not surmise if the units of measurements were in hours, days, or possibly weeks. If this were an accelerated testing procedure, we could easily assume that the failure times were measured in hours; otherwise, the time unit could easily be in days. For simplicity, I will assume that our TTF in this example is measured in hours. It is clear that 8 units were right-censored at the instant t_{22} = 33 hours, i.e., the most conservative life that we can assign to the 8 surviving units is the value of the 22[nd] order-statistic t_{r*} = 33 hours. Note that if a unit is left-censored, then its recorded TTF is actually larger than its true experimental TTF. As in other examples, we 1[st] try to obtain a rough value of the sample cv of TTF. Note that it is impossible to compute the exact sample cv because we do not know the exact values of $t_{23}, t_{24}, ..., t_{30}$, but we do know that their times TF are greater than 33 hours. To this end, let $\bar{t}_u$ and s_u represent the mean and standard deviation of the uncensored part of our data, i.e.,

$\bar{t}_u = \frac{1}{r^*}\sum_{i=1}^{r^*=22} t_i$ = 25.2409 and s_u = 3.7203 $\rightarrow$ cv_u = 0.1474. Now reference to Table 1 on pages 12-14 of my Chapter 2 will reveal that the value of $\hat{\beta}$ will roughly lie within the interval [5, 10]. Please note that $\bar{t} = \frac{1}{n}\sum_{i=1}^{n=30} t_i > \bar{t}_u$ but also the standard deviation, S, of all the 30 times TF, if the last 8 order statistics were available, would also be larger than s_u because the corresponding complete sample would have larger range (however, note that a larger range does not always guarantee a larger S-value). This implies that the value of uncensored cv_u = 0.1474 should be fairly close to cv = S/$\bar{t}$.

To obtain the ML estimates, I will start by assuming that $\hat{\beta} \cong 5.0$ at which equation (139b) yields $\hat{\theta}$ = 30.5811. Inserting these initial estimates into the LHS of equation (140a) yields the value of 0.1288 > 0 for the LHS. Since the LHS of (140a) is a decreasing function of $\hat{\beta}$, we must try a larger value than 5.0, say $\hat{\beta}$ = 5.20. This yields $\hat{\theta}$ = 30.5734 and the LHS of (140a) equal to $-$0.1115. Thus, we have overshot our estimate of $\hat{\beta}$ and we now know that 5 $< \hat{\beta} < 5.2$. Several trial and errors yield $\hat{\beta}$ = 5.1055565545 and $\hat{\theta}$= 30.5764557476 at which the LHS of (140a) reduces to $5.312925655\times10^{-10}$. Note that these MLEs are almost identical to those of Elsayed's given in the middle of his page 299. Further, MS Excel solver should be used to obtain precise values of $\hat{\beta}$ and $\hat{\theta}$.

Reducing the Amount of Bias in the MLEs of Weibull Parameters

As I have repeatedly mentioned before, the MLEs of Weibull parameters β and θ are biased, i.e., $E(\hat{\beta}) > \beta$ and only when $n \geq r > 30$ the amount of overestimation in $\hat{\beta}$ is almost negligible. As Elsayed mentions on his page 297, Bain and Engelhardt (B&E, Statistical Analysis of Reliability and Life-Testing Models, Theory and Methods, 1991, 2nd Edition, Marcel Dekker) provide a bias reducing factor for $\hat{\beta}$ in their Chapter 4, which can be approximated by Elsayed's empirical formula given below.

$$G(n) = 1.00 - 1.346/n - 0.8334/n^2 \qquad \text{(5.42 of Elsayed)}$$

Because I have not done extensive research in this area, I cannot asses how accurate equation (5.42 of Elsayed) for reducing the amount of bias $B(\hat{\beta}) = E(\hat{\beta} - \beta)$ is. However, Elsayed's apparent regression result (5.42) is interesting because the number of observed failures, r, seems to have little impact in determining the bias amount of MLEs than the actual number of units, n, put on test at time zero. Clearly, the larger the value of n, the higher the failure intensity level of the testing process, but if only $r < 10$ failures are observed, the MLEs may not be very precise (i.e., they may have large standard errors). Further, for $r < 10$, the MLEs will most likely be fairly biased. Just to check the accuracy of Elsayed's model (5.42), I regressed the biasing factors, B_n, of Bain and Engelhardt (B&E) given in their Table 2 on page 221 of their text, using Minitab and obtained the following model (with R^2_{Model} = 100%).

The Minitab regression equation is

$$\hat{B}_n = 1.00104 - 1.41518/n + 1.2288/n^2 - 18.1081/n^3 + 47.86/n^4 \qquad \text{(5.42Maghsoodloo)}$$

I also had Minitab compute the fitted values of B_n and all of them were identical to 3 decimals to the tabular values of Bain & Engelhardt's Table 2 on their page 221. R. B. Abernethy (2004) provides the Reduced Bias Adjustment RBA = $(c_4)^{3.52}$, where c_4 is the well-known QC constant $C_4 = \Gamma(r/2)\sqrt{2/(r-1)}/\Gamma[(r-1)/2]$. However, the value of RBA is not close to G(n) and $\hat{B}_n$ when the failed fraction $p = r/n < 0.35$.

<u>The Example 5.19 on pages 299-303 of Elsayed (Continued).</u> We now apply Elsayed's formula (5.42) and then mine to reduce the amount of bias in the MLE $\hat{\beta}$ = 5.1055565545. Inserting n = 30 in equation (5.42Elsayed) yields G(30) = 0.9542 and n = 30 into (5.42Magh) yields $\hat{B}_{30}$ = 0.95462, and hence a less biased estimate of β is given by $\hat{\beta} = 0.9542 \times 5.10556$ = 4.87176. Abernethy's RBA = $(0.988170253316)^{3.52}$ = 0.95897625, which fairly close to G(30) and $\hat{B}_{30}$. This estimate of the parameter β is consistent with that of Elsayed's given at the

bottom of page 299, but I do not believe it is 100% unbiased. We are now in a position to obtain a less biased estimate of θ by inserting $\hat{\beta}$ = 4.87176 into equation (139b). This operation yields $\hat{\theta}$ = 30.58846, which is in agreement with that of Elsayed's atop his page 300.

Approximating the Weibull $se(\hat{\beta})$ for Type II Censoring

Although, we will later use the χ^2 distribution to obtain a more accurate 95% CI for the parameter β, as a 1st approximation I will obtain an approximate 95% asymptotic CI for β from $\hat{\beta} \pm 1.96 \times se(\hat{\beta})$, as we did in the case of uncensored data. Recall that this procedure has to yield an approximate CI for β because all MLEs in the universe, under certain regularity conditions, are asymptotically unbiased and normally distributed. This requires that we 1st compute the $V(\hat{\beta})$. Exact variance of $\hat{\beta}$ cannot be computed, but as a 1st step we can compute the Cramer-Rao's glb for the $V(\hat{\beta})$ from glb$[V(\hat{\beta})] = \dfrac{1}{I(\beta)}$, where $I(\beta) = -E[\partial^2\mathbf{L}(\theta, \beta)/\partial\beta^2] = \dfrac{r^*}{\beta^2} + E\left[\sum_{i=1}^{r^*}[(t_i/\theta)^{\beta} \times \ln^2(t_i/\theta)] + (n - r^*)(t_{r^*}/\theta)^{\beta} \times \ln^2(t_{r^*}/\theta)\right] = I_{22}$. Computing the exact expectation of the rv inside the large brackets is probably too complicated and seems not possible to the ability of this author, and thus, an approximate value of the $V(\hat{\beta})$ is given by

$$\hat{V}(\hat{\beta}) \doteqdot \frac{\hat{\beta}^2}{r + \hat{\beta}^2\left[\sum_{i=1}^{r}[(t_i/\hat{\theta})^{\hat{\beta}} \times \ln^2(t_i/\hat{\theta})] + (n-r)(t_r/\hat{\theta})^{\hat{\beta}}\ln^2(t_r/\hat{\theta})\right]} \cong \frac{1}{I(\beta)} \qquad (142)$$

Substitution of $\hat{\beta}$ = 5.1055565545 and $\hat{\theta}$= 30.5764557476 into equation (142) and the use of Matlab computation yields glb$[V(\hat{\beta})] \cong 0.8334754$ and thus the $se(\hat{\beta}) \geq 0.91295$. Due to the fact that our censoring ratio p = 22/30 = 0.733333, then from Table 5.11 on page 297 of Elsayed simple interpolation yields c_{22} = 1.0577 (note that this is consistent with Elsayed's value of 1.06 given near the bottom of his page 299). Therefore, the $se(\hat{\beta}) \cong 0.91295 \times \sqrt{1.0577}$ = 0.93892 $\rightarrow$ HCIL (half CI length) = 1.959964×0.93892 = 1.84025 $\rightarrow$ β_L = 5.10556 – 1.84025 = 3.26531, and β_U = 5.10556 +1.84025 = 6.94581. Elsayed provides the 90% CI: $3.28 \le \beta \le 6.24$ at the bottom of his page 301, and in order to make a rough comparison I will also obtain an approximate 90% CI for the parameter β from $\hat{\beta} \pm 1.645 \times se(\hat{\beta})$; this leads to $3.56120 \le \beta \le$ 6.64994. Elsayed's CI at the bottom of page 301 is a bit more conservative (wider band) than my approximate 90% CI mainly due to the fact that he is using the unbiased estimator of β instead of the MLE of β. If we use the nearly unbiased estimates $\hat{\beta}$ = 4.87176 and $\hat{\theta}$ = 30.58846, the resulting 90% CI: $3.3274 \le \beta \le 6.4162$ would be in better agreement with that of Elsayed's: ($3.28 \le \beta \le 6.24$).

Approximating the Weibull $se(\hat{\theta})$ for Type II Censoring

As in the case of $\hat{\beta}$, computation of the exact $V(\hat{\theta})$ is intractable mainly due to the fact that there is no closed-form solution for the MLE $\hat{\theta}$. So, the 1st step is to obtain the glb for $V(\hat{\theta})$ from glb[$V(\hat{\theta})$} = $\frac{1}{I(\theta)}$, where $I(\theta) = -E[\partial^2 \mathbf{L}(\theta, \beta)/\partial\theta^2] = -\frac{r^*\beta}{\theta^2} + \beta(\beta+1)\theta^{-\beta-2} E\Big[\sum_{i=1}^{r^*} t_i^{\beta}$ $+ (n - r^*)\, t_{r^*}^{\beta}\Big] = I_{11}$. Again taking the exact mathematical expectation of the rv inside this last large brackets seems almost impossible to carry out (due to the fact that it will require the knowledge of the pdf of all the ith –order, i = 1, 2, …., r, statistics from a Weibull base-line distribution), and thus an estimate of $I(\theta)$, using the fact that $\hat{\theta}$ =

$$\left[[\sum_{i=1}^{r^*} t_i^{\hat{\beta}} + (n - r^*) t_{r^*}^{\hat{\beta}}] / r^* \right]^{1/\hat{\beta}} \text{ is given by } \hat{I}(\theta) = \hat{I}_{11} = \frac{r^* \hat{\beta}^2}{\hat{\theta}^2} \qquad \text{(Type II, 143a)}$$

Since the glb[$V(\hat{\theta})$} = $\frac{1}{I(\theta)}$, then from (143a) we obtain the approximate asymptotic variance

$$V(\hat{\theta}) \cong \frac{\hat{\theta}^2 / \hat{\beta}}{(\hat{\beta}+1)(\hat{\theta})^{-\hat{\beta}}[\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n-r^*)t_r^{\hat{\beta}}] - r^*} \qquad \text{(Type II, 143b)}$$

Combining (143b) with equation (139b) results in $V(\hat{\theta}) \cong \dfrac{\hat{\theta}^2}{r\hat{\beta}^2}$. (143c)

The above approximate asymptotic variance for $\hat{\theta}$ is fairly close to what Elsayed provides in his Eq. (5.40), atop his page 297, in the 1st row and 1st column. For the data of his Example 5.19, equation (143c) yields $V(\hat{\theta}) \cong 1.6303$ and *se*($\hat{\theta}$) = 1.27683, where I have used the actual MLEs in (143c). Thus, a lower one-sided asymptotic 95% CI for θ is given by $\theta_L = \hat{\theta} - 1.645 \times$ *se*($\hat{\theta}$) = 28.4761. This answer should be fairly close to the value of θ_L if we approximate the *se*($\hat{\theta}$) from *se*($\hat{\theta}$) = $\sqrt{c_{11}\hat{\theta}^2/(n\hat{\beta}^2)}$ = $\sqrt{1.4473 \times 30.576456^2/(30 \times 5.105556^2)}$ = 1.3154 (see Table 5.11 of Elsayed repeated below). Note that in the denominator of this last formula for the *se*($\hat{\theta}$) = $\sqrt{c_{11}\hat{\theta}^2/(n\hat{\beta}^2)}$, you must use n and not r. If we use the closely unbiased estimates $\hat{\beta}$ = 4.87176 and $\hat{\theta}$ = 30.58846, we would obtain *se*($\hat{\theta}$) = 1.33863 and θ_L = 28.386413, which is fairly close to the value of θ_L = 28.4761 obtained from (143c).

Table 5.11 of E. A. Elsayed (on his page 297, 1−p = proportion of the sample that is censored; p = failed proportion, p=1→r = n)

p	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
c_{11}	1.1087	1.1517	1.2526	1.4473	1.8120	2.5102	3.9330	7.1904	16.4788	60.5171
c_{22}	0.6079	0.7670	0.9282	1.1225	1.3728	1.7162	2.22474	3.0655	4.7388	9.7447
c_{12}	0.2570	0.1764	0.0493	-.1448	-.4466	-.9358	-1.7855	-3.4386	-7.3753	-22.1872

Bain and Engelhardt (1991) state that the asymptotic SMD of $\hat{a} = \hat{\beta} \times \ln(\hat{\theta}/\theta)$ is approximately Gaussian with zero mean and asymptotic variance equal to c_{11}/n, where the

values of c_{11} are listed in Table 5.11 on page 297 of Elsayed. Using this approximation we obtain

$$V(\hat{a}) = 1.057695/30 = 0.0352565 \rightarrow se(\hat{a}) = 0.187767143 \rightarrow$$

$$P[\hat{a} \le 1.645 \times 0.187767143) = 0.95 \rightarrow$$

$$P[\hat{\beta} \times \ln(\hat{\theta}/\theta) \le 0.30887695) = 0.95 \rightarrow P[\ln(\hat{\theta}/\theta) \le 0.30887695 / 5.10556\ ) = 0.95 \rightarrow$$

$$P[\hat{\theta}/\theta \le e^{0.30887695/5.10556}) = 0.95 \rightarrow P[\hat{\theta}\, e^{-0.30887695/5.10556} \le \theta) = 0.95 \rightarrow \theta_L = 28.78148.$$

If we use the MLEs of β and θ in this last equation, i.e., $\hat{\beta}$ = 5.1055565545 and $\hat{\theta}$ = 30.57645575, we would obtain θ_L = 28.6191, which is in good agreement with the value of θ_L = 28.7814793 that I just calculated using the fact that $\hat{a} = \hat{\beta} \times \ln(\hat{\theta}/\theta)$ is approximately N(0, c_{11}/n).

Before discussing how to use χ^2 to obtain a more accurate CI for β, we must state that a point MLE of the RE function for the Example 5.19 of E. A. Elsayed is given by $\hat{R}(t)$ = $e^{-(t/30.576456)^{5.10556}}$, but obtaining a lower one-sided confidence limit for R(t) is not a simple task because R(t) is a monotonically increasing function of θ but not of β. Recall that if β = 1, i.e., the TTF is exponential, then $R_L(t) = e^{-t/\theta_L}$, where I discussed the development of exponential θ_L and the 95% lower bound for RE on pages 186-213 of my notes. However, in the Example 5.19 of Elsayed the TTF distribution is Weibull [β > 1 and thus an IFR (increasing failure rate)] and hence we need the approximate SMD of the statistic $\hat{R}(t) = e^{-(t/\hat{\theta})^{\hat{\beta}}}$ in order to obtain the approximate lower 95% confidence limit for R(t). I am not well-read in this part of the literature of RE engineering, but I surmise that the exact SMD of $e^{-(t/\hat{\theta})^{\hat{\beta}}}$, where $\hat{\theta}$ is the MLE of θ and $\hat{\beta}$ is the MLE of β, is not mathematically tractable. However, all MLEs in the universe have the following nice properties under certain regularity conditions, which are generally met in life testing situations.

(1) Suppose the statistic $\hat{\theta}$ is the MLE of the parameter θ and let g(θ) be any function of the

parameter θ ; then it can be proven that g($\hat{\theta}$) is also a MLE of the parameter g(θ). For the example 5.19 of Elsayed, this property shows that $e^{-(t/\hat{\theta})^{\hat{\beta}}}$ is a MLE of the parameter R(t) = $e^{-(t/\theta)^{\beta}}$. Note that unbiased estimators, unless the function g(θ) is linear, do not possess this property.

(2) All MLEs in the universe, under certain regularity conditions (see Cramer H., Mathematical Methods of Statistics, Princeton University Press, 1946) are asymptotically unbiased. For practical applications, the size of the sample n should exceed 50 before the amount of bias in the MLE becomes small relative to its *se* of the estimate. For life testing situations, as I have repeatedly pointed out, the number of observed failures, r, also plays a very important role in the accuracy of the MLEs. For practical applications, N_f = r should exceed 20 (or perhaps at least 15) before the amount of bias in a MLE is small relative to the *se* of the estimate.

(3) The SMD of all MLEs approach normality under certain regularity condition (see the 2nd volume of Kendall and Stuart, page 43). However, the approach to normality is agonizingly slow unless the parent population is Gaussian, and unfortunately sometimes so slow that the practical application of this property may be rendered useless! As before, I would refrain from using the normal approximation unless the value of r is reasonably large. Let us relax the required value of r to at least 15 for practical applications, but be cognizant of the fact that the larger r values yield much better normal approximations of confidence intervals.

(4) MLEs are generally consistent and asymptotically efficient if the range of the frequency function f(t | θ) does not depend on θ. By consistency we mean that Lim of $\hat{\theta}$ as n $\to \infty$ is equal to θ, and by asymptotic efficiency we mean that V($\hat{\theta}$) attains its Cramer-Rao's glb as n $\to \infty$.

(5) Unless the underlying population is Gaussian, the exact SMD of ML estimators for most underlying distributions (specifically the Weibull) are not known. Even if the underlying

distribution of the data is Gaussian, the exact SMD of any general g($\hat{\theta}$), is to my knowledge, not tractable. For example, it is well known that the sample mean $\overline{x}$ is the MLE of the population mean E(X) = μ when the parent population is N(μ, σ^2) and the SMD of $\overline{x}$ has been known for well over 100 years to be N(μ, σ^2/n). However, this does not imply that the exact SMD of g($\overline{x}$) = $e^{-(\overline{x})^2}$ is also normal just because $\overline{x}$ is normally distributed. However, since $e^{-(\overline{x})^2}$ is the MLE of $e^{-\mu^2}$, then its asymptotic (n > 100) SMD should be close to normal. I hope the reader understands the spirit of what I am trying to convey from a statistical point of view? Getting now back to the problem at hand, we do know that the Weibull MLEs $\hat{\theta}$ and $\hat{\beta}$ are asymptotically normal, but this does not imply that the SMD of the MLE $e^{-(t/\hat{\theta})^{\hat{\beta}}}$ approaches normality as fast as $\hat{\theta}$ and $\hat{\beta}$ do as n becomes increasingly large. A second obvious problem is the fact that the MLE $\hat{R}(t)$ = $e^{-(t/\hat{\theta})^{\hat{\beta}}}$ contains a bivariate vector $\begin{bmatrix}\hat{\theta}\\ \hat{\beta}\end{bmatrix}$ whose components are highly correlated (see the information matrix inverse on page 253 of my notes) and we have to obtain our lower confidence bound on R(t) without ignoring this correlation (i.e., we must take this correlation into account).

The Bonferroni CI for the Two-Parameter Weibull W(0, δ, β) (Optional)

The 95% CIs for θ and β that we have obtained thus far have ignored the correlation between $\hat{\theta}$ and $\hat{\beta}$, i.e., the correct confidence band for the vector parameter $\begin{bmatrix}\theta\\ \beta\end{bmatrix}$ is an ellipsoid that with 95 % confidence contains the true vector $\begin{bmatrix}\theta\\ \beta\end{bmatrix}$. To avoid such a complicated multivariate analysis, we use the Bonferroni method of obtaining a 95% rectangular confidence region that has at least 95% chance (prior to sampling) of containing the true parameters θ and β simultaneously. This method is valid regardless of the correlation structure of the estimators

and allows the experimenter to control the overall error rate α. It can be shown that $\alpha \le \sum_{i=1}^{m} \alpha_i$, where for our case m = 2 parameters, $1-\alpha_1$ is the confidence coefficient for θ and $1-\alpha_2$ is the confidence coefficient for the parameter β. This implies that if we wish to obtain a joint 95% rectangular confidence region for $\begin{bmatrix}\theta \\ \beta\end{bmatrix}$ so that α = 0.05, then the individual error rates should be set at $\alpha_1 = \alpha_2 \cong 0.025$ because the overall confidence Pr is given by $(0.975)^2$ = 0.950625 > 0.95. If we had three parameters for which we would like to build a cubic 95% confidence region, then the Bonferroni method of obtaining joint CIs tells us that we should set the individual error rates α_i at approximately 0.05/3 = 0.016667 because $(1-0.016667)^3$ = 0.95083. Please note that the larger the value of m is, the more conservative (or smaller) the overall error rate, α, becomes; further, this method will work regardless of the correlation structures behind the confidence statements.

The above discussions tell us that in order to obtain a simultaneous Bonferroni 95% CI for β and θ, we should really obtain the 97.5% individual CI for β and θ. Therefore, the Bonferroni approximate 95% CIs for β and θ for our example are as follows:

$$\hat{\beta} \pm Z_{0.0125} \times se(\hat{\beta}) = 5.10556 \pm 2.2414 \times 0.9391 \rightarrow 3.00066 \le \beta \le 7.21046$$

$$\theta_L = \hat{\theta} - Z_{0.025} \times se(\hat{\theta}) = 30.57646 - 1.960 \times 1.27683 = 28.07387 \rightarrow 28.07387 \le \theta < \infty.$$

Due to the fact that the Weibull reliability, R(t) = $e^{-(t/\theta)^{\beta}}$, is an increasing function of both parameters θ and β up to the characteristic life θ, the Bonferroni 95% lower confidence bound for R(t) is given by $e^{-(t/\theta_L)^{\beta_L}} = e^{-t^{3.00066}/22175.055041}$, which is valid only for

$0 \le t \le \theta \rightarrow e^{-t^{3.00066}/22175.055041} \le R(t) < 1, 0 \le t \le \theta$. For example, the 95% lower confidence limit for R(t) at t = 22.5 hours is given by $R_L(22.5)$ = 0.597664 (22.5 < $\hat{\theta}$). Note that the number of survivors beyond 22.5 hours for the data of Example 5.19 on his page 299 is N_s = 24 so that a direct point estimate of R(22.5) would be roughly equal to $\hat{R}(22.5)$ = 24/30 = 0.80.

It seems that the Bonferroni confidence limit, $R_L(22.5) = 0.597664$, for R(t) is quite conservative, and perhaps almost useless!

Before using the χ^2 for CI estimation, let's try to obtain the 95% greatest lower bound on RE which is valid for the class of IFR distributions (such as the Weibull with $\beta > 1$) that generally will work even if the underlying distribution is not Weibull. The (95%) glb on RE at a specific $t_0 < \tau/n$ is given by

$$R_L(t_0) = \mathrm{Exp}[-\frac{(\lambda_{0.05;r-1})}{\tau}t_0], \qquad \textbf{(144a)}$$

where τ represents the total testing time for all n units and $\lambda_{0.05;\, r-1} = \mu$ is the solution to

$$\sum_{x=0}^{r-1}\frac{\mu^x}{x!}e^{-\mu} \le \alpha = 0.05 \qquad \textbf{(144b)}$$

For the Example 5.19 on page 299 of Elsayed, $\tau = \sum_{i=1}^{22} t_i + 8\times t_r = 555.3000 + 8\times 33 = 819.3000$,

and $\mu = \lambda_{0.05;\,21}$ is the solution to $\sum_{x=0}^{21}\frac{\mu^x}{x!}e^{-\mu} \le 0.05$. Matlab computations yields $\mu = \lambda_{0.05;\,21} =$ 30.24045. Then at $t_0 \equiv 22.5$, we have from (144a), $R_L(22.5) = e^{-(30.24045/819.30)(22.5)} =$ 0.4358412. Note that this 95% glb on RE is too conservative relative to 0.597664 obtained from the Bonferroni method, and perhaps totally useless. The main reason behind this is the fact that the glb in (144a) does not make the assumption of the Weibull baseline distribution; it just provides a glb on RE no matter what the underlying failure distribution is as long as the HZF is of increasing rate. Note that Eq. (144a) is disallowed if $\mu\times t_0 > \tau$.

Bain and Engelhardt (1991) also state on their page 220 that the asymptotic SMD of $\hat{R}(t)$ (for n > 30) is normal with asymptotic mean R(t) and asymptotic variance equal to

$$V[\hat{R}(t)] \cong \hat{R}^2[(\ln(1/\hat{R})]^2\left[c_{11} - 2c_{12}\ln(\ln(1/\hat{R})) + c_{22}[\ln(\ln(1/\hat{R}))]^2\right]/n \qquad \textbf{(145)}$$

From Table 5.11 on page 297 of Elsayed at p = 0.73333, interpolation yields $c_{11} = 1.38237665$, $c_{12} = -0.080121$, and $c_{22} = 1.057695$. Inserting these c-values and $\hat{R}(22.5) =$

$\mathbf{e}^{-(22.5/30.576456)^{5.10556}}$ = 0.81148505 into (145) yields V[$\hat{R}(t)$] ≅ 0.003567906 and *se*[$\hat{R}(t)$] = 0.05973195; as a result, $R_L(22.5)$ = 0.81148505 − 1.645×0.05973195 = 0.713226. This 95% glb on RE is more meaningful than the previous three, as all previous three were too conservative.

More Exact Confidence Interval for the Weibull slope β

For Type II censored data, Minitab uses the fact that that the SMD of $\hat{\beta}\times \ln(\beta/\hat{\beta})$ and $\hat{\theta}\times \ln(\theta/\hat{\theta})$ are approximately normal with mean zero and variances obtained from inverting the Fisher's local Information matrix given by $\hat{\mathbf{F}} = \begin{bmatrix} \hat{\mathbf{I}}_{11} & \hat{\mathbf{I}}_{12} \\ \hat{\mathbf{I}}_{21} & \hat{\mathbf{I}}_{22} \end{bmatrix}$, where $\hat{\mathbf{I}}(\theta) = \hat{\mathbf{I}}_{11} = \frac{\mathbf{r}*\hat{\beta}^2}{\hat{\theta}^2}$,

$\hat{\mathbf{I}}(\beta) = \hat{\mathbf{I}}_{22} = \frac{\mathbf{r}}{\hat{\beta}^2} + \sum_{i=1}^{r}[(t_i/\hat{\theta})^{\hat{\beta}}\times \ln^2(t_i/\hat{\theta})] + (n-r)(t_r/\hat{\theta})^{\hat{\beta}}\ln^2(t_r/\hat{\theta})$, and $\hat{\mathbf{I}}_{12} = \hat{\mathbf{I}}(\theta,\beta) \cong$

$-\mathbf{E}(\frac{\partial^2 \mathbf{L}}{\partial\theta\partial\beta}) = \frac{\mathbf{r}}{\hat{\theta}} - \sum_{i=1}^{r}[\hat{\beta}(t_i/\hat{\theta})^{\hat{\beta}}\times \ln(t_i/\hat{\theta}) + (t_i/\hat{\theta})^{\hat{\beta}}]/\hat{\theta} -$

$(n-r)[\hat{\beta}(t_r/\hat{\theta})^{\hat{\beta}}\ln(t_r/\hat{\theta}) + (t_r/\hat{\theta})^{\hat{\beta}}]/\hat{\theta}$. For the Example 5.19 of Elsayed, my Excel file shows that $\hat{\mathbf{I}}_{11} = \frac{\mathbf{r}*\hat{\beta}^2}{\hat{\theta}^2}$ = 0.6133870, $\hat{\mathbf{I}}_{12}$ = 0.02301614, and $\hat{\mathbf{I}}_{22}$ = 1.1997954. Inverting the matrix $\hat{\mathbf{F}}$, we obtain $\hat{\mathbf{F}}^{-1} = \hat{\mathbf{I}}^{-1}$ = $\begin{bmatrix} 1.6314665 & -0.0312971 \\ & 0.83407581 \end{bmatrix}$. Thus the *se*($\hat{\theta}$) = sqrt(1.6314665) = 1.27728871, CV($\hat{\theta}$) = *se*($\hat{\theta}$) /30.57645575 = 0.0417736, *se*($\hat{\beta}$) = $\sqrt{0.83407581}$ = 0.91327751, and CV($\hat{\beta}$) = 0.17887913. Hence, the 90% confidence limits are β_L = 5.1055566× $e^{-Z_{0.05}\times CV(\hat{\beta})}$ = 3.8041763, and β_U = 5.1055566 ×$e^{1.644854\times 0.17887913}$ = 6.85212930; similarly, θ_L = 28.5460574 and θ_U = 32.7512705. In summary, 3.8041763 ≤ β ≤ 6.8521293 and 28.5460574 ≤ θ ≤ 32.7512705 at the 90% confidence level.

Better Approximate *se*'s for Multiple Censoring

The exact variance of $\hat{\beta}$ cannot be computed, but as in the case of Type II censoring, we compute the 2nd-partial derivatives of the log-likelihood function in order to approximate the Fisher's Information matrix. Form Eq. (147d), $\hat{\mathrm{I}}(\theta) = \dfrac{r\hat{\beta}^2}{\hat{\theta}^2} = \hat{\mathrm{I}}_{11}$, and partially differentiating Eq. (139c) with respect to β yields $\hat{\mathrm{I}}(\beta) = \dfrac{r}{\hat{\beta}^2} + \sum_{i=1}^{r}[(t_i/\hat{\theta})^{\hat{\beta}} \times \ln^2(t_i/\hat{\theta})] + \sum_{j=1}^{n-r}[(t_j^+/\hat{\theta})^{\hat{\beta}} \times \ln^2(t_j^+/\hat{\theta})]$,

and it can be shown that $\hat{\mathrm{I}}_{12} = \dfrac{r}{\hat{\theta}} - \sum_{i=1}^{n}(t_i/\hat{\theta})^{\hat{\beta}}/\hat{\theta} - \sum_{i=1}^{n}[\hat{\beta}(t_i/\hat{\theta})^{\hat{\beta}} \times \ln(t_i/\hat{\theta})]/\hat{\theta}$, where the summation runs over both the failure set S_F and censored set S_C. For the sake of illustration, consider the Example 15.10 on p. 404 of Ebeling, where there are two modes of failure. For Type B mode, we have r = 6 failures, type A failures become censored values of which S_C = 11 and n = 40 units on test. Thus, this is multiple (or arbitrary) censoring, where testing stopped at the end of warranty period of one year with 23 units surviving one year = 365 days. My Excel file shows that the MLEs are $\hat{\beta} = 1.4054271$ (Ebeling LSQE is 1.340) and $\hat{\theta}$ = 1101.4932397 (Ebeling's LSQE is 1037.6). The Same Excel file shows that the Fisher's Local information matrix is given by $\hat{\mathbf{F}} = \hat{\mathbf{I}} = \begin{bmatrix} 9.76797\times10^{-6} & 0.0092075 \\ 0.0092075 & 12.1789734 \end{bmatrix} \rightarrow \hat{\mathbf{F}}^{-1} = \hat{\mathbf{I}}^{-1} =$

$\begin{bmatrix} 356262.4147 & -269.3407 \\ -269.34070 & 0.2857351 \end{bmatrix} \rightarrow se(\hat{\beta})$ = 0.5345419 and $se(\hat{\theta})$ = 569.8772191 $\rightarrow$ CV($\hat{\beta}$) = 0.3803413 and $\mathrm{CV}(\hat{\theta})$ = 0.541880057 $\rightarrow \beta_L = 1.4054271e^{-1.644854\times0.3803413} = 0.7518154$, $\beta_U = 1.4054271e^{1.644854\times0.3803413} = 2.6272744$; similarly, θ_L = 451.74084 and θ_U = 2685.804005. These 90% CIs are identical to those of Minitab's.

Summary of Weibull SI with Censored Samples

1. **For type I censoring, numerically solve the following two Eqs. with 2 unknowns.**

$$\hat{\theta} = \left[[\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n-r)T^{\hat{\beta}}] / r \right]^{1/\hat{\beta}} \qquad ; T = t^*.$$

$$\frac{1}{\hat{\beta}} + \frac{1}{r}\sum_{i=1}^{r} \ln(t_i) - [\sum_{i=1}^{r} t_i^{\hat{\beta}} \ln(t_i) + (n\text{-}r)T^{\hat{\beta}} \ln(T))/ [\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n\text{-}r)T^{\hat{\beta}}] \quad = 0; T = t^*.$$

$$V(\hat{\theta}) \geq \frac{\hat{\theta}^2 / \hat{\beta}}{(\hat{\beta}+1)(\hat{\theta})^{-\hat{\beta}}[\sum_{i=1}^{r} t_i^{\hat{\beta}} + (n-r)T^{\hat{\beta}}] - r} \quad \rightarrow se(\hat{\theta}) \cong \hat{\theta}\sqrt{c_{11} / (n)} / \hat{\beta}$$

$$se(\hat{\beta}) \cong \hat{\beta}\sqrt{c_{22} / n}$$

It is paramount to note that the *se*'s of $\hat{\theta}$ and $\hat{\beta}$ both diminish with increasing n and r. The decreasing relationship wrt n is obvious. However, when r increases, then p increases and Table 5.11 of E. A. Elsayed shows that both c_{11} and c_{22} decrease. A more accurate estimate of *se*($\hat{\beta}$) can be obtained from $se(\hat{\beta}) \cong \hat{\beta}_{Unb}\sqrt{C_n / n}$, where $C_n = 0.617 + 1.8/n + 78.25/n^3$ (Eq. 5.43Elsayed) and $\hat{\beta}_{Unb} = \hat{\beta}_{MLE} \times G_n$, where $G_n = 1 - 1.346/n - 0.8334/n^2$.

(Eq. 5.42Elsayed)

From Bain & Engelhardt $\hat{a} = \hat{\beta} \times \ln(\hat{\theta}/\theta)$ is approximately Gaussian with zero mean and asymptotic variance equal to c_{11}/n, and $c \times r(\beta/\hat{\beta})^{1+p^2}$ approximately follows a χ^2 with c(r –1) *df*, where the constant $c = \dfrac{2}{p(1+p^2)^2 \times c_{22}}$, and p = sample failed fraction.

2. Type II Censoring: $\hat{\theta} = \left[\left\{ \sum_{i=1}^{r^*} t_i^{\hat{\beta}} + (n-r^*)t_{r^*}^{\hat{\beta}} \right\} / r^* \right]^{1/\hat{\beta}}$

$$\frac{1}{\hat{\beta}} + \frac{1}{r^*}\sum_{i=1}^{r^*} \ln(t_i) - \frac{\sum_{i=1}^{r^*} t_i^{\hat{\beta}} \ln(t_i) + (n\text{-}r^*)t_{r^*}^{\hat{\beta}} \times \ln(t_{r^*})}{\sum_{i=1}^{r^*} t_i^{\hat{\beta}} + (n\text{-}r^*)t_{r^*}^{\hat{\beta}}} = 0$$

3. Multiple Censoring: $\hat{\theta} = \left[\left\{ \sum_{i=1}^{r} t_i^{\hat{\beta}} + \sum_{j=1}^{n-r} (t_j^+)^{\hat{\beta}} \right\} / r \right]^{1/\hat{\beta}}$

$$\sum_{i=1}^{r} \left[\frac{1}{\hat{\beta}} + \ln(t_i / \hat{\theta}) - (t_i / \hat{\theta})^{\hat{\beta}} \ln(t_i / \hat{\theta}) \right] - \sum_{j=1}^{n-r} \left[(t_j^+ / \hat{\theta})^{\hat{\beta}} \ln(t_j^+ / \hat{\theta}) \right] = 0$$

4. **In all cases** $\hat{R}(t) = e^{-(t/\hat{\theta})^{\hat{\beta}}}$ **and** $V[\hat{R}(t)] \cong$

$$\hat{R}^2 [(\ln(1/\hat{R})]^2 \left[c_{11} - 2c_{12} \ln(\ln(1/\hat{R})) + c_{22} [\ln(\ln(1/\hat{R}))]^2 \right] / n$$

The 95% glb on R(t) is given by $R_{0.95}^{L}(t) = e^{-(t/\hat{\theta})^{\hat{\beta}}} - 1.645 \times se[\hat{R}(t)]$**, where the** $se[\hat{R}(t)]$ **is the sqrt of the Eq. (145) on p. 286 and is repeated above.**

Note that you may use Minitab's formulas $\beta_L \cong \hat{\beta} e^{-z_{\alpha/2} \times CV(\hat{\beta})}$, $\beta_U \cong \hat{\beta} e^{z_{\alpha/2} \times CV(\hat{\beta})}$**, and similarly for the** $t_c = \theta$ **for all CIs, where** $se(\hat{\beta}) \cong \hat{\beta}\sqrt{c_{22}/n}$ **and** $se(\hat{\theta}) \cong \hat{\theta}\sqrt{c_{11}/n} \;/\hat{\beta}$.

Table 5.11 of E. A. Elsayed (on his page 297, 1−p = proportion of the sample that is censored; p = failed proportion); use c_{11} for θ and c_{22} for β.

p	1	0.9	0.8	0.7	0.6	0.5
c_{11}	1.108665	1.151684	1.252617	1.447258	1.811959	2.510236
c_{22}	0.607927	0.767044	0.928191	1.122447	1.372781	1.716182
c_{12}	0.257022	0.176413	0.049288	−0.144825	−0.446603	−0.935766

p	0.4	0.3	0.2	0.1
c_{11}	3.933022	7.190427	16.478771	60.517110
c_{22}	2.224740	3.065515	4.738764	9.744662
c_{12}	− 1.785525	− 3.438610	−7.375310	− 22.187207