INSY 7380 Reference: Chapters 12 and 13 of Ebeling (2nd)

Statistical Inference Maghsoodloo

Since the entire chapters 12, 13 & 15 deal with parameter estimation and confidence intervals for various reliability parameters, then the chapters are about statistical inference (SI) but mostly in the parametric sense, i.e., the underlying distribution is assumed known, except for the case when the binomial pmf is used for estimation. We will 1st cover SI (statistical inference) when the process is binomial (or Poisson), then when the underlying distribution is exponential, followed by a Weibull density for the TTF distribution.

SI When the Number of Failures has a Binomial Distribution

In this case the experimenter conducts life testing in cycles (or trials) and merely counts whether one cycle leads to a success (N_f = 0), or to a failure (N_f = 1). Thus, the SMD (sampling distribution) of number of failures per trial is Bernoulli with parameters q = Pr(N_f = 1) = failure Pr, and p = Pr(N_f = 0) = Pr(success) = R, where R is the reliability of the item in one cycle. In order to obtain point and interval estimates of R, we must conduct n Bernoulli trials (i.e., we must observe n cycles or have n units on test) and obtain a count of number of failures N_f, and number of successes N_s, where n = N_f + N_s. Thus a point estimate of the reliability function is $\hat{p} = \hat{R}$ = N_s / n. As an example, consider a binomial experiment where in n = 50 cycles N_f = 3 failures are observed; then $\hat{R}$ = 47/50 =0.9400, and a point estimate of unreliability is equal to $\hat{F}$ = $\hat{Q}$ = 3/50 =0.06. Since reliability is a measure that we wish to always maximize , (i.e., R is an LTB type parameter), then we would generally be interested in a lower one-sided CI such as $R_L \le R < 1$. The reader must be cognizant of the fact that in real-life situations a reliability of exactly 1 cannot be attained for any system (unless there are infinite number of standbys), and therefore, the interval $R_L \le R \le 1$ is unrealistic at best and should be written as $R_L \le R < 1$. Instead of presenting the general theory behind the construction of a CI for the reliability R using the binomial distribution, I will illustrate the procedure using the data of the above example. Because the most common confidence

coefficient $1 - \alpha = 0.95$, throughout these notes the confidence level is 95% unless otherwise stated, i.e., $\alpha = 0.05$. Thus, we are looking for an interval for which the $Pr(R_L \le R < 1) = 0.95$ before collecting n data points. For starters one should observe that $R_L < 0.94 = \hat{R}$ (= the point estimate of reliability). For notational convenience we revert back to the binomial notation and observe that the rv N_s, before experimentation, has a binomial Pr mass function with unknown parameter p = R and n = 50 Bernoulli trials, i.e., $Pr(N_s = x) = b(x; 50, p) = {}_{50}C_x\, p^x q^{50-x} = {}_{50}C_x\, R^x (1 - R)^{50-x}$, x = 0, 1, 2, 3, …, 50. For example, suppose that we are testing 50 pressure switches assuming R = 0.90 for each unit, then the pr of obtaining exactly 40 successes in 50 cycles is given by $Pr(N_s = 40)$ = Pr (exactly 40 survivals)= $\binom{50}{40}(0.90)^{40}(0.10)^{10} = 0.0152$, $Pr(N_s = 41) = \binom{50}{41}(0.90)^{41}(0.10)^{9} = 0.0333$ = b(41; 50, 0.90), where $\binom{50}{41} = {}_{50}C_{41}$. Note that at p = 0.90 and n = 50, np = 45 successes (or survivals) must have the highest occurrence Pr because the mean number of successes expected is $n\times p = 50\times 0.90 = 45$. I used Matlab to compute several other Prs such as binopdf(42, 50, 0.9) = b(42; 50, 0.90) =0.0643 , b(43; 50, 0.90) = 0.1076, b(44; 50, 0.90) = 0.1541, and b(45; 50, 0.90) = 0.1849, b(46; 50, 0.90) = 0.1809 and b(50; 50, 0.90) = $(0.9)^{50}$ = 0.0052. Thus we have established the fact that the function b(x; 50, 0.90) is increasing up to and including np (if this is not an exact integer, then rounding to the nearest integer will generally work) and then will be decreasing from np to n. The next question is what sort of function is b(x; 50, p) of p, and specifically for our example what can we assert about the binomial mass pr b(47; 50, p) as a function of the parameter p. I used MS Excel to obtain the needed functional relationship as shown in figure 19 atop the next page. To obtain the lower CI (confidence interval) limit $p_L = R_L$, we must ask ourselves how low can we reduce p from the estimated value of 0.94 such that $Pr(N_s = 47 \mid p = p_L)$ is reduced from 0.94 to roughly 0.05. Figure 19 shows that on the y-intercept the grid line of 0.05 cuts our graph between p = 0.85 and p = 0.90. Hence, $0.85 < p_L = R_L < 0.89$. To be a bit more precise, our objective is to obtain $R_L = p_L$ in such a manner that $Pr(N_s \ge 47 \mid p \downarrow p_L) \le 0.05$; this in turn will assure us with 95% confidence Pr that the true value of R = p lies within the interval $[p_L, 1)$. The Excel file on my website shows that the exact solution to the inequality $Pr(N_s \ge 47 \mid p \downarrow p_L) \le 0.05$ is $p_L = 0.87938 = R_L$. Thus we are 95%

confident that the switches reliability lies in the interval $0.87938 \le R < 1$. This implies that if we hypothesize $H_0 : R = 0.85$ vs $H_1 : p > 0.85$, then this null hypothesis can be rejected at the5% level

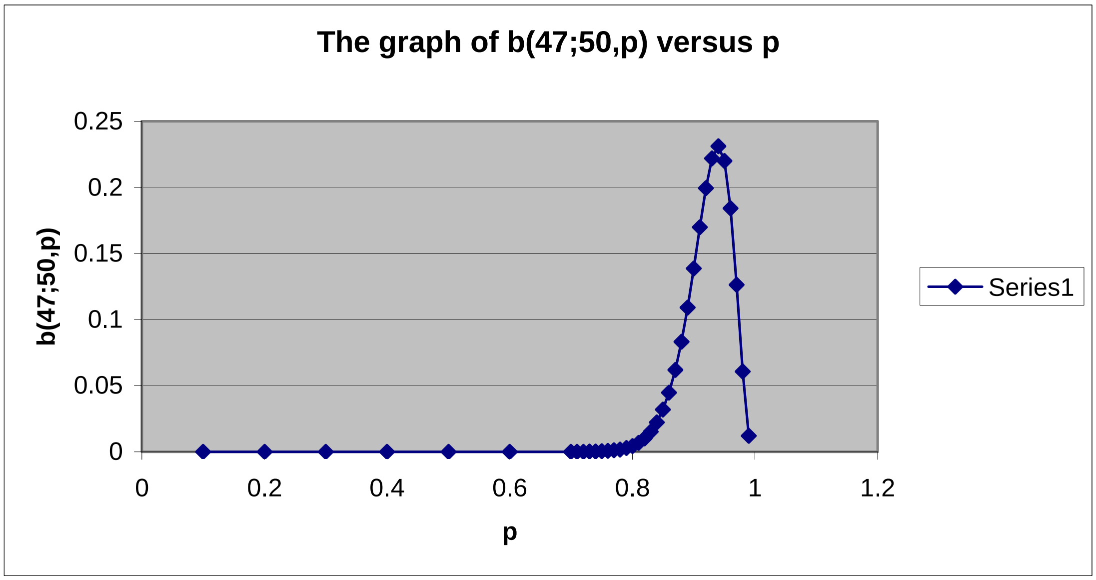

Figure 21. The graph of b(47:50, p) versus p

of significance because 0.85 lies outside the 95% CI : $0.87938 \le R < 1$, but we will not be able to reject $H_0 : R = 0.88$ at the 5% level in favor of H_1: $R > 0.88$.

I now generalize the above procedure for an observed $N_s = x$ in n Bernoulli trials. Simply solve the Pr equation $\Pr(N_s \ge x \mid p = p_L) \le \alpha$, or equivalently solve the equation $\Pr(N_s \le x - 1 \mid p = p_L) = B(x - 1; n, p_L) \ge 1 - \alpha$. If n and N_f are sufficiently large and both R and Q lie within the interval $0.05 < R, Q < 0.95$ and nQ & $nR > 10$, then the binomial mass Prs can be approximated by a normal curve. Note that if Q or $p < 0.05$, then we basically have a (HPP) Poisson process instead of a binomial process, and the needed value of $n > 100$ before a normal approximation is roughly adequate. When these conditions closely hold, then $\hat{R}$ is approximately normal with mean R and variance equal to RQ/n, as depicted in Figure 19 on the next page. From Figure 19 we may easily glean that $P(\hat{R} \le R + 1.645\,\sigma) = 0.95 \rightarrow P(\hat{R} - 1.645\,\sigma \le R) = 0.95 \rightarrow$

P($\hat{R}$ – 1.645 $\sigma \le$ R < 1) = 0.95 $\rightarrow$ $R_L \cong \hat{R}$ –1.645 $\hat{\sigma}$. For the data of above example, R_L = 0.94 – 1.645$\sqrt{\dfrac{0.94\times 0.06}{50}}$ = 0.94 – 1.645×0.0336 = 0.884752, which is not conservative compared to the

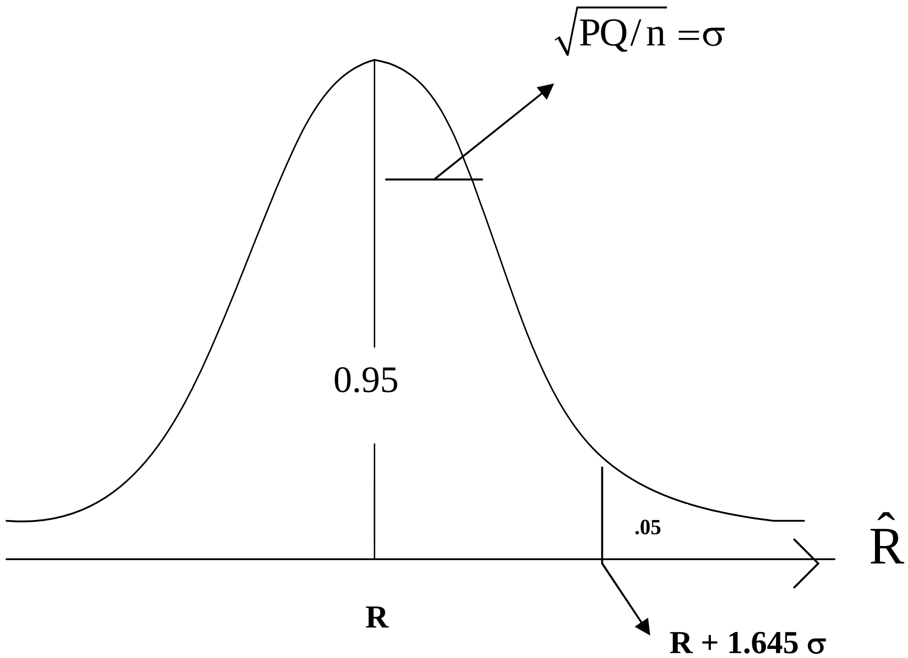

Figure 22. The Normal Approximation to the SMD of $\hat{R}$

exact answer of 0.87938. We can improve this answer by applying a cfc (correction for continuity) of 1/2n = 0.01 yielding $R_L \cong$ 0.87475. Generally, the cfc improves the normal approximation to the binomial.

<u>Example 4.2 of Paul Kales (1998) on his Page 32</u>

Kales provides the results of a meal delivery process in a hospital that is more H-Poisson than binomial because $\hat{Q}$ = 6/1500 = 0.0040 < < 0.05 and n = 1500 is very large. A success is when a patient receives meal on-time and as ordered. Thus the lower 95% confidence limit should be obtained using the Poisson pmf as described below. However, this is difficult to do by direct use of the rv N_s, in other words, instead of solving Pr($N_s \ge$ 1494 | R $\downarrow$ R_L) $\le$ 0.05 for R_L directly, we will solve the equivalent Pr equation Pr($N_f \le 6$ | Q $\uparrow$ Q_U) $\le$ 0.05. Thus, we need to solve the equation

$\sum_{x=0}^{6} \frac{(nQ)^x}{x!} e^{-nQ} \leq 0.05$ **for Q. This inequality can be solved through trail & error by Matlab, which will show that $nQ_U = 11.844 \rightarrow Q_U = 11.844/1500 = 0.007896 \rightarrow R_L \cong 1 - Q_U = 0.9921040 \rightarrow$ $0.992104 \leq R < 1$. If we apply the normal approximation cfc to the binomial pmf, we obtain $R_L \cong \hat{R}$ $- 1.645 \sqrt{\frac{\hat{R}\hat{Q}}{1500}} - 1/3000 = 0.992986$. The exact binomial answer computed by me is R_L = 0.99212043, i.e., B(1493; 1500, 0.99212043) $\geq$ 0.95. Note that Ebeling provides CI limits for the static RE in his Eq. (12.25) on p. 325 from Grosh [1989]. If we apply, Grosh's formula, $F_1 = F_{0.05,14,2988}$ = 1.6951, which leads to $R_L = 1/[1+F_1(7/1496)]$ =0.992131, almost identical to the exact value of 0.99212043.**

Statistical Inference

By statistical inference we mean Estimation and Test of Hypothesis. Estimation consists of point and interval estimation. The entire Chapters 12, 13, 14 & 15 of Ebeling(2nd) are devoted to point estimators, (confidence) interval estimation and prediction. A point estimator of a population parameter θ (such the reliability function R(t), t_c, and the MTTF) is a sample statistic, $\hat{\theta}$, which is a rv with a frequency function that depends on the underlying distribution, f(x; θ), of the parent population, where the vector $\theta = [\theta_1 \quad \theta_2 \quad \theta_3 \ldots \quad \theta_m]'$, i.e., the underlying population has m unknown parameters to be estimated. In RE engineering applications, the value of m rarely exceeds 3. For the exponential density, $\theta = \lambda$ is a scalar, i.e., m = 1. For a normal TTF, $\theta = [\mu \quad \sigma^2]'$ is a 2$\times$1 vector so that m = 2, and for a Weibull TTF $\theta = [\delta \quad \theta \quad \beta]'$ is a 3$\times$1 vector for which m = 3. Our objective in point estimation is to use sample data to obtain an "accurate" vector point estimator of θ, denoted by $\hat{\theta}$. Accuracy of a point estimator can be measured through several properties such as bias, consistency, mean square error (MSE), efficiency and sufficiency. All these properties, with the exception of sufficiency, are well defined on my website under STAT 3600, Chapter 6.

There are several methods of obtaining point estimators :(1) Method of Moments, (2)

Maximum Likelihood estimators (MLEs), (3) Least-squares Estimators, which is applicable when the hazard function can be estimated. In RE engineering the MLEs are often used because all MLEs in the universe have the asymptotic property that their SMD approaches normality as the sample size $n \to \infty$ and they are asymptotically unbiased. This implies that once the *se* (standard error) of a MLE is obtained, then the asymptotic 95% CIs is simply $\hat{\theta} \pm 1.96 \times se(\hat{\theta})$. Further, they have the very nice property that if $\hat{\theta}$ is a MLE of θ, then $h(\hat{\theta})$ is a MLE of $h(\theta)$ for any functional form h. Thus, we will only obtain ML estimates for population parameters in this course.

Maximum Likelihood Estimators

Let $f(x;\theta)$ represent the frequency function of the population from which a random sample of size n is drawn. The occurrence of the sample $(x_1, x_2, ..., x_n)$ has a likelihood (or Pr) of $[f(x_1;\theta)\,dx_1]\times[f(x_2;\theta)\,dx_2]\times ... \times[f(x_n;\theta)\,dx_n] = \prod_{i=1}^{n}[f(x_i;\theta)\,dx_i]$. The product quantity $\prod_{i=1}^{n} f(x_i;\theta)$ in this last Pr statement is called the likelihood function, and after the sample is drawn and the sample values $x_1, x_2, ..., x_n$ are known numbers (no longer rvs), then the likelihood function $\prod_{i=1}^{n} f(x_i;\theta)$ is only a function of the parameter vector θ and is denoted by $L(\theta)$. That is, $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta)$. The maximum likelihood estimator (MLE) of the vector θ is obtained by maximizing $L(\theta)$ with respect to all the m parameters in $L(\theta)$. Further, for notational convenience, let $\mathbf{L}(\theta) = \ln[L(\theta)]$, i.e., $\mathbf{L}(\theta)$ is the natural logarithm of the likelihood function, although some authors just use $\ln[L(\theta)]$. Below I will show that maximizing $\mathbf{L}(\theta)$ is equivalent to maximizing $L(\theta)$.

$\partial\mathbf{L}(\theta)/\partial\theta_k = \frac{1}{L(\theta)}\,\partial L(\theta)/\partial\theta_k$; **Since $0 < L(\theta) < \infty$, i.e., $L(\theta)$ is finite, then $\partial\mathbf{L}(\theta)/\partial\theta_k = 0$ iff $\partial L(\theta)/\partial\theta_k = 0$. Note that most authors, such as Ebeling, use the notation $L(\theta)$ for the likelihood function itself, but do not provide another notation for the $\ln(L)$; further, nearly always it is the**

$\mathbf{L}(\theta)$ = ln[L (θ)] that is maximized instead of L(θ)=$\prod_{i=1}^{n} f(x_i;\theta)$.

MLE for the Exponential Underlying Distribution (Complete Samples)

Suppose the TTF distribution is given by f(t) = $\lambda e^{-\lambda t}$ and as result for a complete sample of size n failures, the likelihood function is given by L(θ) = L(λ) = $\prod_{i=1}^{n} f(t_i;\lambda)$ = $\prod_{i=1}^{n} \lambda e^{-\lambda t_i}$ = $\lambda^n e^{-\lambda \sum_{i=1}^{n} t_i}$ $\rightarrow$ $\mathbf{L}(\theta)$ = $\mathbf{L}(\lambda)$ = ln($\lambda^n e^{-\lambda \sum_{i=1}^{n} t_i}$) = nln $\lambda - \lambda \sum_{i=1}^{n} t_i \rightarrow \rightarrow \partial \mathbf{L}(\lambda)/\partial\lambda$ = n/λ – $\sum_{i=1}^{n} t_i \xrightarrow{\text{Set to}} 0 \rightarrow$ n/$\hat{\lambda}$ = $\sum_{i=1}^{n} t_i \rightarrow \hat{\lambda}$ = n/$\sum_{i=1}^{n} t_i$. This last relationship shows that the MLE of λ given by $\hat{\lambda}$ = n/$\sum_{i=1}^{n} t_i$ = $1/\bar{x}$, and because $\partial^2\mathbf{L}(\lambda)/\partial\lambda^2$ = – n/λ^2 < 0, implying that the MLF is strictly concave. Since the knowledge of $\sum_{i=1}^{n} t_i$ completely specifies the likelihood function L(λ) = $\prod_{i=1}^{n} f(t_i;\lambda)$ = $\lambda^n e^{-\lambda \sum_{i=1}^{n} t_i}$, regardless of individual values of t_i's, the statistic $\sum_{i=1}^{n} t_i$ is said to be sufficient for estimating λ.

<u>Example 15.2 on pages 360-361 of Ebeling.</u> In this example, n = 35 times TF are obtained from a life-testing experiment of 35 components over a 6-month period. Since $\bar{x}$ = 485.1714 and S = 469.8002 then cv(X) = 96.832% is close enough to 100% for the exponential to be a tenable model (recall that all exponential distributions have a CV = 100%). Thus, $\hat{\lambda}$ = 1/$\bar{x}$ = 1/485.1714 = 0.0020611 per hour. If the cv(X) were close to 67%, then we would assume that the data perhaps originated from a Weibull family with β = 1.5 (see Table 1 of Chapter 2 on page 12 of my notes whose values are valid only for $t_0 = \delta = 0$). Note that the MLE of MTTF in this example is simply $\widehat{\text{MTTF}}$ = $\bar{x}$ = 485.1716 hours and that of R(t) is $\hat{R}(t) = e^{-0.0020611t}$.

Sampling Plans With Censoring (and Lifetime Having Underlying Exponential mdf "Mortality Density Function")

(1) Type I Censoring: This occurs when n $\geq$ 1 units are placed on test (at time = 0) for exactly a predetermined fixed length of censoring time T so that the duration of testing is not a random variable but the number of observed failures, N_f, is random. Components may be replaced (or repaired) as they fail (R-Type I, or with replacement censoring) so that there are n units on test in the interval (0, T) at all times, or components may be irreparable or un-repairable (the U-Type I censoring) in which case only (n−r) units will survive beyond the censored termination time T. Assuming that TBF (Time Between Failures) is exponentially distributed, the MTBF (Mean Time Between Failures) for the R-Type is estimated from

(R-Type I): $$\hat{\lambda} = \frac{r}{nT} \quad \rightarrow \quad \hat{\theta} = \frac{nT}{r} \qquad \textbf{(105a)}$$

where r is the observed no. of failures and must exceed zero or else point estimation is not possible. Later, we will show how to obtain a lower one-sided confidence interval for the exponential parameter $\theta = 1/\lambda$ when Type I censoring results in $N_f = 0$ failures during the fixed testing interval (0, T). Generally, the experimenter should a-priori censor (or limit) the total testing time, T, if the failure rate λ is small.

For the case of U-Type I censoring, the Pr of obtaining r failures and (n−r) survivors is given by $L(\lambda) = {}_nP_r \prod_{i=1}^{r} [\lambda e^{-\lambda t_i} dt_i] \times (e^{-\lambda T})^{n-r} = ({}_nP_r)\lambda^r e^{-\lambda \sum_{i=1}^{r} t_i} \times e^{-\lambda(n-r)T} \rightarrow \mathbf{L}(\theta) = \mathbf{L}(\lambda) = \ln[{}_nP_r \lambda^r e^{-\lambda \sum_{i=1}^{r} t_i} \times e^{-\lambda(n-r)T}] = \ln({}_nP_r) + r\ln\lambda - \lambda \sum_{i=1}^{r} t_i - \lambda(n-r)T$, where ${}_nP_r$ stands for permutations of n objects taken r at a time. The partial differentiation of this log-likelihood wrt λ will show that (U-Type I), and equating equal to zero, will show that $\hat{\lambda} = \frac{r}{\sum_{i=1}^{r} t_i + (n-r)T} \rightarrow$

$$\hat{\theta} = \mathbf{MTT\hat{F}} = \frac{\sum_{i=1}^{r} t_i + (n-r)T}{r} \qquad \textbf{(105b)}$$

Eq. (105b) shows that the total survival testing time of all n components is given by $\tau = \sum_{i=1}^{r} t_i$ + $(n - r)$ T, where t_i is the failure instant (or moment of failure) of the i^{th} unit with $t_1 < t_2 < ... < t_r \le T$. Further, the statistic $\sum_{i=1}^{r} t_i + \lambda(n - r)T$ is sufficient for estimating λ because its knowledge completely describes the likelihood function.

Boris Gnedenko (one of the premier mathematician of the 20^{th} century) defines censorship as termination of life testing before the moment (or the instant) t_n, and he refers to T as the moment of censorship (see Gnedenko *et al.*, "Statistical Reliability Engineering", Wiley, ISBN: 0-471-12356-0, the bottom of page 61), where t_n is the failure time of the complete sample. Thus if a sample is not complete (i.e., it is censored from the right so that r < n), then for certain $t_r \le T$ and $t_n > T$. A complete sample is one when all its n units are tested to failure, in which case t_n is a random variable (rv). Again in type I censoring r is a random variable.

<u>Example 10.3</u> (of Kapur & Lamberson, "Reliability in Engineering Design, Wiley & Sons, INC, 1977, pp. 241-242, ISBN:0-471-51191-9). Nine heater switches were placed on different stands and each stand was cycled T = 20000 times (i.e., the testing duration T was censored). As switches failed, they were replaced (R-sampling) with a brand new one instantly. Table 9 reports the result of this life testing (n = 9, R = with replacement, T = 20000 cycles) experiment. Table 9 shows that

Table 9

Stand No.	1	2	3	4	5	6	7	8	9
t_i (Cycles)	6700	4600	4100, 18100, 18950	5400	3100, 8100	2600	No Failures	4700	No Failures

Stands 1, 2, 4, 6, and 8 each had one switch that failed but was immediately replaced. Stand 3 had 3 failures at cycles 4100, 18100, 18950 and all three switches were replaced, and so forth. There were a total of r = 10 failures, where their instances of failures were $t_1 = 2600$, $t_2 = 3100$, $t_3 = 4100$, $t_4 = 4600$, $t_5 = 4700$, $t_6 = 5400$, $t_7 = 6700$, $t_8 = 8100$, $t_9 = 18100$, and $t_{10} = 18950$ cycles. Clearly, this experiment represents an R-Type I censoring plan so that n = 9 units were on test at all times and the estimate of MTBF is obtained from equation (105a) as $\hat{\theta} = \frac{nT}{r} = \widehat{\text{MTBF}}$

$\frac{\text{Total Accumulated Test Time} = 9\times 20000}{10}$ = 18000 cycles $\rightarrow$ $\hat{\lambda} = 1/\hat{\theta} = 0.0000555556$ failures/cycle. $\rightarrow$ The point estimate of the reliability function is given by $\hat{R}(t) = e^{-0.00005555\bar{5}\,t}$. Note that the plan [n, R (with replacement), t*] is the only plan where I would count TBFs because we may start with n= 9 units on test at time zero but end up using 37 units for testing during (0, t*).

To illustrate why the MLE of θ, for the R-Type I censoring, is given by $\hat{\theta} = \frac{nT}{r}$, we must make use of the fact that each unit's failure rate is a constant λ (because of exponential underlying distribution) so that the n units during the interval $(0, t_1)$ create a H-Poisson process at the rate (or traffic intensity) of $n\lambda$ per unit of time. The expected (or mean) number of failures during $(0, t_1)$ is $n\lambda t_1$. Therefore, during the interval $(0, t_1)$ the Pr of exactly one failure is given, from the Poisson pmf, by $n\lambda t_1 e^{-n\lambda t_1}$. Similarly, during the interarrival time $(t_2 - t_1)$ there are still n units on test (because of replacement) so that we still have a H-Poisson process of intensity $n\lambda$ and the Pr of exactly one failure during the interval (t_1, t_2) is given by $n\lambda(t_2 - t_1)e^{-n\lambda(t_2-t_1)}$, and so on up the interval (t_{r-1}, t_r) during which the Pr of one failure is given by $n\lambda(t_r - t_{r-1})e^{-n\lambda(t_r-t_{r-1})}$. Further, there must not be any more failures during the interval (t_r, T), which has a Pr of $e^{-n\lambda(T-t_r)}$. The likelihood Pr mass function for the occurrence of the r failures during (0, T) is, therefore, given by

$$L(\lambda) = (n\lambda t_1 e^{-n\lambda t_1})\times[n\lambda(t_2 - t_1)e^{-n\lambda(t_2-t_1)}]\times\ldots\times[n\lambda(t_r - t_{r-1})e^{-n\lambda(t_r-t_{r-1})}]\times e^{-n\lambda(T-t_r)}$$

$$= (n\lambda)^r \times \prod_{i=1}^{r}(t_i - t_{i-1})\times e^{-n\lambda T} \rightarrow \mathbf{L}(\lambda) = \ln(L(\lambda)) = r\ln(n\lambda) + \sum_{i=1}^{r}\ln(t_i - t_{i-1}) - n\lambda T \rightarrow$$

$\xi(\lambda) = \partial\mathbf{L}(\lambda)/\partial\lambda = r/\lambda + 0 - nT \xrightarrow{\text{Set to}} 0 \rightarrow \hat{\lambda} = r/(nT) \rightarrow \hat{\theta} = nT/r$. **Note that the expression for** $\mathbf{L}(\lambda)$ **clearly shows that the statistics** $\prod_{i=1}^{r}(t_i - t_{i-1})$ **and r are sufficient [i.e., their knowledge defines the** $\mathbf{L}(\lambda)$**] for estimating** θ**, and we have employed the notation** $t_0 = 0$**.**

In retrospect, the above MLE $\hat{\theta} = nT/r$ **could have been obtained from a much simpler argument that now follows. During the entire fixed testing interval [0, T] we have a H-Poisson process of intensity** $n\lambda$ **at all times because failed units are instantly replaced (or minimal-time repaired). Hence, the likelihood Pr mass function to observe r failures is simply** $\mathbf{L}(\lambda) = \frac{(n\lambda T)^r}{r!}e^{-n\lambda T} \rightarrow \mathbf{L}(\lambda) = r\ln(n\lambda T) - \ln(r!) - n\lambda T \rightarrow \xi(\lambda) = \partial\mathbf{L}(\lambda)/\partial\lambda = r/\lambda - nT \rightarrow \hat{\lambda} = r/(nT) \rightarrow \hat{\theta} = n\times T/r$**, as before!**

<u>Example 20.</u> Fifty units are tested for a predetermined duration test-time of T = 30 hours W/O replacement. Six failures were observed at $t_1 = 1.2$, $t_2 = 7.5$, $t_3 = 15.9$, $t_4 = 18.6$, $t_5 = 24.6$, $t_6 = 29.2$ **hours. Clearly this experiment involves an U-Type I censoring denoted by [50, U, 30 hrs] and equation (105b) shows that** $\hat{\theta} = \frac{\tau}{r}$**, where the total test times of all n = 50 units is** $\tau = 1.2 + 7.5 + 15.9 + 18.6 + 24.6 + 29.2 + (50-6)\times 30 = 1417$ **hours. Hence,** $\hat{\theta} = \widehat{\text{MTTF}} = 1417/6 = 236.166667$ **hours, and a point estimate of the survivor function is given by** $\hat{R}(t) = e^{-t/236.166667} = e^{-0.0042343\,t} = e^{-\hat{\lambda}t}$ **.**

<u>Example 5.1 on pages 264-265 of Elsayed (Addison Wesley Longman, INC, ISBN:0-201-63481-3.</u>
Although, it is not clearly stated what type of testing plan was used in this example, I surmise that the life-testing experiment involved an U-Type I censoring for a fixed duration of 100 hours, where $n_0 = 200$ **ceramic capacitors were subjected to a highly accelerated life test. Thus, the test plan is denoted by [200, U, 100 hrs]. However, data were collected on a grouped basis, i.e., the individual times to failure** t_i **were not reported but only the number of failures and censored units during an**

interval of length 10 hours were reported. I am providing Elsayed's data from his Table 5.1 below, where n_c stands for the number of censored units that had to be removed from experimentation for some reason (which I do not know). To illustrate the computations of $\hat{h}(t)$, consider the interval (40, 50 hours] during which there were a total of N_f(during Δt_5) = 6 failures. At the beginning of this interval 19 units had already failed and 28 units had been censored (i.e., removed from testing for some reason that was not stated), and thus the total number of survivors at the beginning of the interval (40, 50 hours] is equal to N_s(at 40 hours) = (200 − 47) = 153 units. Since hazard rate by definition is the proportion of the surviving units failing per unit of time, then

Elsayed's Table 5.1 (Reliability Engineering by E. A. Elsayed, atop his page 265)

Interval	0-10 hrs	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
N_f	0	6	7	6	6	5	4	3	2	1
n_c	3	8	9	8	15	20	18	20	30	29
N_s(at the end)	197	183	167	153	132	107	85	62	30	0
$\hat{h}$ (t)	0	0.0304	0.0382	0.0359	0.0392	0.0373	0.0372	0.0352	0.0322	0.0333

$\hat{h}(45\,\text{hours}) = \dfrac{6/153}{\Delta t_i} = \dfrac{0.039216}{10}$ **= 0.0039216 per hour. Note that this matches Elsayed's answer to 5 decimals, as he states clearly that the hazard rate is constant with the mean failure rate equal to $\bar{\lambda}$ = 0.0319 so that the $\widehat{\text{MTBF}} = \hat{\theta}$ = 1/0.0319 = 31.3578 hours. It may be best to refer to 31.3578 as the MTTf because failed units were not replaced.**

Before we discuss Type II censoring, we need to learn how to test life data for exponentiallity!

Goodness-Of-Fit Test for the Validity of the Exponential Model

The Bartlett's test statistic given below

$$B_r = \frac{2r[\ln(T_r/r) - (\sum_{i=1}^{r} \ln t_i)/r]}{1 + [(1+r)/(6r)]} \qquad (106)$$

is generally used to test the null hypothesis that the exponential failure density is a plausible model for the underlying distribution of a failure data set, where $T_r = \sum_{i=1}^{r} t_i$ stands for the total time to the r^{th} failure. The Bartlett's statistic B_r given in Eq. (106) is chi-square distributed with $r-1$ degrees of freedom (*df*). Further, the goodness-of-fit test is always a two-sided one. Before providing an example as to how to apply the Bartlett's statistic B_r to test for exponentiallity, I may alert you to the fact that for R-Type I censoring [n, R, T], in general the test is to determine if the TBF (time between failures) is exponentially distributed, not the TTF, nor the instances of failures. This is because there are n units on test at all times, even at the censoring time T. We now test the failure data of above Example 10.3 of Kapur & Lamberson to determine if the underlying distribution could be the exponential model at the LOS α = 0.05.

<u>Example 10.3 of Kapur & Lamberson Continued.</u> The times BF in order statistic form are $t_{(1)}$ = 850 hours, $t_{(2)}$ = 2600, $t_{(3)}$ = 3100, $t_{(4)}$ = 4100, $t_{(5)}$ = 4600, $t_{(6)}$ = 4700, $t_{(7)}$ = 5000, $t_{(8)}$ = 5400, $t_{(9)}$ = 6700, $t_{(10)}$ = 14000 hours. Matlab computations give $\ln(T_{10}/10)$ = ln(51050/10) = 8.5379757306 and $\sum_{i=1}^{10} \ln[t_{(i)}]/r$ = 83.323554925/10 = 8.332355493. Hence, the value of Bartlett statistic is

$$B_{10} = \frac{2r[\ln(T_r/r) - \sum_{i=1}^{r} \ln(t_{(i)})/r]}{1+(r+1)/(6r)} = \frac{20\,[8.5379757306 - 8.332355493]}{1+(10+1)/60} = 3.47527163.$$

The acceptance interval for the test consists of AI = $(\chi^2_{0.975;9}, \chi^2_{0.025;9})$ = (2.7003895, 19.0227678). Since B_{10} falls within the critical limits of this decision interval, then the test does not provide sufficient evidence to reject the null hypothesis that the failure data originated from an exponential distribution. The *P-value* of the test is given by $\hat{\alpha}$ = 2×P($\chi^2_9 \le$ 3.47527163) = 2×0.0575540895 = 0.11511. Note the larger the *P-value* is, the stronger our belief will be in the

validity of the null hypothesis that the underlying distribution is exponential. Further, the cv = 68.97632% indicating that a Weibull with $\delta = 0$ and $\beta = 1.48$ may provide a much better fit for the data (see my Table 1 on p. 12). Note that for type 1 censoring, only the observed Times BF should be used to compute the Bartlett's statistic B_r, i.e., the un-failed units should be ignored.

<u>Exercise 26.</u> Test the null hypothesis that the six times TF in the Example 20 can be modeled by the exponential distribution. ANS: B_6 = 3.74080, *P-Value* = 0.8253915.

(2) Type II Censoring: This occurs when n identical items are placed on test (at t = 0) and testing stops as soon as r, $0 < r \le n$, failures occur and failed items are not generally replaced (U-Type) but may be repaired. Note that r > 0 is a predetermined fixed nonnegative integer (i.e., the value of r is censored, or restricted a-priori). Then, the test terminates at the instant t_r, which is a random variable but r is fixed and not random. Therefore, the observable phenomenon in type II censoring are the rvs $t_1, t_2, \ldots, t_r$, i.e., the time to the i^{th} failure, i = 1, 2, ..., r. Type II censoring can also be carried out with or W/O replacement. It can be shown that for the R-Type II censoring plan, [n, R, r], the MLF is given by $L(\lambda) = (n\lambda)^r \times [\prod_{i=1}^{r}(t_i - t_{i-1})] \times e^{-n\lambda t_r}$, which leads to the MLE of θ as $\hat{\theta}$ = nt_r/r. Clearly, for this plan the statistics $\prod_{i=1}^{r}(t_i - t_{i-1})$ and t_r are sufficient for estimating θ. It can also be shown that for the U-Type II plan, [n, U, r], the MLF is $L(\lambda) = {}_nP_r\,(\lambda^r)\,[\prod_{i=1}^{r}(t_i - t_{i-1})] \times e^{-\lambda\sum_{i=1}^{r}[(n-i+1)(t_i-t_{i-1})]}$, and this leads to $\hat{\theta} = [\sum_{i=1}^{r} t_i + (n-r)t_r]/r$. For this plan the sufficient statistics for estimating θ are $\sum_{i=1}^{r} t_i$ and t_r.

<u>Example 10.5 of K&L (see Table 10.9 of Kapur & Lamberson p. 245).</u> Twenty rubber seals designed to prevent dirt from collecting on the wearing surfaces of ball joints were subjected to a life testing experiment to attain r = 10 failures. Testing would stop immediately at the moment

that the 10^{th} failure occurred and units were not replaced as they failed, i.e., the testing plan is [20, U, 10]. The cycles to failure data are as follows: 20400 cycles, 30000, 50700, 57750, 60300, 74100, 78300, 144000, 153500, 166000. Clearly, this is a U-Type II censoring plan because the number of failures were censored prior to the life-testing experiment and thus $\hat{\theta} = [\sum_{i=1}^{r} t_i + (n - r)t_r] / r =$

$\frac{835050 + (20 - 10) \times 166000}{10} = \frac{2495050}{10}$ = 249,505 cycles. I used the Bartlett's statistic to test the Times TF for exponentiallity and obtained B_{10} = 3.3287 giving rise to *P-value* = 0.1003. So, it seems that the *P-value* is not as large as we would like for the exponential to be a tenable model (we generally would like P-value > 0.20). Since testing units were not repaired or replaced, then TTF is a more appropriate measure than TBF. The point estimate of the RE function is given by $\hat{R}(t) =$ $e^{-0.000004007935713t}$. The CV of the ten times to failure is 62.61%, implying that a Weibull with shape β = 1.40 may provide a better fit.

<u>Exercise 27.</u> The following data were obtained from a life-testing experiment where n = 20 heater switches were placed on test, W/O replacement, and it was decided to terminate testing ASA r = 15 failures were observed. An overload voltage was applied to accelerate failures. Assume an acceleration factor of A_f = 25; in a later chapter, we will learn how to estimate A_f using regression analysis. For the time being, it is sufficient to know that the relationship between operating condition TTF, TTF_{OP}, and stress condition TTF, denoted TTF_s, is simply $t_{OP} = A_f \times t_s$. Data: 100 cycles, 340, 1940, 5670, 6010, 7120, 12910, 13670, 19490, 23700, 24110, 28570, 31620, 32800, 34910 cycles. **(a)** Test the accelerated data for exponentiallity, computing the critical level of your test $\hat{\alpha}$. **(b)** Estimate the $MTTF_s$ and $MTTF_{OP}$. **(c)** Obtain the 2-sided 95% CI for $MTTF_{OP}$. ANS: *P-value* = $\hat{\alpha}$ = 0.4045, (b) $MTTF_s$ = 27834, $MTTF_{OP}$ = 695850 cycles.

(3) Random Censoring: In this plan, the test terminates at the instant $Min(t_{r^*}, T)$. Both r* and t* = T are fixed and specified prior to experimentation. The instant of termination is a rv restricted from the right to t* =T and the number of observed failures, N_f, is also random and restricted to r*

from the right. This plan can also be carried out with (R-type) or W/O replacement (U-type). The MLFs are not easy to obtain, but it can be shown that for the R-Type random censoring the MLF is given by

$$L(\lambda) = (n\lambda t_1)\, e^{-n\lambda t_1} \times n\lambda(t_2 - t_1)\, e^{-n\lambda(t_2-t_1)} \times \ldots \times n\lambda(t_r - t_{r-1})\, e^{-n\lambda(t_r-t_{r-1})} \times e^{-n\lambda(T-t_r)} =$$

$$= (n\lambda)^r \prod_{i=1}^{r} (t_i - t_{i-1})\, e^{-n\lambda T}.$$ **Two possibilities may occur: (1) $t_{r*} < t^* = T$ so that $N_f = r^*$ and the observed number of failures are censored to the right, i.e., life testing ends at the moment t_{r*}. In this case the LF becomes $L(\lambda) = (n\lambda)^{r^*} \prod_{i=1}^{r^*} (t_i - t_{i-1})\, e^{-n\lambda t_{r^*}}$. This LF will lead to the estimator $\hat{\theta} = \dfrac{nt_{r^*}}{r^*}$. Case (2) $t_{r*} > t^* = T$ so that the test ends at time T, $L(\lambda) = (n\lambda)^r \prod_{i=1}^{r} (t_i - t_{i-1})\, e^{-n\lambda T}$, and $\hat{\theta} = \dfrac{nT}{r}$, where $N_f = r < r^*$. Note that we have used $t_0 = 0$. Then, in summary for the case of with replacement $\hat{\theta} = \dfrac{\tau}{r}$, where $\tau = \begin{cases} n\mathbf{T}, \text{ if } r < r^* \\ nt_{r^*}, \text{ if } r = r^* \end{cases}$. If the random censoring plan is carried out W/O replacement, then $\hat{\theta} = \dfrac{\tau}{r}$, where $\tau = \begin{cases} (\sum_{i=1}^{r} t_i) + (n-r)\mathbf{T}, \text{ if } r < r^* \\ (\sum_{i=1}^{r^*} t_i) + (n-r^*)t_{r^*}, \text{ if } r = r^* \end{cases}$, where the LF is obtained using the exponential density. Boris Gnedenko (page 68) gives the final result for the LF W/O derivation. The derivation of the LF is as follows, where r is the observed number of failures.**

$$L(\lambda) = ({}_nP_r)\lambda e^{-\lambda t_1} dt_1 \times \lambda e^{-\lambda t_2} dt_2 \times \ldots \times \lambda e^{-\lambda t_r} dt_r \times (e^{-\lambda t^*})^{n-r}$$

where ${}_nP_r = n!/(n-r)!$ and $t^* = \min(t_{r*}, T)$. Again there are two possibilities: (1) $t_r > T$ so that testing stops at the censored moment T and $N_f < r$ is a rv, and $t^* = T$. (2) $t_r < T$ so that testing terminates at the instant of t_{r*}, where r^* is prefixed (or censored) but t_{r*} is a rv so that $t^* = t_{r*}$.

So far, we have discussed point estimation only when the underlying distribution is exponential. After we complete statistical inference for the exponential, we will then return to statistical inference when the underlying distribution is Weibull. Ebeling discusses Multiply censored data where at least two units have different censoring times (see atop his page 307).

Confidence Intervals

As mentioned previously estimation consists of point and interval estimation, and thus far we have completed point estimation for the parameter θ = MTTF when TTF, T, is exponentially distributed. By interval estimation, we mean obtaining a confidence interval (CI) of the type $[\theta_L, \theta_U]$ which has a Pr of $(1-\alpha)$ of containing the true value of θ before the sample has been randomly selected from the population. The experimenter has to be cognizant of the fact that once the values of confidence limits θ_L and θ_U are computed from the sample statistics, then no longer the interval (θ_L, θ_U) has a Pr of $(1-\alpha)$ of containing θ because this Pr reduces to the deterministic values of 0 or 1 after data collection and ensuing computation of $\hat{\theta}$. In other words, the interval $[\theta_L, \theta_U]$ is random only prior to sampling. Note that confidence intervals are simply Tests of Hypotheses in disguise! Before the formulation of parametric CIs, I must assert that the characteristic life (or mean life in the exponential case) θ is definitely an LTB type parameter and hence a CI of the type $[\theta_L, \infty)$ would be more informative from an engineering standpoint than a 2-sided CI = $[\theta_L, \theta_U]$, as there are no concerns on the high side. However, I checked on more than 10 different texts by well-known authors, most of them discussed only a 2-sided CI for θ. Two authors discussed a lower confidence bound on reliability R(t). Since I do not wish to deviate much from RE literature, I will discuss both the lower one-sided CI and the 2-sided CI for θ. However, in this course you will be more often asked to obtain the lower 95% CI bound for θ.

In order to perform SI on an exponential parameter, we need to obtain the sampling distribution (SMD) of the statistic $T_r = \sum_{i=1}^{r} t_i$, where $t_1 < t_2 < ... < t_r$, and the rvs t_i are iid

with an underlying exponential pdf $f(t_i) = \frac{1}{\theta}e^{-t_i/\theta} = \lambda e^{-\lambda t_i}$. I will first show that the SMD of the rv $2\lambda t_i = 2t_i/\theta$ follows a Chi-square (χ^2) distribution with ν = 2 degrees of freedom (*df*).

The pdf of a χ^2 rv, say W, with any ν *df* is given by

$$g(w) = Cw^{(\nu/2)-1}e^{-w/2}, \quad 0 \le w < \infty \tag{107}$$

where the normalizing constant $C = 1/[2^{\nu/2}\Gamma(\nu/2)]$. If $C \ne \frac{1}{2^{\nu/2}\Gamma(\nu/2)}$, then g(w) in (107) will not integrate to 100%. Form (107) the pdf of a χ^2 rv with 2 *df* is given by $g(\chi_2^2) = \frac{1}{2}\, e^{-w/2}$ which is an exponential failure density with the intensity λ = ½ failures per unit of time. Now, let $w/2 = \lambda t_i$; then the SMD of $w = 2\lambda t_i$ follows a χ_2^2, i.e., a chi-square rv with 2 *df*. Using the additive property of χ^2, then it follows that the SMD of $2\lambda T_r = 2\lambda[\sum_{i=1}^{r} t_i + (n - r)\, t_r] = 2\lambda[\sum_{i=1}^{r-1} t_i + (n - r + 1)t_r]$ is chi-square with 2r *df*, i.e., $2\lambda T_r = 2r\hat{\theta}/\theta \sim \chi_{2r}^2$. Before, we proceed further, henceforth in these notes, I will generally be developing a 95% CI (either one- or two-sided), i.e., unless otherwise specified the value of confidence level (or confidence coefficient) is always $1 - \alpha = 0.95 = 95\%$. Our author, Ebeling, generally uses $1 - \alpha = 0.95$ nearly all the times. The second choice for $1 - \alpha$ is 0.90.

For type II censoring (i.e, the number of failures is censored to r), it follows that for a 2-sided CI the Pr statement from a χ_{2r}^2 distribution is $P[\chi_{0.975;\,2r}^2 \le (2T_r/\theta) \le \chi_{0.025;\,2r}^2] = 0.95$, where $\chi_{0.025;\,2r}^2$ is the 2.5 percentage point (or the 0.975 quantile) of a χ^2 rv with (2r) *df*. This Pr statement leads to the following 2-sided CI for the exponential MTTF θ.

$$\frac{2T_r}{\chi^2_{0.025;2r}} \le \theta \le \frac{2T_r}{\chi^2_{0.975;2r}} \qquad \textbf{(108a)}$$

might be best to rewrite the CI in (108a) as

$$\frac{2r\hat{\theta}}{\chi^2_{0.025;\,2r}} \le \theta \le \frac{2r\hat{\theta}}{\chi^2_{0.975;\,2r}}, \qquad \textbf{(108b)}$$

which will hold for any Type II censoring. Since the E(χ^2_ν) = ν, then it follows that E($2T_r/\theta$) = 2r $\rightarrow$ E(T_r/θ) = r $\rightarrow$ E(T_r / r) = θ $\rightarrow$ Thus, T_r /r is an unbiased estimator of θ. Further, because the V(χ^2_ν) = 2ν, then V($2T_r/\theta$) = 4r $\rightarrow$ V(T_r /r) = V($\hat{\theta}$) = θ^2/r = 1/($r\lambda^2$).

<u>Example 10.5 of Kapur & Lamberson (continued).</u> Our objective is to use the failure data of Example 10.5 of K&L to obtain a 2-sided CI followed by a one-sided CI for θ. The requisite percentage points of χ^2 are $\chi^2_{0.975;20}$ = 9.5908 and $\chi^2_{0.025;20}$ = 34.1696, recalling that

$\hat{\theta}$ = [$\sum_{i=1}^{r} t_i$ + (n – r)t_r] = 249,505 cycles. Thus, from equation (108b), θ_L = $\frac{2r\hat{\theta}}{\chi^2_{0.025;20}}$ = $\frac{20\times 249505}{34.1696}$ = 146039.13982, and similarly θ_U = $\frac{2r\hat{\theta}}{\chi^2_{0.975;20}}$ = $\frac{20\times 249505}{9.5908}$ = 520301.93131 $\rightarrow$ we are 95% confident that the true value of MTTF lies in the interval 146039.13982 $\le \theta \le$ 520301.93131 cycles. Since the failure rate λ = 1/ θ, then the 95% CI for λ is given by 1/146039.13982 $\ge$ 1/θ $\ge$ 1/520301.93131 $\rightarrow$ 0.000001921961 $\le \lambda \le$ 0.0000068474794. Because the survivor function, R(t), is an increasing function of θ, then $R_L(t)$ = e^{-t/θ_L} = $e^{-0.00000684748t}$, and similarly $R_U(t)$ = $e^{-0.000001921961t}$. Therefore, we are 95% confident that the RE function lies in the interval

$$e^{-0.00000684748t} \le R(t) \le e^{-0.000001921961t}.$$

<u>Exercise 28.</u> For the data of Example 10.5 of K&L above, obtain the 95% lower one-sided CIs for θ and R(t). Then obtain the upper one-sided 95% CI for λ. (b) Obtain the point estimate of

R(t) at t = 120,000 cycles and determine if your point estimate is unbiased. (c) Recall the relationship between the p^{th} percentile of the exponential and its RE function from Chapter 2. From Equation (9a, p. 10 of my notes) with δ = 0 and β = 1, $x_p = \theta \times \ln[1/(1-p)]$. Therefore, for the exponential failure law, the 10% life L_{10} for which RE is 0.90 is given by $x_{0.10} = \theta \times \ln(1/0.90) = 0.105360515660\theta$. Clearly the 10% life is an increasing function of θ. Use this fact to obtain a 2-sided followed by a lower 1-sided CI for $x_{0.10} = L_{10}$ for the data of K&L Example 10.5.

Statistical Inference for Type I censoring

Recall that Type I sampling plans involve testing n identical units with failure rate λ on a fixed pre-assigned interval [0, T =t*], where T = t* is the censored testing time. Therefore, what is random is the number of failures, N_f, observed during the testing interval [0, T]. Of course, the testing can be carried out with (R-Type) or W/O replacement (U-Type). Further, you must have guessed it by now as to what problem the experimenter will encounter at the end of the testing interval? Zero failures! I will restrict the rest of the following developments to the case of with replacement (R-type sampling, where we should count TBFs). What do we do next if we observe zero number of failures, i.e., r = 0. Well, first we will not be able to obtain a point estimate of MTBF, $t_c = \theta$, which is given by $\hat{\theta} = n\times T/r$; as you know division by zero is not kosher! However, the length of our testing time, T, is not completely wasted either! Because now we will be in a position to obtain a lower one-sided (95%) CI bound for the parameter θ in the form $\theta_L \le \theta < \infty$, i.e., we will be able to state with 95% confidence that the value of θ exceeds the lower bound θ_L. I will go thru the developments in great details, but you will be responsible only for the use of the result! A good reference on this is Kapur & Lamberson pp. 276-290.

Consider a Poisson process of intensity $n\lambda$ (remember that we have n units on test at all times each with a failure rate λ resulting in a collective traffic intensity $n\lambda$ in an R-type sampling plan). As before, let X(T) denote the number of failures occurring during [0, T] so that $E[X(T)] = n\lambda T$; I am using X(T) in lieu of $N_f(T)$ only for convenience. Therefore, from the H-Poisson pmf we have:

$$P[X(T) \le r] = \sum_{k=0}^{r} \frac{(n\lambda T)^k}{k!} e^{-n\lambda T} \qquad (109)$$

Just to understand the problem a bit better, assume that the test duration is fixed at T = 200 hours and there are n = 10 units on test at all times (i.e., with the replacement case) each with a failure rate λ = 0.0006 per hour. Then we now have a HPP over the fixed interval [0, 200 hours] at the intensity level of 0.006 failures/ hour. Suppose we are looking for the occurrence Pr of r = 3 or less failures. Equation (109) gives $P[X(10) \le 3] = \sum_{k=0}^{3} \frac{(1.20)^k}{k!} e^{-1.20}$ = 0.9662; if we keep all parameters fixed but increase λ to 0.0008 per hour, then the $P[X(10) \le 3] = \sum_{k=0}^{3} \frac{(1.60)^k}{k!} e^{-1.60}$ = 0.9212. You may try many more values of λ, then you will easily discover that the cumulative Pr: $P[X(10) \le 3] = \sum_{k=0}^{3} \frac{(200\lambda)^k}{k!} e^{-200\lambda}$ is a decreasing function of λ, i.e., as λ is increases, the cumulative Pr of observing r or less failures decreases. We are trying to obtain a one-sided upper CI for λ of the form $0 < \lambda \le \lambda_U$, which is the same as a lower one-sided CI for θ. So the question now is how large we have to make λ so that the cdf in Eq. (109) is reduced all the way down to 0.05? Therefore we have to solve the Pr equation

$$P[X(T) \le r] = \sum_{k=0}^{r} \frac{(n\lambda_U T)^k}{k!} e^{-n\lambda_u T} = 0.05 \qquad (110)$$

for λ_U; this value of λ_U gives the 95% upper bound for λ. Clearly, the two events $[X(T) \le r]$ and $[T_{r+1} > T]$, where T_{r+1} is the time to the r+1 failure, are equivalent. Recall from Chapter 2 that the time to the (r+1) failure has a gamma pdf with parameters (r+1) and $n\lambda$. Hence, from Eq. (110)

$$\sum_{k=0}^{r} \frac{(n\lambda_U T)^k}{k!} e^{-n\lambda_u T} = P(T_{r+1} > T) = \int_T^{\infty} \frac{(n\lambda_u)(n\lambda_u t)^r}{\Gamma(r+1)} e^{-n\lambda_u t}\, dt = 0.05 \qquad (111)$$

In equation (111), we now make the transformation $n\lambda_u t = w/2$, and $n\lambda_u dt = dw/2$. Then,

$$\int_{T}^{\infty}\frac{(n\lambda_u)(n\lambda_u t)^r}{\Gamma(r+1)}e^{-n\lambda_u t}\,dt = \int_{2n\lambda_u T}^{\infty}\frac{(w/2)^r}{\Gamma(r+1)}e^{-w/2}\frac{dw}{2}$$

$$= \int_{2n\lambda_u T}^{\infty}\frac{w^{[2(r+1)/2]-1}}{2^{r+1}\Gamma(r+1)}e^{-w/2}\,dw = 0.05 \qquad (112)$$

The last integrand in equation (112), g(w) = $\frac{w^{[2(r+1)/2]-1}}{2^{r+1}\Gamma(r+1)}e^{-w/2}$, is the pdf of a χ^2 rv with 2(r+1) *df*, and thus equation (112) can be written as $\int_{2n\lambda_u T}^{\infty} g(\chi^2_{2(r+1)})\,dw$ = 0.05. The only way this Pr can hold true is for the lower limit of this last integral to equal to $\chi^2_{0.05;2(r+1)}$, i.e., we must require that

$2n\lambda_U T = \chi^2_{0.05;2(r+1)} \rightarrow \lambda_U = \chi^2_{0.05;2(r+1)}/(2nT) \rightarrow 0 < \lambda \le \chi^2_{0.05;2(r+1)}/(2nT) \rightarrow$

$$1/0 > 1/\lambda \ge 1/[\chi^2_{0.05;2(r+1)}/(2nT)] \rightarrow \frac{2nT}{\chi^2_{0.05;2(r+1)}} \le \theta < \infty \qquad (113a)$$

Eq. (113a) now shows that if the experimenter observes 0 failures during the fixed interval [0, T = t*] then s\he can be 95% confident that the value of MTTF is at least as large as $\frac{2nt^*}{\chi^2_{0.05;2}} = \frac{2nT}{\chi^2_{0.05;2}}$.

<u>Example 21.</u> Four electrical components were places on life test for a duration of 500 hours at an accelerated load with the (linear) loading factor A_f = 15. Our objective is to obtain the 95% lower CI for θ, for both the accelerated load and normal operating conditions. The number of observed failures under accelerated conditions during the 500 hours was zero. Then from equation (113a), $\frac{2nT}{\chi^2_{0.05,2}} = \frac{2\times4\times500}{5.9915}$ = 667.6164 $\rightarrow$ 667.6164 $\le \theta_s < \infty \rightarrow$ 150213.690313 hours $\le \theta_o < \infty$, where θ_s represents stressed characteristic life and θ_o represents normal conditions operating characteristic life.

It can be shown in a similar fashion that a 2-sided 95% CI for θ for type I censoring is given by (t* = T)

$$\frac{2nT}{\chi^2_{0.025,2(r+1)}} \le \theta < \frac{2nT}{\chi^2_{0.975,2r}} \qquad \textbf{(113b)}$$

The upper limit for the above CI is obtained by considering the probability $P[X(T) \ge r] = \sum_{k=r}^{\infty} \frac{(n\lambda T)^k}{k!} e^{-n\lambda T} = \alpha/2$ (= 0.025 for a 95% CI). Clearly, this last exceeding Pr at r −1 is an increasing function of λ simply because as failure rate increases, the Pr of observing r or more failures must increase. This implies that the Pr = $\sum_{k=r}^{\infty} \frac{(n\lambda T)^k}{k!} e^{-n\lambda T}$ keeps decreasing as λ gets smaller and smaller. Therefore, we make λ as small as λ_L at which the $P[X(T) = N_f(t^*) \ge r]$ is reduced down to 0.025. The corresponding value of $\theta_U = 1/\lambda_L$.

Finally, if testing is for Type I censoring W/O replacement, then approximate CI for θ in (113b) is modified to

$$\frac{2r\hat{\theta}}{\chi^2_{0.025,2r+2}} \le \theta < \frac{2r\hat{\theta}}{\chi^2_{0.975,2r}} \qquad \textbf{(113c)}$$

Note that equation (113c) is only an approximation for the case of W/O replacement (U-type) because in this case we have a simple binomial process with n Bernoulli trials (because n units are on test) and we are looking for the occurrence of $r \ge 0$ failures. The exact Pr of any one unit not failing during the interval [0, T] is $e^{-\lambda T}$, and hence the binomial parameter $q = 1 - e^{-\lambda T}$, i.e., the Pr of any one unit failing is $q = 1 - e^{-\lambda T}$. The exact 95% confidence limits on q are obtained from the following Pr statements.

$$\sum_{k=0}^{r} {}_nC_k (q_u)^k (1-q_u)^{n-k} = 0.025, \text{ and } \quad \sum_{k=r}^{n} {}_nC_k (q_L)^k (1-q_L)^{n-k} = 0.025.$$

Because q is an increasing function of λ, then $q_u = 1 - e^{-\lambda_u T}$ and $q_L = 1 - e^{-\lambda_L T}$. Further, the function $\sum_{k=0}^{r} {}_nC_k (q)^k (1-q)^{n-k}$ is a decreasing function of q, while $\sum_{k=r}^{n} {}_nC_k (q)^k (1-q)^{n-k}$ is an increasing function of q. Rearranging these confidence limits for q = F(t) results in an exact 2-sided

CI for θ given below.

$$\frac{-\mathbf{T}}{\ln(1-q_u)} \le \theta \le \frac{-\mathbf{T}}{\ln(1-q_L)} \qquad \textbf{(113d)}$$

Using the normal approximation to the binomial, an approximate 95% upper confidence limit on Q(t) = 1–R(t) is given by q_u = $\hat{q}$ + 1.645×*se*($\hat{q}$) + (1/2n), where $\hat{q}$ = r/n, the *se*($\hat{q}$) = $\sqrt{\hat{p}\hat{q}/n}$, and $\hat{p}$ = 1 – (r/n). Eq. (113d) can be modified to yield the 95% lower confidence bound θ_L = $\frac{-T}{\ln(1-q_u)}$

where q_u is the solution to $\sum_{k=0}^{r} {}_nC_k\,(q_u)^k(1-q_u)^{n-k}$ = 0.050.

<u>Exercise 29.</u> Use equation (113a) to obtain the lower one-sided CIs for R(t) and L_{10}, when Type I censoring results in r = 0 failures during the testing interval [0, T].

<u>The Example 5.9 on pages 276-278 of Elsayed.</u> The data from a Type I censoring (I assume W/O replacement as the author does not specify) experiment with n = 10, T = t* =50,000 minutes and observed times TF are t_1 = 3000 minutes, t_2 = 7000, t_3 = 12000, t_4 = 18000, t_5 = 20000, t_6 = 30000 minutes. The Times BF are given in the last column of his Table 5.9 at the bottom his page 277, which are 3000, 4000, 5000, 6000, 2000, 10000. I first applied the Bartlett's statistic to test the underlying distribution for exponentiallity with the following results: For the TTF, I obtained B_6 = 2.3970 with P-value = 2×P(χ^2_5 ≤ 2.3970) = 0.4162, and for Times BF I obtained B_6 = 1.2973 with *P-value* = 2×P(χ^2_5 ≤ 1.2973) = 0.1296. From equation (105b), $\hat{\theta}$ = $\frac{\tau}{r}$, where τ = $\sum_{i=1}^{r} t_i$ + (n–r) T is the total testing times of all n units. Thus, $\hat{\theta}$ = $\frac{90000+(10-6)\times 50000}{6}$ = 48333.3333333 = MTTf = $\widehat{\mathrm{MTTF}}$. This point (unbiased) estimate of θ, 48333.3333333 minutes, does not match Elsayed's answer of $\mathrm{MTB\hat{F}}$ = $\hat{\mu}$ = 38333.333 minutes atop his page 278 because he used TBFs, although this plan did not involve with replacement. Assuming that my point estimate of the MTTF, $\hat{\theta}$ = 48333.3333333, is correct, the approximate 2-sided 95% CI for θ from equation (113c) is given by

$\frac{2\times 6\times 48333.33333}{\chi^2_{0.025,14}} \le \theta \le \frac{2\times 6\times 48333.33333}{\chi^2_{0.975,12}}$ $\rightarrow$ **22206.1011 $\le \theta \le$ 131704.78080. This CI does not match that of Elsayed's for two obvious reasons: (1) My confidence coefficient is 1 $-$ α = 0.95 (hence should provide a wider band than Elsayed's) while the CI 25,127 $\le \mu \le$ 116,750 in the middle of his page 278 is for 1 $-$ α = 0.90. (2) It seems that the *df* used to obtain 25,127 $\le \mu \le$ 116,750 is 10, which is not accurate because the correct *df* for χ^2 is 2r = 2$\times$6 =12. My point estimate of RE at t = 60,000 minutes is given by $\hat{R}(60000)$ = $e^{-60000/\hat{\theta}}$ = 0.28898534. Since the reliability function R(t) = $e^{-t/\theta}$ is an increasing function of t_c = θ, then $R_L(t)$ = e^{-t/θ_L} and $R_u(t)$ = e^{-t/θ_u} .**

<u>Example 5.8 on pages 273-275 of Elsayed.</u> This experiment involves a fatigue test to estimate the life of rods (made of steel) by subjecting 25 rods to an axial load of 9000 psi. Because r = n = 25, this was a complete sample. The 25 cycles to failure are 200, 280. 340, 460, 590, 720, 850, 990, 1200, 1420, 1950, 2460, 2590, 3520, 4560, 5570, 6590, 7600, 8630, 9650, 10660, 11670, 12680, 13685, 14690. I tested the TTF for exponentiallity and obtained B_{25} = 29.2550960142 with a *P-value* of $\hat{\alpha}$ = 2$\times$P(χ^2_{24} $\ge$ 29.2550960142) = 0.42135585, which implies a good fit. Therefore, T TF can be assumed to have an exponential distribution, i.e., the failure rate can be assumed to be a constant.

Next we obtain a point estimate of θ and λ. Using the equation $\hat{\theta}$ = [$\sum_{i=1}^{r} t_i$ + (n $-$ r)t_r] /r the point estimator of θ is given by $\hat{\theta}$ = $\frac{\tau}{r}$, where τ = $\sum_{i=1}^{r} t_i$ + (n$-$r)t_r and t_i is the failure instant (or the moment of failure) of the i^{th} unit with $t_1 < t_2 < ... < t_r$. For this case n = r, the MTTF is estimated from $\hat{\theta}$ = $\frac{\tau}{r}$ = $\sum_{i=1}^{25} t_i$ /25 = 123555 /25 = 4942.200 cycles $\rightarrow$ $\hat{\lambda}$ = 1 / $\hat{\theta}$ = 0.00020234; this is quite different from Elsayed's answer of 25/14690 = 0.00170184 failures per cycles given in the middle of his page 275. Since the testing was conducted at 10 cycles per minute, then $\hat{\theta}$ = 4942.200 cycles =

4942.200 cycles $\times \frac{1\ \text{minute}}{10\,\text{cycles}}$ = 494.22 minutes $\rightarrow \hat{R}(\text{t minutes}) = e^{-t/494.22} \rightarrow \hat{R}(\text{10 hours})$ =

$\hat{R}(\text{600 minutes}) = e^{-600/494.22}$ = 0.29699670463. This RE is substantially different from Elsayed's answer of 0.3676×10^{-4} because our estimates of MTTF are different. The 95% confidence limits for the characteristic life are $\theta_L = \frac{2r\hat{\theta}}{\chi^2_{0.025,50}} = \frac{50\times 494.22}{71.4202}$ = 345.9946 minutes, and similarly $\theta_U = \frac{2r\hat{\theta}}{\chi^2_{0.975,50}} = \frac{50\times 494.22}{32.3574}$ = 763.6893 minutes $\rightarrow$ R_L(600 minutes) =

$e^{-600/345.9946}$ = 0.1766 and R_u(600 minutes) = $e^{-600/763.6893}$ = 0.4558 $\rightarrow$ we are 95% confident that the true value of the reliability function at 10 hours lies within the interval 0.1766 $\leq$ R(at 600 minutes) $\leq$ 0.4558.

Testing for Abnormally too Short (or too long) Failures

Consider the data on page 270 for the Example 5.4 of Elsayed that represent cycles to failure of turbine blades. Our objective is to determine if the 1st failure, t_1 = 120 cycles, is abnormally too short (i.e., an outlier) relative to the other 19 times to failures t_2 = 1300, t_3 = 1680, t_4 = 1990, 2010, 2112, 2192, 2215, 2290, 2581, 2689, 2892, 2999, 3565, 3873, 4256, 4368, 4657, 4933, and t_{20} = 5832. This is a special Type II censoring where the required a-priori value of r = n = 20, and termination test time t_{20} is random. I will go thru the testing procedure step-by-step.

Step 1. State the null hypothesis that t_1 is not too short relative to other (r − 1) failures, i.e., H_0: t_1 is not too short. Then alternative hypothesis is that H_1 : t_1 is too short relative to other failures (i.e., an outlier relative to other failures).

Step 2. Develop a statistic for testing H_0 at a pre-assigned LOS (level of significance) α, say α = 0.05.

Step 3. Recall that SMD of the rv $2t_1/\theta$ follows a χ^2 with ν_1 = 2 *df* , and due to the additive nature of χ^2, the rv $[\sum_{i=2}^{20} 2t_i/\theta]$ has a χ^2 distribution with 2×19 = 38 *df*.

Step 4. Recall from STAT 3610 that the ratio of 2 scaled χ^2, $\frac{\chi^2_{\nu_1}/\nu_1}{\chi^2_{\nu_2}/\nu_2}$, has the Fisher's F distribution with *df* of the numerator equal to ν_1 and *df* of the denominator equal to ν_2. Applying this principal to our example, we deduce that the rv $\frac{(2t_1/\theta)/2}{[\sum_{i=2}^{20} 2t_i/\theta]/38}$ has an F distribution with 2 df for the numerator and 38 df for the denominator. As a result, the SMD of the statistic F_0 = $\frac{t_1}{(\sum_{i=2}^{20} t_i)/19} = \frac{\bar{x}_1}{\bar{x}_2}$ is $F_{2,38}$, where $\bar{x}_2$ is the mean of the last 19 times TF.

Step 5. Determine the critical (rejection) region of the test statistic F_0. Note that we should reject H_0 in favor of H_1 : "t_1 is too short relative to the other 19 failures" only if the value of $F_0 = \frac{t_1}{(\sum_{i=2}^{20} t_i)/19}$ is too small. This implies that the rejection region of the test statistic will locate in the left tail of the $F_{2,38}$ distribution. Therefore, the rejection region of the test consists of the values in the interval (0, $F_{0.95;2,38}$), where $F_{0.95;2,38}$ is the 5th percentile (or the 95th percentage point, or the 0.05 quantile) of the F distribution with ν_1 = 2 *df* and ν_2 = 38 *df*. Henceforth, the acronym AI will stand for the acceptance interval of a test, and thus, $\overline{\mathrm{AI}}$ will represent the rejection interval, i.e., for our example $\overline{\mathrm{AI}}$ = (0, 1/$F_{0.05;38,2}$) = (0, 1/19.4694218) = (0, 0.0513626).

Step 6. The test procedure consists of computing the test statistic F_0 and comparing it to the rejection region (or the AI). For our example, $F_0 = \frac{t_1}{(\sum_{i=2}^{20} t_i)/19} = \frac{120}{58434/19}$ = 0.03901838.

Since our test statistic F_0 = 0.03901838 falls well inside the critical region $\overline{AI}$ = (0, 0.0513626), then we have sufficient evidence at the 5% LOS to reject H_0 and conclude the 1st failure is indeed abnormally too short. Since our test is left-tailed, then the *P-value* of the test is given by $\hat{\alpha}$ = P($F_{2,38} \le F_0$) = P($F_{2,38} \le$ 0.03901838) = 0.03822848837 < α = 0.05 as expected.

Example 5.5 on pages 271-272 of Elsayed. **I will deviate from Elsayed's objective in this example, and test to see if the last 4 times to failure are abnormally too long relative to the 1st 16 failures at the LOS α = 0.05. The life data are given in his table 5.5 as 30000, 34500, 37450, 39950, 43760, 46585, 49970, 54430, 57600, 59990, 63200, 66600, 70000, 73120, 75690, 77990, 80330, 84450, 88960, and 99550 hours. The experiment has no specific censoring and if I had to classify it, I would have to say it is of Type II with r = n = 20. Since the $F_0 = \dfrac{\sum_{i=17}^{20} t_i/4}{(\sum_{i=1}^{16} t_i)/16} = \dfrac{\bar{x}_1}{\bar{x}_2}$ has an F distribution with ν_1 = 8 *df* and ν_2 = 32 *df* and the test is right-tailed, the rejection region of the test is $\overline{AI}$ = ($F_{0.05;8,32}$, ∞) = (2.24439614, ∞). The value of the test statistic is $F_0 = \dfrac{\bar{x}_1}{\bar{x}_2}$ =**

$\dfrac{353290/4}{880835/16} = \dfrac{88322.50}{55052.18750}$ **= 1.60434134. Since this test statistic value falls well inside the AI = [0, 2.24439614], then there is not sufficient evidence to conclude that the last 4 failures are too long relative to the first 16 failure times. The Pr level of the test is given by $\hat{\alpha}$ = P($F_{8,32} \ge F_0$) = P($F_{8,32} \ge$ 1.60434134) = 0.16265838 > α = 0.05 as expected.**

Because of so many different censoring situations, below I will summarize the MLEs and their corresponding CI estimates.

(1) U-Type I: Test-plan: [n, U, t* =T hrs]; t_i = time to the ith failure $\rightarrow$

$$L(\lambda) = {}_nP_r \prod_{i=1}^{r} [\lambda e^{-\lambda t_i} dt_i] \times (e^{-\lambda T})^{n-r} \quad \rightarrow \hat{\theta} = \frac{\sum_{i=1}^{r} t_i + (n-r)T}{r}$$ **(where r is a rv.)**

The exact CI limits are $\frac{-T}{\ln(1-q_u)} \le \theta \le \frac{-T}{\ln(1-q_L)}$, where $q = 1-e^{-\lambda t}$.

The approximate 95% CI in this case is given by

$$\frac{2r\hat{\theta}}{\chi^2_{0.025,2(r+1)}} \le \theta \le \frac{2r\hat{\theta}}{\chi^2_{0.975,2r}}$$

(2) R-Type I: [n, R, t* =T hrs] $\rightarrow t_0 = 0$, t_i = time between the i^{th} and $(i-1)^{th}$ failure

$L(\lambda) = \frac{(n\lambda T)^r}{r!}e^{-n\lambda T} \quad \rightarrow \hat{\theta} = n\times T/r$

$$\frac{2nT}{\chi^2_{0.025,2(r+1)}} \le \theta \le \frac{2nT}{\chi^2_{0.975,2r}}$$

Note that if r > 0, then $\theta_L = \frac{2nT}{\chi^2_{0.025,2(r+1)}}$.

(3) U-Type II: [n, U, r* failures] $\rightarrow$ $L(\lambda) = {}_nP_{r^*} (\lambda^{r^*}) [\prod_{i=1}^{r^*}(t_i - t_{i-1})]$

$\times e^{-\lambda \sum_{i=1}^{r^*}[(n-i+1)(t_i - t_{i-1})]} \quad \rightarrow \quad \hat{\theta} = [\sum_{i=1}^{r^*} t_i + (n-r)t_{r^*}]/r^*$, and $\frac{2r^*\hat{\theta}}{\chi^2_{0.025,2r^*}} \le \theta \le \frac{2r^*\hat{\theta}}{\chi^2_{0.975,2r^*}}$

(4) R-Type II: [n, R, r*] $\rightarrow$ $L(\lambda) = (n\lambda)^{r^*} \times [\prod_{i=1}^{r^*}(t_i - t_{i-1})] \times e^{-n\lambda t_{r^*}} \rightarrow \hat{\theta} = nt_{r^*}/r^*$

Use the Poisson distribution to obtain the 95% CI for θ by solving the following two equations for λ_L and λ_u:

$\Pr[N_f(T_{r^*}) \ge r^* | n\lambda_L t^*] \le 0.025$ and $\Pr[N_f(T_{r^*}) \le r^* - 1 | n\lambda_u t_{r^*}] \le 0.025$. These exact CI limits should be close to those obtained from the $\chi^2_{2r^*}$ approximation given by

$$\frac{2T_{r*}}{\chi^2_{0.025,2r*}} \le \theta \le \frac{2T_{r*}}{\chi^2_{0.975,2r*}}, \text{ where } T_{r*} = nt_{r*}.$$

(5) U-Type Random Censoring: [n, U, (r*, t*=T)] $\rightarrow$ $\hat{\theta} = \frac{\tau}{r}$, **where** $\tau =$

$$\begin{cases} (\sum_{i=1}^{r} t_i) + (n-r)T, & \text{if } N_f < r^* \\ (\sum_{i=1}^{r^*} t_i) + (n-r^*)t_{r*}, & \text{if } r = r^* \end{cases}$$

, where the LF is obtained using the exponential density. Boris Gnedenko (page 68) gives the final result for the LF W/O derivation. The derivation of the LF is as follows, where r is the observed number of failures.

$$L(\lambda) = ({}_nP_r)\, \lambda e^{-\lambda t_1} dt_1 \times \lambda e^{-\lambda t_2} dt_2 \times \ldots \times \lambda e^{-\lambda t_r} dt_d \times (e^{-\lambda t^*})^{n-r}$$

where ${}_nP_r = n!/(n-r)!$ **and** $t^* = \min(t_r, T)$**. Again there are two possibilities: (1)** $t_r > T$ **so that testing stops at the censored moment T and** $N_f < r$ **is a rv, and** $t^* = T$**. (2)** $t_r < T$ **so that testing terminates at the instant of** t_{r*}**, where r* is prefixed (or censored) but** t_{r*} **is a rv so that** $t^* = t_{r*}$**. Gnedenko's expression for the LF is** $L(\lambda) = C\lambda^r e^{-\lambda\tau}$**, where C is a constant free of** λ**. If** $N_f < r^*$**, the confidence limits can be obtained using the binomial pmf. However, when** $N_f = r^*$ **so that the testing stops at** $t_{r*} < T$**, then CI limits derivations are much more complicated and it will be best to approximate them as in the plan [n, U, (r, T)] summarized next.**

(6) R-Type Random Censoring: [n, R, (r*, t* =T)]. $L(\lambda) = (n\lambda)^r \prod_{i=1}^{r} (t_i - t_{i-1})\, e^{-n\lambda T} \rightarrow$

Two possibilities may occur: (1) $t_{r*} < T$ **so that** $N_f = r^*$**, and the observed number of failures are censored to the right, i.e., life testing ends at the moment** $t^* = t_{r*}$**. In this case the MLE becomes** $\hat{\theta} = \frac{nt_{r*}}{r^*}$**. Case (2)** $t_{r*} > T$ **so that the test ends at time** $t^* = T$ **and** $\hat{\theta} = \frac{nT}{r}$**, where** $r < r^*$**. Note that**

we have used $t_0 = 0$. Then, in summary for the case of with replacement $\hat{\theta} = \dfrac{\tau}{r}$, where $\tau = \begin{cases} nT, \text{ if } N_f < r^* \\ nt_{r^*}, \text{ if } N_f = r^* \end{cases}$, and r = the realized value of N_f. The CI limits for λ are obtained using the Poisson distribution as in the case of [n, R, r]. As an example, consider the plan [40, R, (6, 200)], where t_6 = 187 hours. Then $\hat{\lambda} = \dfrac{\hat{q}}{t_6} = \dfrac{6/40}{187}$ = 0.00080214 per hour. In order to obtain λ_u, we need to increase $n\lambda t_6$ to the point that the P[X(187)≥ 6 failures] = $1 - \sum_{k=0}^{5} \dfrac{(n\lambda t_6)^k}{k!} e^{-n\lambda t_6} \geq 0.975 \rightarrow$ $n\lambda_u t_6 = 11.669 \rightarrow \lambda_u = \dfrac{11.669}{40(187)} = 0.00156 \rightarrow \theta_L = 1/0.00156 = 641.0147$ hours. Similarly, to obtain λ_L, we need to decrease $n\lambda t_6$ to the point that the P[X(187)≤ 5 failures] = $\sum_{k=0}^{5} \dfrac{(n\lambda t_6)^k}{k!} e^{-n\lambda t_6} \geq 0.975 \rightarrow n\lambda_L t_6 = 2.201 \rightarrow \lambda_L = \dfrac{2.201}{40(187)} = 0.000294 \rightarrow \theta_u = 1/0.000294 =$ 3397.0835 hours. The ML point estimate of θ is given by $\hat{\theta} = \dfrac{nt_6}{6} = \dfrac{7480}{6}$ = 1246.6667 hours and the corresponding approximate CI limits are given by

$$\theta_L = \frac{2T_{r^*}}{\chi^2_{0.025,2r^*}} = \frac{2\tau}{\chi^2_{0.025,12}} = \frac{2(7480)}{23.3366642} = 641.0513473 \text{ hours, and } \theta_u = \frac{2T_{r^*}}{\chi^2_{0.975,2r^*}} = \frac{2(7480)}{4.403788507} =$$

3397.0750358 hours. Thus, the 95% CI on θ is $641.05135 \leq \theta \leq 3397.075036$ hours.

(7) Multiply Censored Data (see p. 307 of Ebeling)

$L(\theta) = \prod_{i=1}^{S_F} f(t_i;\theta)dx_i \prod_{j=1}^{S_C} R(t_j^+)$, where S_F represents the failure data set and S_C represents the censored data so that $S_F + S_C = n$. The Table of Example 12.8 on page 321 of Ebeling shows a multiply censored data set where $r = S_F = 6$ failures and $S_C = 4$ censored test units so that n = 10.

Thus for the case of CFR, $L(\theta)$ = $\prod_{i=1}^{S_F}\frac{1}{\theta}e^{-t_i/\theta}\prod_{j=1}^{S_C}e^{-t_j^+/\theta}$ = $(\frac{1}{\theta})^r e^{-\sum_{i=1}^{r}t_i/\theta}e^{-\sum_{j=1}^{n-r}t_j^+/\theta}$, where r = S_F and S_C = n−r. → $\mathbf{L}(\theta)$ = $\ln(\theta^{-r})\times -\theta^{-1}\sum_{i=1}^{r}t_i - \theta^{-1}\sum_{j=1}^{n-r}t_j^+$ = $-r\ln(\theta) - \theta^{-1}\sum_{i=1}^{r}t_i - \theta^{-1}\sum_{j=1}^{n-r}t_j^+$ →

$\frac{\partial \mathbf{L}(\theta)}{\partial\theta}$ = $-\frac{r}{\theta}$ + $\theta^{-2}\sum_{i=1}^{r}t_i$ + $\theta^{-2}\sum_{j=1}^{n-r}t_j^+$ $\xrightarrow{\text{Equate to 0}}$ $-r\hat{\theta}$ + $\sum_{i=1}^{r}t_i$ + $\sum_{j=1}^{n-r}t_j^+$ = 0 → $\hat{\theta}$ = $\frac{\sum_{i=1}^{r}t_i+\sum_{j=1}^{n-r}t_j^+}{r}$. For the data of Example 12.8 on page 321 of Ebeling, $\sum_{i=1}^{6}t_i$ = 15350 hours and $\sum_{j=1}^{4}t_j^+$ = 8150^+ hours → $\hat{\theta}$ = $\frac{15350+8150^+}{6}$ = 3916.6666667^+ hours. The next problem is how we convert this point estimator $\hat{\theta}$ = 3916.6666667 into the 95% CI for θ. I have not looked in the literature to find the correct CI limits for the multiply censored data. However, I will take a stab at obtaining one but I am not sure how accurate what follows is?

From statistical theory, it can be shown using the Cramer-Rao inequality that the variance of a MLE has the glb given by $-1/E[\frac{\partial^2\mathbf{L}(\theta)}{\partial\theta^2}] \le V(\hat{\theta}) < \infty$. Thus as a first rough estimate of the variance, I will obtain this variance glb. $-\frac{\partial^2\mathbf{L}(\theta)}{\partial\theta^2}$ = $-r\theta^{-2}+2\theta^{-3}\sum_{i=1}^{r}t_i+2\theta^{-3}\sum_{j=1}^{n-r}t_j^+$ =

→ $E[-\frac{\partial^2\mathbf{L}(\theta)}{\partial\theta^2}]$ = $-r\theta^{-2}+2\theta^{-3}E(\sum_{i=1}^{r}t_i+\sum_{j=1}^{n-r}t_j^+)$ = $-r\theta^{-2}+2\theta^{-3}(r\theta)$ = $r\theta^{-2}$ → glb[$V(\hat{\theta})$] =

$\frac{1}{r\theta^{-2}}$ = $\frac{\theta^2}{r}$ → glb[*se*($\hat{\theta}$)] = $\frac{\hat{\theta}}{\sqrt{r}}$ → *se*($\hat{\theta}$) ≅ $\frac{3916.6666667}{\sqrt{6}}$ = 1598.972471→ CV($\hat{\theta}$) = 0.4082483,

θ_L = $\hat{\theta}e^{-Z_{0.025}\times CV(\hat{\theta})}$ =1759.604391; θ_U = $\hat{\theta}e^{Z_{0.025}\times CV(\hat{\theta})}$ = 8718.0265391. This CI based on the normal approximation cannot be very accurate because we used the minimum variance that $\hat{\theta}$ can have,

although Minitab provides the same identical answers. The 2nd method of obtaining the CI is to use the χ^2 **approximation for type II censoring, i.e.,** $\frac{2r\hat{\theta}}{\chi^2_{0.025;2r}} \le \theta \le \frac{2r\hat{\theta}}{\chi^2_{0.975;2r}}$**, where** $\chi^2_{0.025;2r}$ **= 23.3366642 and** $\chi^2_{0.975;12}$ **= 4.40378851 gives** $\rightarrow \frac{12\times 3916.66667}{23.3366642} \le \theta \le \frac{47000}{4.40378851} \rightarrow$ **2013.99822** $\le \theta \le$ **10672.62878894. This last CI, which is more conservative than 1759.604391** $\le \theta \le$ **8718.0265391, is also an approximation because the sample is heavily censored and the CV being much less than one implies that the data may not be exponentially distributed.**