Let the rv $Y$ represent the strength of a component and the rv $X$ represent the stress applied on the component. Then the component reliability is given by

$$ R = P(Y > X) = \begin{cases} P(Y - X > 0), & \text{or } P(Y/X > 1) \\ P(Y/X > 1) & \text{or } P(\eta > 1) \end{cases} $$

(89)

where the random variable $\eta = Y/X$ is called the Factor of Safety (or Safety Factor). We assume that both $X$ and $Y$ are independent continuous rvs with pdfs $f(x)$ and $g(y)$, respectively. To obtain a general expression for $R$, we can use any one of the following two approaches depending on the expressions for $f(x)$ and $g(y)$.

1. Suppose that the stress acting on a device is in the neighborhood of $x_0$. Then the device incremental $R$ is given by

$$ P[(x_0 - dx \leq x \leq x_0 + dx) \cap (Y > x_0)] = P(x_0 - dx \leq x \leq x_0 + dx) \times P(Y > x_0 | x_0 - dx \leq x \leq x_0 + dx) = f(x_0)dx \int_{x_0}^{\infty} g(y)dy . $$

where $P(Y > x_0 | x_0 - dx \leq x \leq x_0 + dx) = P(Y > x_0)$ because $Y$ is independent of $X$. Since the value of stress can range from $-\infty$ (in reality from 0) to $\infty$, we can remove the condition on $x_0$ by integrating over all possible values of $x_0$, i.e., the unconditional $R$ is given by

$$ R = \int_{-\infty}^{\infty} [f(x_0)] \int_{x_0}^{\infty} g(y)dy dx = \int_{-\infty}^{\infty} [f(x)] \int_{x}^{\infty} g(y)dy dx = \int_{-\infty}^{\infty} G_y(x)f(x)dx $$

(90a)

where $G_y(x) = 1 - F_y(x)$ and $F_y(x)$ is the cdf of $Y$ at the stress value $x$ so that $G_y(x)$ is the exceeding pr of strength at the stress value $x$.

2. Suppose the strength of a device is around the value $y_0$. Then the device will be $R$ iff the stress acting on it is less than $y_0$. Letting $F_x(y)$ represent the cdf of stress at the strength $y$ and applying similar logic as above, we deduce that

$$ R = \int_{-\infty}^{\infty} F_x(y)g(y)dy $$

(90b)
Example 13. Suppose a component's strength, \( Y \sim N(100 \text{ MPa}, 100) \) and the stress acting on the component is exponentially distributed with mean \( \mu_x = 50 \text{ MPa} \). Since the exponential cdf is directly invertible while the Gaussian is not, then it is best to use Eq. (90b) because the cdf of the exponential is given by \( F_X(y) = 1 - e^{-y/50} \), while the normal cdf is an integral. Hence, from (90b)

\[
R = \int_{-\infty}^{\infty} (1 - e^{-y/50}) g(y) \, dy = \int_{-\infty}^{\infty} (1 - e^{-y/50}) \frac{1}{10\sqrt{2\pi}} \left( y - 100 \right)^2 e^{-y^2/100} \, dy = 1 - \int_{-\infty}^{\infty} e^{-200y^2 + 200y - 10000} \, dy = \int_{-\infty}^{\infty} e^{-200y^2 + 200y - 10000} \, dy = 1 - \int_{-\infty}^{\infty} e^{-y^2 - 196y + 104} \, dy = 1 - \int_{-\infty}^{\infty} e^{-y^2 - 196y + 104} \, dy = 1 - \int_{-\infty}^{\infty} e^{-y^2 - 196y + 104} \, dy = 1 - \int_{-\infty}^{\infty} e^{-y^2 - 196y + 104} \, dy = 1 - e^{-1.98} = 0.861931.
\]

The central factor of safety is given by \( n_c = \frac{\mu_y}{\mu_x} = E(Y)/E(X) = 2 \), while the mean factor of safety \( \bar{n} = E(Y/X) = E(\eta) \) cannot be directly computed; however it can be shown that \( \bar{n} \approx n_c(1 + CV_x^2) \), where \( n_c = \frac{\mu_y}{\mu_x} \) is called the central factor of safety and \( CV_x \) is the coefficient of variation of stress. For our example, \( CV_x = \frac{\sigma_x}{\mu_x} = 1 \) (or 100%) so that the expected factor of safety is approximately \( \bar{n} \approx 2(1 + 1^2) = 4 \).

Suppose now in the above problem the strength \( Y \) was deterministic at a constant value of 100 MPa but stress was still exponential. Then the component RE reduces to \( R = P(X < 100) = F_X(100) = 1 - e^{-100/50} = 0.864665 \). Conversely, if stress were deterministic at the constant value of \( x = 50 \text{ MPa} \), then \( R = P(Y > 50) = P(Z > \frac{50 - 100}{10}) = P(Z > -5) = 0.99999971334843 \).

Normal Stress and Strength

Suppose the strength of a device \( Y \sim N(\mu_y, \sigma_y^2) \) and is subject to the stress \( X \) which is also \( N(\mu_x, \sigma_x^2) \). Then from Eq. (89), we deduce that \( R = P(Y > X) = P(Y - X > 0) = P(W > 0) \), where \( W = \)
Y - X is a LC of normally and independently distributed rvs and hence itself is Gaussian with $\mu = E(W) = \mu_y - \mu_x$ and variance $V(W) = V(Y) + V(X) = \sigma_y^2 + \sigma_x^2$, depicted in Figure 18. Figure 18 shows that

$$R = P(W > 0) = P(Y - X > 0) = P(Z > \frac{0 - \mu}{\sigma_w}) = P(Z > \frac{-\mu}{\sqrt{\sigma_y^2 + \sigma_x^2}}) = P(Z > \frac{- (\mu_y - \mu_x)}{\sqrt{\sigma_y^2 + \sigma_x^2}}) =$$

$$P(Z > -Z_0) = \Phi(Z_0), \text{ where } Z_0 = \frac{\mu_y - \mu_x}{\sqrt{\sigma_y^2 + \sigma_x^2}}.$$

**Example 14.** The strength of a component is $N(\mu_y, 400 \text{ MPa}^2)$ and the stress acting on it is also Gaussian $N(\mu_x, 625 \text{ MPa}^2)$. Determine the value of central factor of safety, $n_c = \frac{\mu_y}{\mu_x}$, such that the component RE is at least 0.999 if $CV(Y) = 0.10$.

Solution: $CV(Y) = 0.10 \rightarrow 0.10 = \frac{\sigma_y}{\mu_y} = 20/\mu_y \rightarrow \mu_y = 200 \text{ MPa}$. \(\sigma_w = \sqrt{\sigma_y^2 + \sigma_x^2} = \sqrt{1025} = \)

**Figure 18. The SMD of $W = Y - X$**
32.10562; \Phi(Z_0) = 0.999 \rightarrow Z_{0.001} = \frac{\mu_y - \mu_x}{\sqrt{\sigma_y^2 + \sigma_x^2}} \rightarrow 3.090232 = \frac{200 - \mu_x}{32.01562} \rightarrow \mu_x = 101.0643 \rightarrow n_c = \frac{200/101.0643}{1.9789383} = 1.026.

Thus, on the average the mean strength has to be nearly at least twice the mean stress to attain a RE of at least 0.999.

In the normal case, the device RE can also be expressed in terms of CV_y and CV_x as follows:

\[ Z_0 = \frac{\mu_y - \mu_x}{\sqrt{\sigma_y^2 + \sigma_x^2}} = \frac{\mu_y / \mu_x - 1}{\sqrt{\sigma_y^2 / \mu_x^2 + \sigma_x^2 / \mu_x^2}} \rightarrow n_c - 1 \]

\[ = \frac{n_c - 1}{\sqrt{CV_y^2 (n_c^2) + CV_x^2}} \rightarrow R = P(Z > -Z_0) = \Phi(Z_0). \]

Further, it can be shown that given the values of \( \bar{n} = E(Y/X) = E(\eta) \) and \( \sigma_\eta \), the maximum UNRE value, regardless of the underlying distributions, is given by

\[ Q = \bar{R} \leq \frac{(\bar{n})^2 CV_\eta^2}{(\bar{n})^2 CV_\eta^2 + (\bar{n} - 1)^2} \]

where \( CV_\eta = \sigma_\eta / \bar{n} \). Conversely, if the desired value of \( R \) and the \( CV_\eta \) are given, then the minimum required mean factor of safety, \( \bar{n} \), is given by

\[ \bar{n} \geq \frac{1}{1 - CV_\eta \sqrt{R/Q}} \]

where \( Q = \bar{R} \) and \( CV_\eta = \frac{\sqrt{CV_y^2 + CV_x^2}}{1 + CV_x^2} \). For example, if \( r = 0.999 \) is required and it is known that \( CV(Y/X) = 0.03 \), then the minimum required factor of safety is given by

\[ \bar{n} \geq \frac{1}{1 - 0.03 \sqrt{0.999 / 0.001}} = 19.3081. \]

Note that this required \( \bar{n} \geq 19.3081 \) makes no assumptions about \( f(x) \) and \( g(y) \); for example, if both underlying distributions are normal as in Example 14, then \( \bar{n} \approx n_c(1 + CV_x^2) = 1.9789383 \times (1 + 0.2473673^2) = 2.100031. \)
On the other hand, if no assumptions are made about \( f(x) \) and \( g(y) \), and the design value of \( R \) is given along with \( CV_y \) and \( CV_x \), then the minimum required value of central safety factor is given by

\[
nc = \frac{\mu_y}{\mu_x} \geq \frac{1}{1 + CV_x^2 - \sqrt{R(CV_y^2 + CV_x^2)}}/R
\]

For example, suppose the design value of \( R = 0.999 \), and it is known that \( CV_y = 2.4\% \) while \( CV_x = 1.8\% \), then the minimum central safety factor is given by \( nc \geq \)

\[
\frac{1}{1 + (0.018)^2 - \sqrt{0.999(0.024^2 + 0.018^2)}}/0.001 = 19.1883.
\]

**Exponential Strength and Uniform Stress**

As an example suppose \( Y \sim \text{Exp}(\mu_y = 1000 \text{ MPa}) \) and stress acting on the device is \( U(200, 500 \text{ MPa}) \). Our objective is to compute component \( RE \).

\[
R = \int_{-\infty}^{\infty} g(y)dy \times F_X(y) = \int_{-\infty}^{\infty} 0.001e^{-0.001y} \frac{y-200}{300} dy = \int_{-\infty}^{0} 0 \times 0 dy + \int_{0}^{200} 0.001e^{-0.001y} \times 0 dy + \int_{200}^{500} 0.001e^{-0.001y} \times 0 dy + \int_{500}^{\infty} 0.001e^{-0.001y} (1) dy =
\]

\[
\frac{1}{300} \left[ -(y-200)e^{-0.001y} \right]_{200}^{500} + \int_{200}^{500} e^{-0.001y} dy + \left[ -e^{-0.001y} \right]_{500}^{\infty} = \frac{1}{300} [-300e^{-0.5} + (e^{-0.20} - e^{-0.50})/0.001] + 0.60653066 = 0.100803 + 0.60653066 = 0.707333645.
\]

We could also arrive at the same \( RE \) as follows:

\[
R = \int_{-\infty}^{\infty} G_y(x)f(x)dx = \int_{-\infty}^{\infty} e^{-0.001x} dx/300 = \int_{200}^{500} e^{-0.001x} dx/300 = \frac{1}{0.30}e^{0.001x} \bigg|_{200}^{500} = \frac{1}{0.30}(e^{-0.20} - e^{-0.50}) = 0.707333645, \text{ as before.}
\]
**Exercise 25.** Suppose stress acting on a component has an exponential distribution with mean \( \mu_x = 100 \) MPa and its strength is Weibull with minimum strength \( \delta = 600 \) MPa, characteristic strength \( \theta = 900 \) MPa and slope \( \beta = 2.00 \). Compute the device RE. ANS: \( R = R_1 + R_2 = P(X < 600) \times P(Y \geq 600) + P(X \geq 600) \times P(Y \geq x) = 0.9996405635 \).

**Lognormal Stress and Strength**

Suppose \( X \) and \( Y \) are independent and lognormally distributed with parameters \( x_{0.50} = e^{\mu_x} \), \( \sigma_x \), \( y_{0.50} = e^{\mu_y} \) and \( \sigma_y \). Then, \( R = P(Y/X > 1) = P(\ln Y - \ln X > 0) = P(W > 0) \), where \( W = \ln Y - \ln X \) is normally distributed with mean \( E(W) = E(\ln Y) - E(\ln X) = \mu_y - \mu_x = \ln(y_{0.50}) - \ln(x_{0.50}) = \ln(y_{0.50}/x_{0.50}) \) and \( V(W) = V(\ln Y - \ln X) = \sigma_y^2 + \sigma_x^2 \). Hence, \( R = P(W > 0) = P[Z > \frac{0 - \ln(y_{0.50}/x_{0.50})}{\sqrt{\sigma_y^2 + \sigma_x^2}}] \)

\[
= 1 - \Phi\left[-\frac{0 - \ln(y_{0.50}/x_{0.50})}{\sqrt{\sigma_y^2 + \sigma_x^2}}\right] = \Phi\left[\frac{\ln(y_{0.50}/x_{0.50})}{\sqrt{\sigma_y^2 + \sigma_x^2}}\right].
\]

As an example consider the Example 7.11 on page 134 of Ebeling where \( y_{0.50} = 8.1, \sigma_y = 0.07, x_{0.50} = 5.5, \) and \( \sigma_x = 0.15 \). Then, \( R = \Phi\left[\frac{\ln(8.1/5.5)}{\sqrt{0.0049 + 0.0225}}\right] = 0.990323303 \), which match’s that of Ebeling’s to 2 decimal accuracy.

**RE Estimation from Strength/Stress Data**

Recall from Eq. (90a) that \( R = \int_{-\infty}^{\infty} f(x)dxG_y(x) \); making the transformation \( f(x) = dF_x(x)/dx \) in this integral results in \( f(x)dx = dF_x(x) \) and \( R = \int_{0}^{1} G_y(x)dF_x \); thus, if we graph \( G_y(x) \) versus \( F_x(x) \), the area under \( G_y(x) \) and the abscissa \( F_x(x) \)-axis from zero to 1 will give the approximate reliability.

As an example suppose that stress and strength analysis performed on a randomly selected component resulted in the following simulated data.

**Stress (MPa):** 8 15 12 13 14 17 15

**Strength (MPa):** 14 10 17 18 20 19 23 22 25 19
We first tabulate the estimates of $F_x(x)$ and $G_y(x)$ as shown below.

<table>
<thead>
<tr>
<th>KPa</th>
<th>0</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_x(x)$</td>
<td>0</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{2}{7}$</td>
<td>$\frac{3}{7}$</td>
<td>$\frac{4}{7}$</td>
<td>$\frac{6}{7}$</td>
</tr>
<tr>
<td>$G_y(x)$</td>
<td>1</td>
<td>1</td>
<td>$\frac{9}{10}$</td>
<td>$\frac{9}{10}$</td>
<td>$\frac{9}{10}$</td>
<td>$\frac{8}{10}$</td>
<td>$\frac{8}{10}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>KPa</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>22</th>
<th>23</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_x(x)$</td>
<td>$\frac{7}{7}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$G_y(x)$</td>
<td>$\frac{7}{10}$</td>
<td>$\frac{6}{10}$</td>
<td>$\frac{4}{10}$</td>
<td>$\frac{3}{10}$</td>
<td>$\frac{2}{10}$</td>
<td>$\frac{1}{10}$</td>
<td>0</td>
</tr>
</tbody>
</table>

The above Figure shows that the approximate value of RE is given by

$$\hat{R} = \frac{1}{7} \left[ \frac{1 + 0.95}{2} + \frac{0.95 + 0.90}{2} + 0.90 + 3(0.8) + 0.8 / 2 \right] = 0.80.$$
Random Loads (Section 7.3.2 on pp. 157-158 of Ebeling)

Because random loads represent realistic life situations, we will illustrate the use of the Poisson distribution through the Exercise 7.8 on p. 163 of Ebeling (2nd Ed.). The stated parameters are maximum strength capacity of the dam at $y_U = 20$ feet, and the distribution of flood levels, when they do occur, is $\exp(\lambda_x = 0.25/\text{feet})$ so that the mean flood levels is $1/0.25 = 4$ feet. The Poisson occurrence rate, $r$, of floods is $0.50$ floods/year, i.e., $r = 0.50/\text{year}$. In order to compute the RE over 10 years, we first compute the $\mu = \mathbb{E}(N_{\text{floods}}) = 10 \times 0.50 = 5$ floods. Thus, the Pr that exactly $i$ floods occur in the next 10 years is given by $P_i = \frac{\mu^i}{i!} e^{-\mu}$, $i = 0, 1, 2, 3, 4, 5, 6, \ldots$. Secondly, each time a flood occurs the Pr that its level does not exceed 20 feet is given by $R = 1 - e^{-0.25 \times 20} = 0.993262053$, and hence the collapse Pr is given by $Q = 1 - R = 0.006737947$. Therefore, the system RE over $t$-years is given by

$$R_{\text{Sys}}(t) = P_0 + P_1 \times R + P_2 \times R^2 + P_3 \times R^3 + \ldots = e^{-\mu} + \mu e^{-\mu} \times R + \frac{\mu^2}{2!} e^{-\mu} \times R^2 + \frac{\mu^3}{3!} e^{-\mu} \times R^3 + \ldots$$

$$= e^{-\mu} \left[ 1 + \mu R + \frac{(\mu R)^2}{2!} + \frac{(\mu R)^3}{3!} + \ldots \right] = e^{-\mu} \sum_{i=0}^{\infty} \frac{(\mu R)^i}{i!} = e^{-\mu} e^{\mu R} = e^{-\mu(1-R)} = e^{-\mu Q},$$

where $Q = e^{-\lambda_x y_U}$. Hence, $R_{\text{Sys}}(10\text{ years}) = e^{-10 \times 0.006737947} = 0.9668714445$. For a 20-year-span, $R_{\text{Sys}}(20) = e^{-10 \times 0.006737947} = 0.9348404$.

Suppose now we wish to have a Sys RE of least 99% at 20 years instead of 0.9348404. How much larger the dam’s maximum capacity must be in order to attain this Sys RE, i.e., $R_{\text{Sys}}(20)$

$$= e^{-20RQ} \geq 0.99 \rightarrow -20RQ \geq \ln(0.99) \rightarrow -20 \times 0.50Q \geq -0.01005034 \rightarrow Q \leq 0.10005034 \rightarrow e^{-0.25 y_U} \leq 0.10005034 \rightarrow y_U = 27.611 \text{ feet.}$$