Reference : Chapter 15 of Ebeling (The simple case of r = n) Maghsoodloo

Statistical Inference (SI) When the Underlying Distribution is Weibull

Recall that by SI we mean estimation and test of hypothesis. We will cover only the case of minimum life $\delta = 0$, and therefore, the underlying TTF pdf is given by:

$$f(t) = \frac{\beta}{\theta^{\beta}} t^{\beta-1} e^{-(t/\theta)^{\beta}} = \frac{\beta}{\theta}(t/\theta)^{\beta-1} e^{-(t/\theta)^{\beta}} = (\beta\lambda)\lambda^{\beta-1} e^{-(\lambda t)^{\beta}}$$

The experimenter should bear in mind that if data shows that the minimum life is not zero (for example if the value of the 1st-order statistic were equal to 23000 cycles to failure, then obviously the minimum life is not zero), then such data can easily be converted to $t_0 = \delta = 0$ by subtracting, say $0.90 \times t_{(1)}$ = 20700 cycles (you may even try $0.85 \times t_{(1)}$), from all times to failure and treating the resulting data with minimum life equal to zero in order to use the above 2-parameter Weibull pdf. For more accurate estimate of minimum life t_0 see pp. 412 and 413 of Ebeling, where for the exponential (β =1) $\hat{t}_0$ = Max $[2t_{(1)} - t_{(2)}, 0]$, this is basically the Eq. 15.25 of Ebeling on p. 413. For $\beta > 1$ (IFR), use Eq. (15.23) of Ebeling on his page 412, where $t_n = t_{(n)}$ is the nth order-statistic and t_j for the Weibull is jth-order statistic, where $j = \lceil np \rceil$ and np is always rounded up to the next higher positive integer. The value of p is given in Ebeling's Eq. (15.24) on his p. 413 as $p = 0.8829n^{-0.3437}$. In almost all cases, the results of these more accurate formulas come very close to $0.85 \times t(1) \le \hat{t}_0 \le 0.91 \times t_{(1)}$. Further, Ebeling (p. 421) states that the MLE of $t_0 = \delta$ =Minimum-life is equal to $t_{(1)}$.

Failure Data Without Censoring (Complete Samples, i.e., r = n)

In this case we have n units on test and our objective is to test all of them to failure in order to obtain the ordered failure times $t_1, t_2, ..., t_n$ and then use these observed failure instances to obtain the MLEs of θ and β. Note that this testing situation is applicable only to components that have moderately large increasing hazard rate (function); otherwise, the testing duration will become cost prohibitive. I will provide the MLF and its log and the resulting two partial derivatives wrt θ and β, but I am doing it only for the record. You will be responsible only for knowing how to apply the results, which I will outline in a step-by-step

procedure.

ML Estimation Procedure

Recall that the Pr element of the ith failure time, t$_i$, is given by $\frac{\beta}{\theta^\beta} t_i^{\beta-1} e^{-(t_i/\theta)^\beta}$ dt$_i$ and hence the LF (likelihood function) is given by

$$L(\theta, \beta) = \prod_{i=1}^{n} \frac{\beta}{\theta^\beta} t_i^{\beta-1} e^{-(t_i/\theta)^\beta} = \left(\frac{\beta}{\theta^\beta}\right)^n \times \left[\prod_{i=1}^{n} t_i^{\beta-1}\right] \times e^{-\sum_{i=1}^{n}(t_i/\theta)^\beta} \qquad \textbf{(123a)}$$

Taking the natural log of (123a) leads to

$$\mathbf{L}(\theta, \beta) = n[\ln(\beta) - \beta \ln(\theta)] + (\beta - 1)\sum_{i=1}^{n} \ln(t_i) - \sum_{i=1}^{n} (t_i/\theta)^\beta \qquad \textbf{(123b)}$$

The partial derivative of the Log-likelihood, $\mathbf{L}(\theta, \beta)$, wrt θ is given by

$$\partial \mathbf{L}(\theta, \beta)/\partial\theta = -n\beta/\theta - \frac{\partial}{\partial\theta}\left[\theta^{-\beta}\sum_{i=1}^{n}(t_i)^\beta\right] = -n\beta/\theta + \beta\,\theta^{-\beta-1}\sum_{i=1}^{n}(t_i)^\beta \xrightarrow{\text{Set to}} 0 \qquad \textbf{(124)}$$

The solution to equation (124) is the MLE of θ which is given below.

$$\hat{\theta} = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{\hat{\beta}}\right]^{1/\hat{\beta}} \qquad \textbf{(125a)}$$

Equation (125a) is the result that we need to obtain the point MLE of the characteristic life t$_c$ = θ, but the difficulty lies in the fact that unless we 1st obtain the point ML estimate of the slope β, then we will not be able to compute $\hat{\theta}$ from Eq. (125a). Next we will try to obtain the MLE of β by partially differentiating $\mathbf{L}(\theta, \beta)$ wrt β.

$$\xi(\beta) = \partial \mathbf{L}(\theta, \beta)/\partial\beta = \frac{n}{\beta} - n\ln(\theta) + \sum_{i=1}^{n}\ln(t_i) - \left[\sum_{i=1}^{n}\frac{\partial}{\partial\beta}(t_i/\theta)^\beta\right]$$ **; bearing in mind that**

$\frac{\partial}{\partial\beta}(t_i/\theta)^\beta = \frac{\partial}{\partial\beta} e^{\ln(t_i/\theta)^\beta} = \frac{\partial}{\partial\beta} e^{\beta\ln(t_i/\theta)}$ **then it follows that**

$$\partial \mathbf{L}(\theta, \beta)/\partial\beta = \frac{\mathbf{n}}{\beta} - \mathrm{n}\ \ln(\theta) + \sum_{i=1}^{n} \ln(t_i) - \sum_{i=1}^{n}\left[(t_i/\theta)^{\beta} \times \ln(t_i/\theta)\right] \xrightarrow{\text{Set to}} 0 \qquad \textbf{(125b)}$$

Equations (125a&b) will have to be solved simultaneously in order to obtain the ML estimates of θ and β. Unfortunately, no closed-form solution will ever exist for $\hat{\theta}$ and $\hat{\beta}$. Therefore, the solutions have to be obtained thru trial/and error that will make both partial derivatives $\partial \mathbf{L}(\theta, \beta)/\partial\theta$ in (124) and $\partial \mathbf{L}(\theta, \beta)/\partial\beta$ in (125b) almost equal to zero. I will now go thru the Example 5.17 on pages 291-295 of Elsayed in a step-by step procedure. I do not know if I will obtain the same identical solution because Elsayed's Weibull characteristic life is defined differently from mine.

Step 1: For the Example 5.17 on pages 291-295 of E. A. Elsayed, n = 10 diodes were tested to failure under accelerated conditions. The instances to failure in his example are t_i = 31000, 36000, 40000, 44000, 50000, 51000, 51500, 54000, 57000, and 63000 minutes (i = 1, 2, …, 10) . It seems Elsayed states at the bottom of his page 291 that TTFs are measured in minutes but all subsequent units are in terms of hours. If TTF were measured in hours, then it actually took 63000/(24×365) = 7.191781 years to collect the failure data in his Example 5.17, which does not seem realistic. It is possible that the accelerated life testing procedure that was used had a load factor of A_f = 60; I am just guessing! Since roughly 44 days of testing is more reasonable, then I am going to have to assume that the TTFs were actually measured in minutes. Therefore, I have surmised that the units for this problem are in terms of minutes.

Matlab computations yield $\bar{x} = \frac{1}{10}\sum_{i=1}^{n=10} t_i$ = 47750 minutes, $S = \sqrt{\frac{1}{9}\sum_{i=1}^{10}(t_i - \bar{x})^2}$ = $\sqrt{\frac{1}{9}\mathrm{CSS}}$ = $\sqrt{\frac{1}{9}(879625000)}$ = 9886.1575504 minutes. (Notice that Elsayed lists hours for the units of $\bar{x}$ and s.) Thus, the sample coefficient of variation of TTF is given by $cv = S/\bar{x}$ = 20.704% < < 1, which clearly implies that the underlying TTF distribution is not exponential.

<u>Step 2.</u> Since the hazard function of a Weibull is decreasing only during the early-life cycle of products (i.e., only during the RE growth cycle or burn-in period the slope lies in the interval $0 < \beta < 1$), then for our example $\beta > 1$. Equation (5.32) on page 292 of E. A. Elsayed

gives a rough approximation of the Weibull slope in terms of the corresponding sample cv.

$$\hat{\beta} = \frac{1.05}{\mathrm{cv}} \qquad \text{(Equation 5.32 of Elsayed)}$$

The above equation gives us a starting point for ML estimation, i.e., we estimate our initial slope as $\hat{\beta}$ = 1.05/0.20704 = 5.0715. I have used the values in my Table 1 in Chapter 2 to obtain a better regression model that relates the Weibull slope to its cv. The Minitab output is given below.

```
Regression Analysis: Beta versus x = 1/CV

The regression equation is

Beta = - 0.270 + 1.20 x, where x = 1/CV, 1 ≤ β ≤ 5.

Predictor      Coef       SE Coef       T          P

Constant    -0.27015    0.02082    -12.97     0.000

x             1.19841      0.00715     167.71    0.000

S = 0.03010    R-Sq = 99.9%    R-Sq(adj) = 99.9%

Analysis of Variance

Source              DF      SS          MS            F          P

Regression         1     25.486     25.486    28125.57    0.000

Residual Error   15      0.014      0.001

Total                16     25.500

Unusual Observations

Obs       x     Beta        Fit     SE Fit     Residual      St Resid

 1      1.00    1.00000    0.92826    0.01435    0.07174       2.71R

R denotes an observation with a large standardized residual

Results for: WBeta.MTW

Regression Analysis: Beta versus x = 1/CV

The regression equation is

Beta = - 0.591 + 1.28 x,   5 ≤ β ≤ 20

Predictor      Coef         SE Coef         T          P
```

```
Constant   -0.590770   0.008170    -72.31    0.000

x              1.27540     0.00075    1698.72   0.000

S = 0.01085    R-Sq = 100.0%    R-Sq(adj) = 100.0%

Analysis of Variance

Source             DF        SS        MS       F       P

Regression          1    340.00     340.00 2.886E+06    0.000

Residual Error   14       0.00       0.00
Total              15    340.00

Unusual Observations

Obs        x      Beta        Fit      SE Fit   Residual    St Resid

  1      4.4    5.0000     4.9762    0.0052     0.0238       2.50R

 16     16.1   20.0000    19.9801    0.0052     0.0199       2.08R

R denotes an observation with a large standardized residual
```

The estimate of β from the above 2 regression models are, respectively, $\hat{\beta}$ = –0.27015 + 1.19841 /0.20704 = 5.5182, or $\hat{\beta}$ = –0.59077 +1.2754 /0.20704 = 5.5694. These are fairly close to the value of 4.83 obtained from Cohen (1965), Technometrics 7, pp. 579-588, given in equation (5.32) of Elsayed. So, I will settle for a rough estimate of $\hat{\beta}$ = 5.00 and will use this to obtain a rough value of $\hat{\theta}$ from my equation (125a), which is $\hat{\theta}$ = 50938.5117215225 minutes. These two estimates ($\hat{\beta}$ = 5.00) make the RHS of equation (125b) equal to 0.504003453. Since $\partial \mathbf{L}(\theta, \beta)/\partial\beta$ in equation (125b) is a decreasing function of β, then we must increase the value of β in order to reduce the RHS of (125b) close to zero. Through trial & error, I found that at $\hat{\beta}$ = 5.996972780 and $\hat{\theta}$= 51559.62598478 minutes, the RHS of (125b) is close to 5×10^{-15}. As a matter of fact, the estimate $\hat{\beta}$ = 5.996972780 is very close to that of Elsayed's in the middle of his page 294. Therefore, the MLE of β and θ to 6 decimals are $\hat{\beta}$ = 5.996973 and $\hat{\theta}$= 51559.625985, which lead to $\hat{R}(t) = e^{-(t/51559.625985)^{5.996973}}$. As a result, a point ML estimate of the RE function at 40000 minutes is given by $\hat{R}(40000\ \text{min})$ = 0.80397255, which is consistent (to 3 decimals) with Elsayed's answer on his page 295.

We have completed point ML estimation for a $W(0, \theta, \beta)$ for a complete sample, and now it is time to perform further SI by obtaining a 95% CI for both parameters θ and β. Again since the Weibull t_c (characteristic life) is definitely a LTB type parameter (and thus no concern on the high side), my personal engineering preference would be to obtain the lower one-sided CI for θ, ($\theta_L \le \theta < \infty$), and a 2-sided CI for the parameter β: $\beta_L \le \beta \le \beta_U$. The reader must be cognizant that throughout these notes, we are setting the confidence coefficient, for the sake of convenience, at either $1 - \alpha = 0.90$, or 0.95; Minitab's default confidence level is 0.95. Further, the statistical entities θ_L, β_L, and β_U are indeed (correlated) random variables before experimentation for obtaining failure data. However, after experimental data are gathered and the needed sample statistics are computed, then no longer θ_L, β_L, and β_U are random variables but they are simply real-valued numbers and hence the deterministic intervals no longer have 95% Pr of containing the true values of θ and β. Once these two CIs are obtained, then they may be used to conduct tests of hypotheses about the parameters θ and β. For example, if the data of the example 5.17 of Elsayed provides the 95% CI : $3.5 \le \beta \le 8.5$, then we cannot reject the null hypothesis $H_0 : \beta = 6.5$ at the LOS $\alpha = 0.05$ because the hypothesized value of $\beta_0 \equiv 6.5$ lies inside the 95% CI : $3.5 \le \beta \le 8.5$.

Confidence Interval for the Weibull Slope β

It is well known that an exact CI for a parameter can be obtained only if the frequency function (or SMD) of the corresponding point estimator is exactly known. Equation (125b) clearly shows that no closed-form solution exists for the point estimator $\hat{\beta}$. I am almost certain that the exact SMD (or pdf) of $\hat{\beta}$ for small to moderate values of r (r < 50) is intractable, but from statistical theory the frequency function of all ML estimators (MLEs) asymptotically (in terms of n) approach normality with asymptotic mean equal to the corresponding parameter and the asymptotic variance equal to $1/I(\beta)$, where the sample information $I(\beta)$ is described below. Therefore, as Elsayed mentions atop his page 291, the asymptotic 95% CI for the parameter β is given by $\hat{\beta} \pm Z_{0.025} \times se(\hat{\beta})$, where $Z_{0.025} = 1.96$. The

exact *se*($\hat{\beta}$) cannot be computed, and as a result all of the following developments are simply somewhat conservative approximations, and only for moderately large values of r failures the approximate CIs are expected to be fairly accurate. Thus, our next objective is to obtain a rough estimate of the $\sqrt{V(\hat{\beta})}$ = *se*($\hat{\beta}$). This will require understanding the concept of statistical efficiency.

Efficiency

We first show that $E[\partial \mathbf{L}(x;\theta)/\partial\theta]$ is always zero for any parameter θ, where $x = [x_1\ x_2\ \ldots\ x_n]'$ represent a complete sample of n failure times. $E[\partial \mathbf{L}(x;\theta)/\partial\theta] =$

$$\int [\partial \mathbf{L}(x;\theta)/\partial\theta] f(x;\theta)\,dx = \int \frac{1}{f(x;\theta)}[\frac{\partial f(x;\theta)}{\partial\theta}] f(x;\theta)\,dx = \frac{d}{d\theta}\int f(x;\theta)\,dx = \frac{d}{d\theta}(1) = 0 \rightarrow$$

$V[\partial \mathbf{L}(x;\theta)/\partial\theta] = E\{[\partial \mathbf{L}(x;\theta)/\partial\theta]^2\}$, where we are using the fact that the joint pdf of $x = [x_1\ x_2\ \ldots\ x_n]'$ is given by $f(x;\theta) = \prod_{i=1}^{n} f(x_i;\theta)$. Let $\hat{\theta}$ be a MLE of θ (or any estimator of θ); then, the

$$COV[\hat{\theta}, \partial \mathbf{L}(x;\theta)/\partial\theta] = E[\hat{\theta} \times \partial \mathbf{L}(x;\theta)/\partial\theta] = \int_{R_x} \hat{\theta}[\partial \mathbf{L}(x;\theta)/\partial\theta] f(x;\theta)]\,dx = \int \hat{\theta}\,[\frac{d}{d\theta} f(x;\theta)]\,dx$$

$= \frac{d}{d\theta}E(\hat{\theta}) = \frac{d}{d\theta}[\theta + B(\hat{\theta})] = 1 + B'(\hat{\theta})$, where $B(\hat{\theta})$ is the amount of bias in the estimator $\hat{\theta}$ and $B'(\hat{\theta}) = \frac{d}{d\theta}B(\hat{\theta})$. For convenience of notation, let $\xi = \partial \mathbf{L}(x;\theta)/\partial\theta$ for which we just have shown that $E(\xi) = 0$ and $COV(\hat{\theta}, \xi) = E(\hat{\theta} \times \xi) = 1 + B'(\hat{\theta})$. It is well known that $0 \le \rho^2 \le 1$, where ρ is the correlation coefficient between any two random variables. Thus, $0 \le \rho^2(\hat{\theta}, \xi) \le 1 \rightarrow 0 \le \frac{COV^2(\hat{\theta},\xi)}{V(\hat{\theta})V(\xi)} \le 1 \rightarrow$

$$\frac{[COV(\hat{\theta},\xi)]^2}{V(\xi)} \le V(\hat{\theta}) < \infty \qquad (126a)$$

The quantity $V(\xi) = V[\partial \mathbf{L}(x;\theta)/\partial\theta] = E(\xi^2) = I(\theta)$ in the denominator of (126a) is called the information in the sample (about θ); since, $COV(\hat{\theta}, \xi) = E(\hat{\theta} \times \xi) = 1 + B'(\hat{\theta})$, then (126a) reduces to

$$\frac{[1+B'(\hat{\theta})]^2}{I(\theta)} \le V(\hat{\theta}) < \infty \qquad (126b)$$

The information inequality (126b) is called the Cramer-Rao inequality in the field of statistics. It provides the greatest lower bound (glb) for the variance of any estimator in the universe. Simply put, there exists no estimator in the universe whose variance is less than the Cramer-Rao's glb $\frac{[1+B'(\hat{\theta})]^2}{I(\theta)}$. Further, only those estimators whose variance is equal to the Cramer-Rao's glb are called efficient. Before discussing an Example, we need to show that V(ξ) = V[$\partial\mathbf{L}$(x; θ)/∂θ] = E(ξ^2) = I(θ) is also equal to –E(∂ξ/∂θ) as illustrated below.

$\frac{\partial\xi}{\partial\theta} = \frac{\partial}{\partial\theta}[\frac{\partial}{\partial\theta}\ln(f(x;\theta))] = \frac{\partial}{\partial\theta}[\frac{f'(x;\theta)}{f(x;\theta)}]$, where $f'(x;\theta) = \frac{\partial}{\partial\theta}[f(x;\theta)]$. Thus,

$$\frac{\partial\xi}{\partial\theta} = \frac{ff''-(f')^2}{f^2} = \frac{f''}{f} - \xi^2 \qquad (127)$$

Applying the expected-value operator to both sides of (127), and using the fact that

$E(\frac{f''}{f}) = \int(\frac{f''}{f})f\,dx = \frac{d^2}{d\theta^2}\int f(x;\theta)\,dx = \frac{d^2}{d\theta^2}(1) = 0$, we obtain V(ξ) = –E(∂ξ/∂θ).

This last result also shows that

$$I(\theta) = V(\xi) = -E[\frac{\partial^2}{\partial\theta^2}\ln(f(x;\theta))] = -E[\frac{\partial^2}{\partial\theta^2}\mathbf{L}(x;\theta)] = I_{11} \qquad (128)$$

In general, if $\hat{\theta}_i$ (i = 1, 2, ..., m) are the MLEs of m parameters with the log likelihood function $\mathbf{L}(x;\theta_1,\theta_2,\ldots,\theta_m)$, then the (i, j)th element of the information matrix, I, is given by $I_{ij} = -E[\partial^2\mathbf{L}(x;\theta_1,\theta_2,\ldots,\theta_m)/\partial\theta_i\partial\theta_j]$. It can then be shown that the asymptotic covariance matrix of the vector $\hat{\theta} = [\hat{\theta}_1 \quad \hat{\theta}_2 \ldots \quad \hat{\theta}_m]'$ is given by the inverse of the information matrix I, i.e., COV(

$\hat{\theta}$) = COV[$\hat{\theta}_1$ $\hat{\theta}_2$ ... $\hat{\theta}_m$]′ = $\mathbf{I}^{-1}$.

I will start the procedure by 1st obtaining the exact Cramer-Rao's glb for the V($\hat{\beta}$); recall from inequality (126b) of my notes that $\frac{[1+B'(\hat{\beta})]^2}{I(\beta)} \le$ V($\hat{\beta}$) < ∞. From statistical theory it is well known that MLEs are asymptotically unbiased, and hence for large n, $\frac{1}{I(\beta)} \le$ V($\hat{\beta}$) < ∞, where I(β) = –E[$\partial^2\mathbf{L}(\theta, \beta)/\partial\beta^2$]. I partially differentiated equation (125b) to obtain $\partial^2\mathbf{L}(\theta, \beta)/\partial\beta^2$, which is provided below.

$$\partial^2\mathbf{L}(\theta, \beta)/\partial\beta^2 = -n/\beta^2 - \sum_{i=1}^{n}\left[(t_i/\theta)^{\beta} \times [\ln(t_i/\theta)]^2\right] \rightarrow$$

$$-\partial^2\mathbf{L}(\theta, \beta)/\partial\beta^2 = n/\beta^2 + \sum_{i=1}^{n}\left[(t_i/\theta)^{\beta} \times [\ln(t_i/\theta)]^2\right] \rightarrow$$

$$I(\beta) = -E[\partial^2\mathbf{L}(\theta, \beta)/\partial\beta^2] = n/\beta^2 + E\sum_{i=1}^{n}\left[(t_i/\theta)^{\beta} \times [\ln(t_i/\theta)]^2\right] = I_{22} \qquad (129)$$

Eq. (129) clearly shows that the information on β contained in the sample is directly proportional to the number of failures n because as n increases, the amount of information about β increases. Further, in involves a very complicated mathematical expectation on the RHS of equation (129), given by $\theta^{\beta}\sum_{i=1}^{n} E\left[(t_i)^{\beta} \times [\ln(t_i/\theta)]^2\right]$. Applying the Expected-Value operator to the term inside this last summation in order to obtain an exact general result is impossible, at least to the capability of this author. Even simulation may not help because both parameters β and θ are unknown and therefore the simulation procedure has to start with specified values of these parameters of the Weibull and hence the simulation result for $\theta^{\beta}\sum_{i=1}^{n} E\left[(t_i)^{\beta} \times [\ln(t_i/\theta)]^2\right]$ would depend on the inputted values of β and θ. Having stated the simulation problems, there is hope because once failure data are obtained, then the MLEs of β and θ can be obtained, and hence the simulation can get started with these MLEs. This last computer simulation procedure in order to estimate the *se* (of any estimate) is called Bootstrapping in the field of Statistics.

Bootstrap Estimation Procedure

Step 1: Recall that the MLEs of β and θ (to 3 decimals) for the Example 5.17 are $\hat{\beta}$ = 5.997 and $\hat{\theta}$ = 51559.626 minutes. First assume that the underlying distribution of the failure data is W(0, θ = 51559.626, β = 5.997), i.e., assume that the cdf is actually $F(t) = 1 - e^{-(t/51559.626)^{5.997}}$.

Recall from statistical theory that all continuous cdfs in the universe are uniformly distributed over the interval [0, 1], i.e., F(t) ~ U(0, 1). This is because $\int_0^1 dF$ = 1 for all continuous F(t).

Step 2: Use the cdf from step 1 with the aid of a computer to generate a random bootstrap sample of n, denoted as t_{11}, t_{12}, ... t_{1n}. Use these sample results to obtain the point MLE estimate of β and θ, denoted by $\hat{\beta}_1^*$ and $\hat{\theta}_1^*$, as outlined above.

Step 3: Repeat step 2 roughly B = 500 times (i.e., a simulation run size of at least 500, where B = 500 is my recommendation; in practice, I would recommend that the simulation starts with B= 500, and then 600, 700, etc until two successive runs provide practically the same results). By now we will have roughly 500 bootstrap estimates ($\hat{\beta}_1^*$, $\hat{\beta}_2^*$, ..., $\hat{\beta}_{500}^*$) and ($\hat{\theta}_1^*$, $\hat{\theta}_2^*$, ..., $\hat{\theta}_{500}^*$). Next compute the bootstrap averages $\overline{\beta^*} = \frac{1}{B}\sum_{i=1}^{B}\hat{\beta}_i^*$ and $\overline{\theta^*}$.

Step 4: Then Bootstrap estimate of the *se*($\hat{\beta}$) is given by $S_{\hat{\beta}} = \sqrt{\frac{1}{B}\sum_{i=1}^{B}[\hat{\beta}_i^* - \overline{\beta^*}]^2}$ and similarly for the *se*($\hat{\theta}$), i.e., $S_{\hat{\theta}} = \sqrt{\frac{1}{B}\sum_{i=1}^{B}[\hat{\theta}_i^* - \overline{\theta^*}]^2}$. You may wish to replace B in the divisors by B – 1, if that is your preference. Note that the above Bootstrap procedure can be applied to all estimates whether the underlying distribution family is known or not. When the underlying distribution family is unknown, it will become more difficult. In the above example, the underlying family is W(0, θ, β).

Now that I have outlined how to obtain reliable point estimates of the standard errors by a computer simulation, we have to get back to the problem at hand where we have to obtain rough estimate of *se*($\hat{\beta}$) W/O the use of a computer. Eq. (126b & 129) show that the glb for the V($\hat{\beta}$) is given by

$$\frac{[1+B'(\hat{\beta})]^2}{\frac{n}{\beta^2} + \sum_{i=1}^{n} E\left[(t_i/\theta)^{\beta} \times [\ln(t_i/\theta)]^2\right]} \le V(\hat{\beta}) < \infty \qquad (130)$$

If we assume that n is sufficiently large in order to ignore the amount of bias in the MLE $\hat{\beta}$, then the numerator of (130) reduces to almost 1. To get the procedure for estimation of V($\hat{\beta}$) started from inequality (130), we first assume that V($\hat{\beta}$) $\doteqdot \hat{\beta}^2/n$, where the two dots on the equality imply very rough approximation, and n = r when the failure data are uncensored. Note that V($\hat{\beta}$) $\doteqdot \hat{\beta}^2/r$ is consistent with equation (5.40) atop page 297 of Elsayed with the element in the 2nd row and 2nd column of information matrix inverse, I^{-1}, except for the multiplier c_{22}, which for the case of r = n is equal to 0.607927 from Table 5.11 on page 297 of Elsayed, reproduced atop the next page. Therefore, for the Example 5.17 data on pp. 291-292 of Elsayed, we have a better approximation for V($\hat{\beta}$) $\cong c_{22} \times \hat{\beta}^2/n$ =0.6079×$(5.997)^2$/10 = 2.18623→ *se*($\hat{\beta}$) ≅ 1.4786. Therefore, our half-interval asymptotic 95% confidence length for the parameter β is given by 1.96×*se*($\hat{\beta}$) = 2.8980, and as a result our requisite CI for β is given by $3.0989 \le \beta \le 8.8950$. Unfortunately, Elsayed does not provide a CI for the Weibull slope is of his Example 5.17, and hence I am not sure how accurate my CI is? We can run a simple

check on the value of the *se*($\hat{\beta}$) by simply computing the Cramer-Rao's glb of the inequality (130) using the given data, which I will now proceed to do. From Eq. (130),

Table 5.11 of E. A. Elsayed (on his page 297, 1−p = proportion of the sample that is censored; p = failed proportion, p=1→r = n)

p	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
c_{11}	1.1087	1.1517	1.2526	1.4473	1.8120	2.5102	3.9330	7.1904	16.4788	60.5171
c_{22}	0.6079	0.7670	0.9282	1.1225	1.3728	1.7162	2.22474	3.0655	4.7388	9.7447
c_{12}	0.2570	0.1764	0.0493	-.1448	-.4466	-.9358	-1.7855	-3.4386	-7.3753	-22.1872

$$V(\hat{\beta}) \doteq \frac{1}{\dfrac{n}{\beta^2} + \sum_{i=1}^{n} E\left[(t_i/\theta)^{\beta} \times [\ln(t_i/\theta)]^2\right]} \overset{\cdot\cdot}{=} \frac{\hat{\beta}^2/n}{1 + \dfrac{\hat{\beta}^2}{n} \sum_{i=1}^{n} \left[(t_i/\hat{\theta})^{\hat{\beta}} \times [\ln(t_i/\hat{\theta})]^2\right]} =$$

= 2.063142867; this Cramer-Rao glb (CRGLB) is very close to the actual V($\hat{\beta}$) ≅ 2.1866.

We can now use the 95% CI: $3.0989 \le \beta \le 8.8950$ to perform further SI on the parameter β. For example, suppose we wish to test the null 2-sided hypothesis $H_0 : \beta = 4.00$ at the pre-assigned LOS $\alpha = 0.05$. Since the hypothesized value of β, $\beta_0 \equiv 4$, is inside this 95% CI, then we cannot reject $H_0 : \beta = 4.00$ at the 5% LOS. However, if we were to test $H_0 : \beta = 10.00$ at $\alpha = 0.05$, then we have sufficient evidence to reject $H_0 : \beta = 10.00$ because $\beta_0 \equiv 10.00$ lies outside the 95% CI $3.0989 \le \beta \le 8.8950$. In other words, our 95% CI : $3.0989 \le \beta \le 8.8950$ has provided all possible 5%-level tests of hypotheses regarding the parameter β.

Our next objective is to use the MLE point estimator of θ, which is given by $\hat{\theta}$ =

$\left[\dfrac{1}{n}\sum_{i=1}^{n} t_i^{\hat{\beta}}\right]^{1/\hat{\beta}}$, and its *se* to obtain a lower one-sided 95% CI for the characteristic life $t_c = \theta$. To

his end, we 1st need to compute the value of the sample information on θ given by $I_{11} = I(\theta) = -$

$E[\partial^2 \mathbf{L}(\theta, \beta)/\partial\theta^2] = \beta(\beta+1)\theta^{-\beta-2}\ E(\sum_{i=1}^{n} t_i^{\beta}) - \dfrac{n\beta}{\theta^2} = \dfrac{\beta}{\theta^2}\left[(\beta+1)\theta^{-\beta} E(\sum_{i=1}^{n} t_i^{\beta}) - n\right].$

Therefore, from Cramer-Rao's inequality the glb[V($\hat{\theta}$)] = $\dfrac{\theta^2}{\beta\left[(\beta+1)\theta^{-\beta} E(\sum_{i=1}^{n} t_i^{\beta}) - n\right]}$.

Equation (124) shows that $\partial\mathbf{L}(\theta, \beta)/\partial\theta = -n\beta/\theta + \beta\,\theta^{-\beta-1}\sum_{i=1}^{n}(t_i)^{\beta} \xrightarrow{\text{Set to}} 0$, or

$-n + (\hat{\theta})^{-\hat{\beta}}\sum_{i=1}^{n}(t_i)^{\hat{\beta}} = 0$. Inserting $(\hat{\theta})^{-\hat{\beta}}\sum_{i=1}^{n}(t_i)^{\hat{\beta}} = n$ into glb[V($\hat{\theta}$)] =

$\dfrac{\theta^2}{\beta\left[(\beta+1)\theta^{-\beta} E(\sum_{i=1}^{n} t_i^{\beta}) - n\right]}$, we obtain V($\hat{\theta}$$\le \dfrac{\hat{\theta}^2}{\hat{\beta}\left[(\hat{\beta}+1)n - n\right]} = \dfrac{(\hat{\theta}/\hat{\beta})^2}{n}$. (131a)

It is interesting to notice that $E\sum_{i=1}^{n} t_i^{\beta} = \sum_{i=1}^{n} E(t_i^{\beta}) = n\int_0^{\infty} t^{\beta}\,\dfrac{\beta}{\theta}(t/\theta)^{\beta-1}e^{-(t/\theta)^{\beta}}dt = n\theta^{\beta}$, which is consistent with ML estimation. The V($\hat{\theta}$) in Equation (131a) is identical to the term in the 1st row and column of the $\mathbf{I}^{-1}$ matrix given in equation (5.40) atop page 297 of Elsayed, except for the multiplier c_{11}, which is equal to 1.108665 listed in his Table 5.11. Thus, for the case of uncensored data (i.e., complete sample),

$$V(\hat{\theta}) \cong 1.109 \times \frac{(\hat{\theta}/\hat{\beta})^2}{n} \qquad \textbf{(131b)}$$

For the failure data of the Example 5.17 on pages 291-292 of Elsayed, V($\hat{\theta}$) ≅ 1.1087× $(8597.6088104)^2/10 = 8195127.205356 \rightarrow$ *se*($\hat{\theta}$) = 2862.7133 hours. Hence, the value of the asymptotic 95% lower confidence limit is θ_L = 51559.6259832 – 1.645× *se*($\hat{\theta}$) = 46850.4627 → $46850.4627 \le \theta < \infty$. This 95% CI provides uncountably infinite number of right-tailed tests on

the parameter θ of the type $H_0 : \theta = \theta_0$ versus the alternative $H_1 : \theta > \theta_0$. For example, if we were to test $H_0 : \theta = 50000$ hours vs $H_1 : \theta > 50000$, then our CI does not provide sufficient evidence at the 5% level to reject H_0 and hence we will be unable to conclude that $\theta > 50000$. This is due to the fact that the hypothesized value $\theta_0 \equiv 50000$ hours lies inside the 95% CI: $46850.4627 \le \theta < \infty$. On the other hand, our 95% CI will provide sufficient evidence to conclude, at the 5% level, that the value of $t_c = \theta$ exceeds 45000 hours. Although we are dealing with 100% samples, however, if a sample is 70% completed to failure, or 30% censored (i.e., p = r/n = 0.70), then Table 5.11 of Elsayed shows that $V(\hat{\theta}) \cong 1.4473 \times \frac{(\hat{\theta}/\hat{\beta})^2}{n}$, $V(\hat{\beta}) \cong c_{22} \times \hat{\beta}^2 / n = 1.1225 \times \hat{\beta}^2 / n$, and $COV(\hat{\theta}, \hat{\beta}) = -0.1448\hat{\theta}/n$. The above $\mathbf{I}^{-1}$ clearly shows that the two estimators $\hat{\theta}$ and $\hat{\beta}$ are not independent, i.e., they are correlated. These estimates of *se*'s are generally conservative because they discount the correlation between $\hat{\theta}$ and $\hat{\beta}$.

In fact, Minitab, first obtains the local Fisher's information matrix by estimating the elements of the actual FIM (Fisher's Information Matrix) defined as

$$\mathbf{F} = \begin{bmatrix} \mathbf{I}_{11} & \mathbf{I}_{12} \\ \mathbf{I}_{12} & \mathbf{I}_{22} \end{bmatrix} = \begin{bmatrix} -E(\partial^2 \mathbf{L} / \partial\theta^2) & -E(\partial^2 \mathbf{L} / \partial\theta\partial\beta) \\ -E(\partial^2 \mathbf{L} / \partial\theta\partial\beta) & -E(\partial^2 \mathbf{L} / \partial\beta^2) \end{bmatrix}. \qquad (131c)$$

From Eq. (131a) we have $\hat{\mathbf{I}}_{11} = \frac{n}{(\hat{\theta}/\hat{\beta})^2} = \frac{n\hat{\beta}^2}{\hat{\theta}^2}$, and Eq. (129) shows that $\hat{\mathbf{I}}_{22} = n/\hat{\beta}^2 + \sum_{i=1}^{n}\left[(t_i/\hat{\theta})^{\hat{\beta}} \times [\ln(t_i/\hat{\theta})]^2\right]$. Further, I have obtained the expectation of the mixed-2nd-partial derivative of the Log-Likelihood function of the two-parameter Weibull and is given below:

$E(\partial^2 \mathbf{L} / \partial\theta\partial\beta) = -n(\beta/\theta)\ln(\theta) + (\beta/\theta) \times E\sum_{i=1}^{n}\left[(t_i/\theta)^{\beta} \times \ln(t_i)\right] \rightarrow \hat{\mathbf{I}}_{12} = n(\hat{\beta}/\hat{\theta})\ln(\hat{\theta}) - (\hat{\beta}/\hat{\theta}) \times \sum_{i=1}^{n}\left[(t_i/\hat{\theta})^{\hat{\beta}} \times \ln(t_i)\right]$. The direct inverse of the F-matrix in (131c), after estimating $\mathbf{I}_{ij}$, gives the MLE estimate of the $COV(\hat{\theta}, \hat{\beta})$. The diagonal elements of the 2×2 matrix,

COV($\hat{\theta}, \hat{\beta}$), provide the variances of $\hat{\theta}$ and $\hat{\beta}$. For the Example 5.17 of Elsayed, the local Fisher's Information matrix is

$$\hat{\mathbf{F}} = \hat{\mathbf{I}} = \begin{bmatrix} \hat{\mathbf{I}}_{11} & \hat{\mathbf{I}}_{12} \\ \hat{\mathbf{I}}_{12} & \hat{\mathbf{I}}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{0.000000135283} & \mathbf{-0.000080405433} \\ & \mathbf{0.484697408} \end{bmatrix}.$$

Upon inverting this last matrix, we obtain $\hat{\mathbf{I}}^{-1} = \begin{bmatrix} \mathbf{8200408.0038111} & \mathbf{1360.348435} \\ \mathbf{1360.348435} & \mathbf{2.288808208} \end{bmatrix}$. **Thus, the** ***se***$(\hat{\theta})$ **=** $\sqrt{\mathbf{8200408.0038111}}$ **= 2863.6354523 and** ***se***$(\hat{\beta})$ **= 1.51288076. These estimates of** ***SE*****'s are identical to those of Minitab's.**

The Example 5.18 on pages 298-301 of Elsayed. **This example provides a complete sample with times to failure of n = 10 identical units. Since the unit of TTF is not specified, I will assume the times TF were measured in hours. The failure data are 20, 22, 24, 25, 26, 27, 30, 35, 42, 52 hours, which give $\bar{t}$ = 30.3000, S = 10.0117, and cv_t = 33.04% < < 1. The Minitab output on page 236-7 of my notes yields $\hat{\beta}$ = − 0.270 + 1/0.3304 = 2.7566 and substituting this into (125a) yields $\hat{\theta}$ = 32.9335; these estimates may not be ML because they may not make equation (125b) ≅ 0. Inserting these initial estimates into (125b) yields the LHS = 1.0399 > > 0; since $\partial \mathbf{L}(\theta, \beta)/\partial\beta$ is a decreasing function of β, then we have to increase $\hat{\beta}$ = 2.7566 to, say, $\hat{\beta}$ = 3.20. This yields $\hat{\theta}$ = 33.6383 and the value of $\partial \mathbf{L}(\theta, \beta)/\partial\beta$ = 0.1373. After many iterations, we obtain $\hat{\beta}$ = 3.2755032, $\hat{\theta}$ = 33.7585 and $\partial \mathbf{L}(\theta, \beta)/\partial\beta = 1.370356153\times10^{-6}$. These MLEs are consistent with those of Elsayed's listed in the middle of his page 298. Using V($\hat{\beta}$) ≅ 0.6079× $(\hat{\beta})^2$/n yields V($\hat{\beta}$) = 0.65232 (Elsayed's answer is 0.6521) and the *se*($\hat{\beta}$) = 0.80766. Thus, the asymptotic 95% HCIL (half confidence interval length) is equal to 1.96×0.80766 = 1.58302 → β_L = 1.692485 and β_U = 4.85852. The 95% CI: 1.692485 ≤ β ≤ 4.85852 is not in close agreement with Elsayed's (1.788, 4.032) atop his page 301 because his 1 − α = 90%. As before, we should expect a wider confidence band simply because my 1 − α = 0.95 > 90%. Further, Elsayed is using the more exact SMD of (c×r)$(\beta/\hat{\beta})^{(1+p^2)}$ which is $\chi^2_{c(r-1)}$, where p = proportion of the**

sample that has failed (1 − p represents the proportion of the sample that has been censored), and c = $2[(1+p^2)^2\, p\, c_{22}]^{-1}$. Table 5.11 on page 297 of Elsayed gives the values of c_{ij} for p = 0.10(0.10)1. The 95% lower confidence limit for the characteristic life is given by $\hat{\theta}$ − 1.645×*se*($\hat{\theta}$), where *se*($\hat{\theta}$) = $\sqrt{c_{11}\dfrac{(\hat{\theta}/\hat{\beta})^2}{n}}$ = $\sqrt{1.1087\times\dfrac{(33.7585/3.2755032)^2}{10}}$ = 3.4317 → θ_L = 33.7585 −1.645×3.4317 = 28.1133 → $28.1133 \le \theta < \infty$ at the 95% confidence level.

Obtaining More Exact 95% Lower One-Sided CI for the Weibull W(0, θ, β) Characteristic Life and Reliability Function

Since the characteristic life, $t_c = \theta$, is an LTB type population parameter, then there is absolutely no concern on the high side and hence, as before, we will obtain only a lower 95% CI for θ of the form $\theta_L \le \theta < \infty$. As always, before experimentation θ_L is a rv (random variable), but after data have been gathered and a numerical value computed for θ_L, then the deterministic interval $\theta_L \le \theta < \infty$ is no longer random and its Pr of containing the true value of θ reduces to 0 or 1. I will discuss the CI estimation only for the uncensored sample in this Chapter. This estimation is easier than CI estimation for a censored data. We will first obtain the lower one-sided CI for θ, followed by a lower 1-sided CI for R(t).

Since there does not exist a closed-form solution for the ML estimator $\hat{\theta}$, then it will always be impossible to use mathematical statistics to obtain the precise SMD of the statistic $\hat{\theta}/\theta$. However, Bain, L. J. and Engelhardt, M. (1991), 2nd Edition, New York, Marcel Dekker, used simulation to obtain the approximate percentiles of the SMD of the statistic U = $\hat{\beta}\sqrt{n}\,\ln(\hat{\theta}/\theta)$, which they tabulate on their page 230 for different sample sizes and confidence levels $\gamma = 1-\alpha$ = 0.02, 0.05, 0.10, 0.25, 0.50, 0.75, 0.90, 0.95, and 0.98. I do not know why the tabulation was not done for $1-\alpha$ = 0.005, 0.99, and 0.995. Since, we are interested only in developing a 95% lower one-sided CI for θ, then our interest centers only at the value of $\gamma = 1-\alpha$ = 0.95. I took Bain and Engelhardt's 95th percentiles of U, for n = 5 to n = 120, given in their Table 4A on their page 230 and used Minitab to obtain the following models

for the percentiles $u_{0.95}$ and $u_{0.975}$.

$$\hat{u}_{0.95}(n) = 1.7093 + 3.8522/n - 11.3533/n^2 + 56.3650/n^3 \tag{132a}$$

$$\hat{u}_{0.975}(n) = 2.1043 + 6.3308/n - 30.9180/n^2 + 153.95/n^3 \tag{132b}$$

Minitab reported an $R^2_{\text{Model}} \geq 99.6\%$ for the above models and I tried to improve them to obtain nearly a 100% value of R^2_{Model} by adding $1/n^4$, but the models would not improve any further. I tested the values of $\hat{u}_{0.95}(n)$ from equation (132a) against those given in equation (5.51) of Elsayed on his page 302 as $U_{0.95} = 1.720 + 3.163/n + 18.25e^{-n}$, both Elsayed's and mine are very close (to 3 decimals) to those listed in Table 4A of Bain and Engelhardt (on their page 230) for n = 5 to n = 120.

The Example 5.18 on page 298 of Elsayed provides a complete sample for n = 10 with times to failure as 20, 22, 24, 25, 26, 27, 30, 35, 42, 52 hours (note that units are not provided in this example and I am guessing that it is hours to failure). Inserting n = 10 in equation (132a) gives $\hat{u}_{0.95}(10) = 2.03733$ (B&E report $U_{0.95} = 2.037$ in their Table 4A on page 230 for n = 10). Therefore, the $\Pr[\hat{\beta}\sqrt{n}\,\ln(\hat{\theta}/\theta) \leq 2.03733] \cong 0.95$. Rearranging the inequality inside this last brackets yields

$$\theta_L(0.95) = \hat{\theta}e^{-u_{0.95}/(\hat{\beta}\sqrt{n})} \tag{133}$$

Inserting the values of $\hat{\theta} = 33.7585$, $\hat{\beta} = 3.2755032$, and $\hat{u}_{0.95}(10) = 2.03733$ into equation (133) yields $\theta_L(0.95) = 27.7307$, which is fairly close to the value of $\theta_1^L = 26.24$ reported near the bottom of Elsayed's page 302, where Elsayed is using the uncommon superscript to denote the lower confidence limit. The discrepancy lies in the fact that E. A. Elsayed is using the unbiased estimates of β and θ, while I have used the MLEs of β and θ from Eqs. (125). Thus, we are 95% confident that the true value of θ lies in the interval 27.63201 hours $\leq \theta < \infty$. This CI is not as conservative as that of Elsayed's 26.24 hours $\leq \theta < \infty$, given near the bottom of page 302, because of the fact that Elsayed used the unbiased estimates of β and θ, which he computed to be $\hat{\beta} = 2.81$ and $\hat{\theta}_1 = 33.01$. Since his 95% lower bound for θ is more

conservative, then you may wish to also use the unbiased estimates of β and θ, as he did. Using equations (132b & 133), $\theta_L(0.975)$ = $\hat{\theta}e^{-u_{0.975}/(\hat{\beta}\sqrt{n})}$ = 26.30984. A point MLE of R(t) is given by $\hat{R}(t)$ = $e^{-(t/33.7585)^{3.2755}}$. For example, $\hat{R}(25\,\text{hours})$ = 0.6881. Due to the fact that the RE function of a Weibull is not a monotonically increasing function of the slope β (recall that the Weibull RE increases with increasing β up to the characteristic life θ and then decreases with increasing β). Hence, $R_L(t)$ will not equal to $e^{-(t/\theta_L)^{\beta_L}}$ and as a result in order to obtain a lower 95% CI for $R(25\,\text{hours})$, we need to have some idea about the SMD of $\hat{R}(25\,\text{hours})$. As stated before, the SMD of all MLEs approach normality (as n → ∞) with mean equal to the corresponding parameter, namely R(25), and a variance whose asymptotic value is equal to 1/I[R(t)]. I surmise that I[R(t)] = $-E[\partial^2\mathbf{L}(\theta, \beta)/\partial R^2]$ cannot be computed directly. Thus, Bain and Engelhardt (p. 217) provide an empirical formula to compute the asymptotic variance of $\hat{R}(t)$, but their formula does not provide a conservative glb for R(t) as shown in their Table 7A (pp. 236-243). Thus, I took the liberty, through trial/error, to provide a revised version of their formula, given below, so that the normal approximation will provide a more conservative value of $R_L(t)$ for almost all n. $V[\hat{R}(t)] \cong \hat{R}^2(t) \times [\ln(1/\hat{R}(t)]^2 \times \{1.70 - 0.70 \ln(\ln(1/\hat{R}(t)))+ 0.70[\ln(\ln(1/\hat{R}(t)))]^2\}/n$ (134)

Inserting the value of $\hat{R}(25\,\text{hours})$ = 0.6881 into equation (134) gives $V[\hat{R}(25)]$ = 0.0203, and the *se*$[\hat{R}(25)]$ = 0.142449 and 1.645× se$(\hat{R}(25))$ = 0.23433, and hence $R_L(25\text{ hours}) \cong \hat{R}(25) - Z_{0.05} \times se[\hat{R}(25)]$ = 0.45373 → 0.45373 ≤ R(25) < 1. Thus, we are 95% confident that the reliability function at t = 25 hours exceeds 0.45373.

Estimation of Weibull Parameters using Weibull Graph Paper (WGP)

Recall that because R(t) = $e^{-(\frac{t-\delta}{\theta-\delta})^{\beta}}$, $t>\delta$, then ln[R(t)] = $-(\frac{t-\delta}{\theta-\delta})^{\beta}$ → ln[1/R(t)] = $(\frac{t-\delta}{\theta-\delta})^{\beta}$ → ln{ln[1/R(t)]} = $\beta\ln(\frac{t-\delta}{\theta-\delta})$ → ln[-lnR(t)] = $\beta[\ln(t-\delta)-\ln(\theta-\delta)]$ = $\beta\ln(t-\delta)-\beta\ln(\theta-\delta)$ = $\beta\ln(t-\delta)+C$, where C = $-\beta\ln(\theta-\delta)$ is a constant . Letting y = ln[ln(R^{-1})] , x = $\ln(t-\delta)$ yields y = $\beta x+C$, which represents a line with slope β and Y-intercept C. This is why the shape parameter β is also called the slope of the Weibull distribution. For the case of zero minimum life, x = $\ln(t)$ and the y-intercept is C = $-\beta\ln(\theta)$ = $\beta\ln(1/\theta)$. Because all Weibull distributions have a RE value of e^{-1} at their characteristic life $t_c = \theta$, the Weibull graph paper (WGP) has two ordinates, where the right scale gives ln[–ln(R)] and the opposing left scale gives F(t) = 1–R(t). Thus, y = ln[ln(R^{-1})] is the right coordinate scale and its value at t = θ is equal to y(θ) = ln[– ln(e^{-1})] = ln[ln(e)] = ln(1) = 0 while the corresponding left ordinate is obtained from 0 = ln[−ln(R)] = $\ln[\ln(\frac{1}{1-F})]$ → $\ln(\frac{1}{1-F})$ = e^0 = 1 → $\frac{1}{1-F}$ = e → 1 – F = e^{-1} ≅ 0.3679 → F = 0.6321 → The 63.21% failure point on the left scale corresponds to the characteristic life θ on the abscissa because ln[ln(1/R(θ))] = 0 [see Figure 11.2 of Kapur and Laberson (1977) on their p. 296] reproduced on the next page]. Yet as another example, –1 = ln[ln(1/R)] on the right scale of figure 11.2 implies that $\ln(\frac{1}{1-F})$ = e^{-1} = 0.3679 → $\frac{1}{1-F}$ = 1.4447 →1–F = 0.6922 → F = 0.3078 → the WGP left scale will correspond to roughly 30.78% cumulative failure, etc. Finally, the RE function of the Weibull for the same values of δ and θ is an increasing function of the slope β up to the characteristic life θ and then becomes a decreasing function of β for t values beyond θ. As an example of using the WGP to estimate the parameters of a W(0, θ, β), see the Example 15.5 on pages 395-6 of Ebeling and my Excel file on my website that obtains the LSE of β followed by θ. The 95% glb on RE at 40 hours for the example 15.5 of Ebeling is roughly given by 1–0.52 = 0.48, while we are 90% confident that 30% of the units will fail within the interval (20, 87 hours).

The estimation procedure using the WGP requires that we estimate its left-ordinate $p_j = F(t_j)$ = proportion of the population failing by the j^{th} order-statistic. It can be shown (see Kapur & Lamberson, pp. 297-303) that the pdf of $p_j = F(t_j)$ is given by

$$g(p_j) = \frac{n!}{(j-1)!(n-j)!}(p_j)^{j-1}(1-p_j)^{n-j} \qquad (135)$$

which is the Beta density with parameters a = j and b = n–j+1. The use of Eq. (135) provides two estimates of $p_j = F(t_j)$: (1) Based on Mean-Rank, (2) Based on Median-Ranks. However, note that in Pr plots, some Statistical packages estimate $F(t_j)$ as (j–0.5)/n.

Estimates Based on Mean Ranks

Recall from my Chapters 2,3&4 that if X has a Beta distribution, then E(X) = a/(a+b), and as a result $E(p_j)$ = j(j+n–j+1) = j/(n+1). As an example, for the data of Example 12.3 on p. 312 of Ebeling, the mean-rank estimate of the $p_3 = F(t_3)$ = F3.5 hrs) is equal to 3/9 = 0.33333, i.e., we expect that 33.33% of the units to get repaired by the 3^{rd} –order statistic 3.50 hrs. Similarly, 6/(8+1) = 66.67% of the population are expected to get repaired by 5.40 hrs.

Estimates Based on Median Ranks

Let $\tilde{p}_j$ be the median of p_j; then from Eq. (135) $\int_0^{\tilde{p}_j} g(p_j)dp_j$ =0.50. Because the Beta pdf is not in general directly invertible, then there is no closed-form solution for $\tilde{p}_j$ for all n and j. Fortunately, an approximate solution to $\int_0^{\tilde{p}_j} g(p_j)dp_j$ = 0.50 is given by $\tilde{p}_j = \hat{F}(t_j)$ = (j–0.3)/(n+0.4). For the data of Example, 15.5 on p. 396 of Ebeling $\tilde{p}_3$ = (3–0.3)/(8+0.4) =0.3214, i.e., we are 50% confident that 32.14% of the population fail by 74 hrs. From statistical theory, it can be shown that for any underlying distribution the solution to $\int_0^{p_{1-\alpha}} g(p_j)dp_j = 1-\alpha$ is given by $K/[K + F_{1-\alpha,2(n+1-j),2j}]$, where K = j/(n–j+1). The reader has to

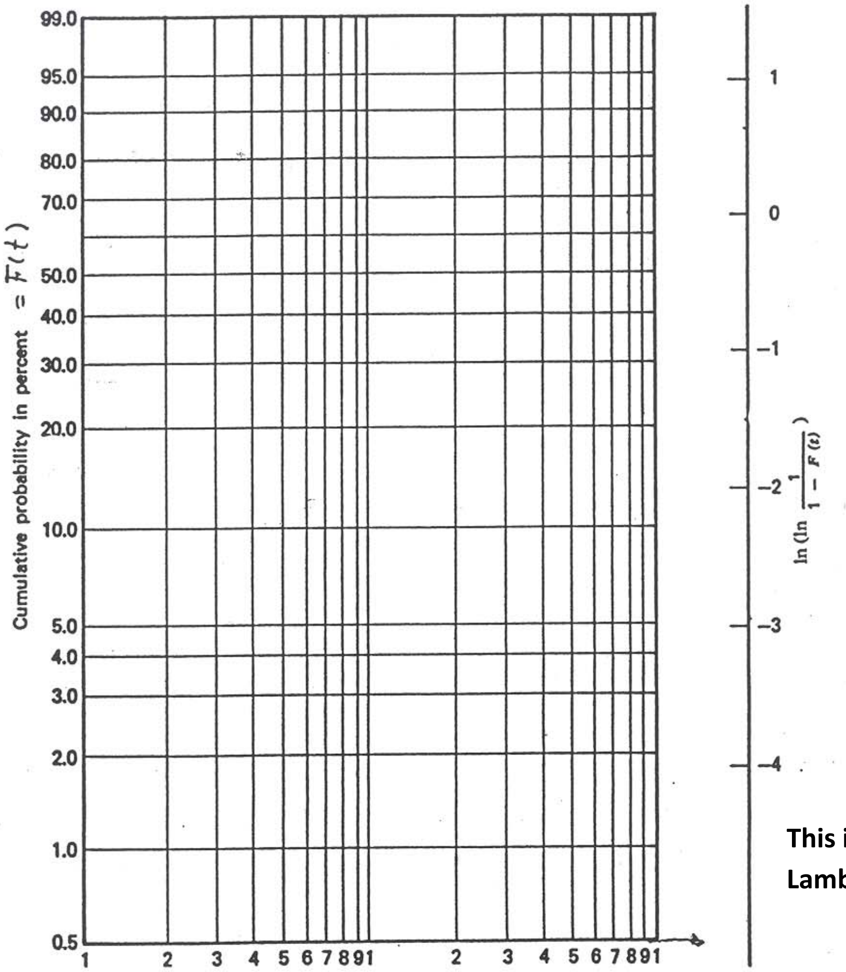


Figure 11.2 Weibull probability paper.

This is Figure 11.2 of Kapur & Lamberson (1977) on their page 296

Example 15.5 on p.396 of Ebeling (2nd)

296 **reliability estimation: weibull distribution** page 396 of Ebeling 2nd

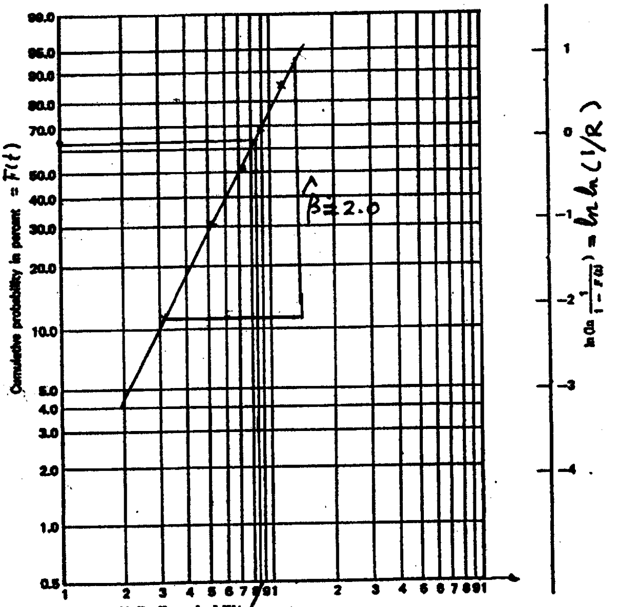


Figure 11.2 Weibull probability paper.

$\hat{\theta} \doteq 85.0$ hrs , $n = 5$

Figure 11.2 of Kapur and Lamberson (1977) from their page 296

A point estimate of RE at 80 hrs $\doteq 1 - 0.60 = 0.40$, while the 90% CI for R(80) is given by $0.15 \leq R(80) \leq 0.72$

$$w_{\alpha;j} = \frac{K}{K + F_{\alpha}(2n-2j+2, 2j)}, \quad K = j/(n-j+1)$$

Data: 32, 51, 74, 90, 120 hr

Complete Sample

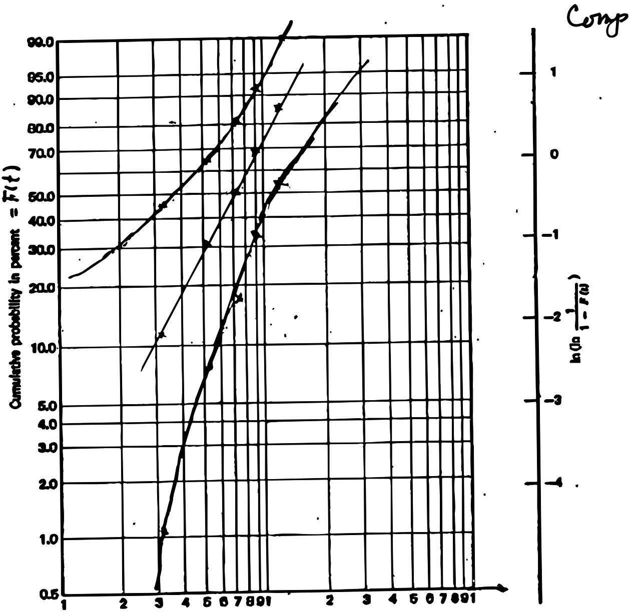


Figure 11.2 Weibull probability paper.

Example 15.5 on page 396 of Ebeling ; $n = 5$

Figure 11.2 of Kapur and Lamberson (1977) from their page 296

At $j = 1$, $w_{.05;1} = \frac{0.20}{0.20 + F_{.05;10,2}} = \frac{0.20}{0.20 + 19.3959} =$

$= 0.0102$

At $j = 4$, $w_{0.95;4} = \frac{2}{2 + F_{.95;4,8}} = \frac{2}{2 + 0.1655} = 0.9236$

be careful not to confuse the value of $F_{1-\alpha,2(n+1-j),2j}$ on the left tail of Fisher's F-distribution, which is the α^{th} quantile of F_{ν_1,ν_2} with $\nu_1 = 2(n+1-j)$ and $\nu_2 = 2j$ *df*. Statistical Tables do not tabulate the left-tail of F because $F_{1-\alpha,2(n+1-j),2j} = 1/F_{\alpha,2j,2(n+1-j)}$. As an example, suppose we wish to compute the 95% rank-estimate for $t_3 = 74$ hrs on p. 396 of Ebeling. Then, K = 3/(8−3+1) = 0.50, $F_{0.05,6,12} = 2.9961 \rightarrow F_{0.95,12,6} = 1/2.9961 = 0.3338 \rightarrow p_{0.95} = 0.50/(0.50+0.3338) = 0.5997$; thus, we are 95% confident that at most 59.97% of the population fails by $t_3 = 74$ hrs. On the other hand, similar calculation will show that $0.50/(0.50+ F_{0.05,12,6}) = 0.50/(0.50+ 3.9999) = 0.11111$; thus, we are only 5% confident that at most 11.111% of the population fails by 74 hrs. I have provided a Table rank-estimates for $\alpha = 0.05$, 0.50, and 0.95 on my website.

The Graphical Estimation of Weibull Slope (β) and Characteristic Life θ

I will illustrate the procedure using the data given in Elsayed's Example 5.18 on his pp. 298-299. Although, Ebeling also provides an example of the GWP for n = 5 failure times on his Example 15.5 on pp. 396-397. Elsayed's data provide a more interesting situation where the WG shows the underlying data distribution is probably not Weibull. The n = 10 times to failure are 20, 22, 24, 25, 26, 27, 30, 35, 42, and 52 hrs. I used the Excel sheet on my website to obtain the 5%, 50%, and 95% ranks. For example, the Median-ranks to 3 decimals are 0.067, 0.162, 0.259, 0.355, 0.452, 0.548, 0.645, 0.741, 0.838, and 0.933, respectively. Note that the WGP on the next page graphs only the (population) 50% ranks because the 50% line is concave downward (note that if data is roughly Weibull, then the median ranks always will plot nearly exactly a straight line where roughly half the points lie below and the other half will lie above the 50%-rank line). For this reason I decided that the problem may be that the Min-life t_0 is not zero and subtracted 90%, or even 80% and re-graphed the ensuing data in an attempt to straighten out the 50% population line. All attempt were futile, and thus stopped analyzing the data. In fact I used Minitab to assess the goodness-of-fit to Weibull, and Minitab reports an AD-statistic = 2.30 with a P-value << 0.001. I have also analyzed the Example 15.5 of Ebeling using a WGP, which is given on pp. 261-262.

t_i : 20, 22, 24, 25, 26, 27, 30, 35, 42, 52

296 reliability estimation: weibull distribution

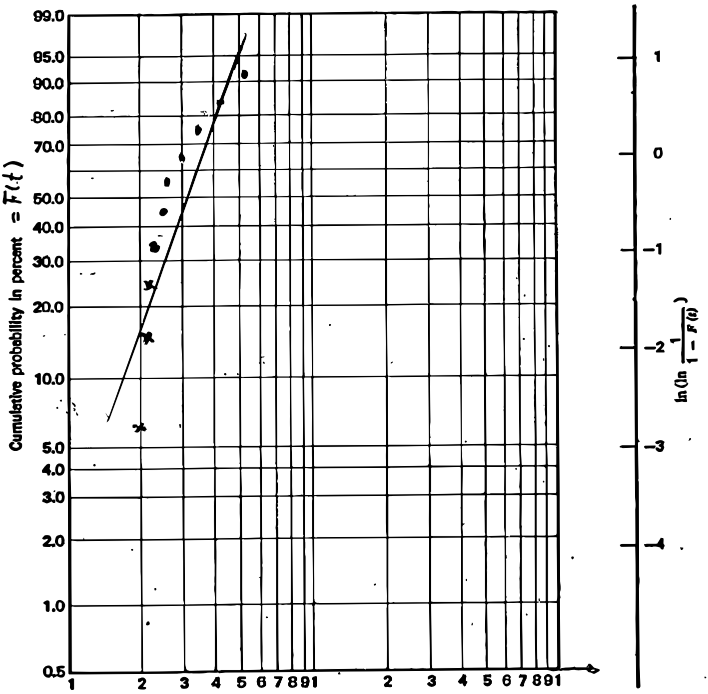


Figure 11.2 Weibull probability paper.

Figure 11.2 of Kapur and Lamberson (1977) from their page 296

Data from Example 5.18 on pp. 298-301 of E. A. Elsayed.

Maximum Likelihood Estimation when the Underlying Distribution is Gaussian

We use the data of Example 15.8 and 15.16 on pages 402-410 of Ebeling to illustrate ML estimation in the normal case. In these examples n (= 24) repair-times are given from either a N(μ, σ^2) or Lognormal distribution. The objective is to obtain the MLEs of μ and σ^2 and to determine if the estimates are efficient (i.e., do they achieve the Cramer-Rao inequality glb). The 24 TTRs are given atop page 402 of Ebeling. W/O much writing I will go thru the procedure step by step for the Normal ML estimation.

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-(\frac{x_i-\mu}{\sigma})^2/2} = (\sigma\sqrt{2\pi})^{-n}\, e^{-\frac{1}{2}\sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma})^2} \rightarrow$$

$$\mathbf{L}(\mu, \sigma^2) = -\,n\ln(\sigma\sqrt{2\pi}) - \frac{1}{2}\sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma})^2 \rightarrow \xi(\mu) = \partial\mathbf{L}(\mu, \sigma^2)/\partial\mu \rightarrow$$

$$\xi(\mu) = -\sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma})(\frac{-1}{\sigma}) \xrightarrow{\text{Set to}} 0 \rightarrow \sum_{i=1}^{n}(x_i-\hat{\mu}) = 0 \rightarrow \sum_{i=1}^{n} x_i - n\hat{\mu} = 0$$

$\rightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$ **→ The sample mean, $\bar{x}$, is the MLE of μ. In order to apply equation (126b) with $\theta = \mu$, we have to take the 2nd partial derivative of $\mathbf{L}(\mu, \sigma^2)$ wrt (with respect to) μ.**

$$\partial^2\mathbf{L}(\mu, \sigma^2)/\partial\mu^2 = \frac{\partial}{\partial\mu}\xi(\mu) = \frac{\partial}{\partial\mu}[\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)] = \frac{\partial}{\partial\mu}[\frac{1}{\sigma^2}(\sum_{i=1}^{n} x_i - n\mu)] = -n/\sigma^2$$

From equation (128), $I(\mu) = -E[\partial^2\mathbf{L}(\mu, \sigma^2)/\partial\mu^2] = n/\sigma^2$. The relation $I(\mu) = n/\sigma^2$ is intuitively appealing because it states that larger sample sizes provide more information about μ. Since the MLE of μ is $\hat{\mu} = \bar{x}$, then the Cramer-Rao inequality (126b) now shows that $V(\hat{\theta} = \bar{x}) \geq \frac{[1+B'(\bar{x})]^2}{I(\mu)}$. However, it is well known that $\hat{\mu} = \bar{x}$ is an unbiased estimator of the population mean μ for any random sample of size n and hence $B(\bar{x}) = 0$ showing that $\frac{d}{d\mu}B(\bar{x}) = B'(\bar{x}) = 0$

and hence $V(\overline{x}) \geq \frac{[1+0]^2}{I(\mu)} = \sigma^2/n$. It is also well known that for a random sample of size n from an infinite population (not finite), the $V(\overline{x}) = \sigma^2/n$ so that the estimator $\overline{x}$ is efficient because its variance attains the glb of the Cramer-Rao's inequality. For a random sample of size n, there does not exist another estimator of μ (other than $\overline{x}$) whose variance is smaller than σ^2/n. Further, the reader should note that if a MLE $\hat{\theta}$ is unbiased, then its variance, for large n, is almost equal to $I^{-1}(\theta)$.

To obtain the MLE of σ and its information $I(\sigma)$, we have to partially differentiate $\mathbf{L}(\mu, \sigma)$ =

$-n\ln(\sigma\sqrt{2\pi}) - \frac{1}{2}\sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma})^2 = -n\ln(\sigma) - n\ln(\sqrt{2\pi}) - \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\sigma^{-2}$ with respect to σ.

$\rightarrow \xi(\sigma) = \partial\mathbf{L}(\mu, \sigma)/\partial\sigma = -n/\sigma + \sum_{i=1}^{n}(x_i-\mu)^2\sigma^{-3} \xrightarrow{\text{Set to}} 0 \rightarrow$

$-n + \sum_{i=1}^{n}(x_i-\hat{\mu})^2\hat{\sigma}^{-2} = 0 \rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2$ = The sample variance. Thus, the sample variance $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2$ is the MLE of σ^2, and hence the sample standard deviation $\hat{\sigma}$ = $\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2}$ is the MLE of σ. Since, $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x})^2$ is an unbiased estimator of σ^2 for a random sample of size n from an infinite population, then the amount of bias in the ML estimator $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2$ is given by $B(\hat{\sigma}^2) = E(\hat{\sigma}^2) - \sigma^2 = E[(n-1)S^2/n] - \sigma^2 = \frac{n-1}{n} E(S^2) - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = -\sigma^2/n$. The exact amount of bias in the MLE $\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2}$ of σ is not clearly equal to $\sqrt{B(\hat{\sigma}^2)} = \sqrt{-n/\sigma^2}$, where this last square root yields an imaginary number. It is known in the field of QC that only for a normal universe $E(S) = c_4\sigma$,

where the well-known QC constant $c_4 = \sqrt{\frac{2}{n-1}} \times \frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}$. Therefore, E($\hat{\sigma}$) = E(S $\sqrt{\frac{n-1}{n}}$) = $\sqrt{\frac{n-1}{n}}$ $c_4\sigma$ = $\sqrt{\frac{2}{n}}\,\frac{\Gamma(n/2)\sigma}{\Gamma[(n-1)/2]}$ giving rise to B($\hat{\sigma}$) = E($\hat{\sigma}$) – σ = $\sqrt{\frac{2}{n}}\,\frac{\Gamma(n/2)\sigma}{\Gamma[(n-1)/2]}$ – σ = [$\sqrt{\frac{2}{n}} \times \frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}$ – 1] σ = $b_4\sigma$, where the constant b_4 = $\sqrt{\frac{2}{n}} \times \frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}$ – 1. The value of $b_4 < 0$ for all n, and its limit as $n \to \infty$ is exactly zero. This implies that the MLE $\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$ is asymptotically an unbiased estimator of σ.

Further, dB($\hat{\sigma}$)/dσ = b_4. Similarly, for a normal universe B(S) = E(S) – σ = $(c_4 - 1)\sigma$.

We now obtain the I(σ) = –E[∂ξ(σ)/∂σ]. Since ξ(σ) = –n/σ + $\sum_{i=1}^{n}(x_i - \mu)^2\,\sigma^{-3}$, then

∂ξ(σ)/∂σ = $n/\sigma^2 - 3\sum_{i=1}^{n}(x_i - \mu)^2\,\sigma^{-4} \to$ E[∂ξ(σ)/∂σ] = $n/\sigma^2 - 3\,\sigma^{-4}\sum_{i=1}^{n}E(x_i - \mu)^2$ =

$n/\sigma^2 - 3\,\sigma^{-4}\,(n\,\sigma^2) = n/\sigma^2 - 3\,n/\sigma^2 = -2\,n/\sigma^2 \to$ I(σ) = $2n/\sigma^2$.

Therefore, the Cramer –Rao glb (CROGLB) for the V($\hat{\sigma}$) is equal to $\frac{(1+b_4)^2}{I(\sigma)}$ = $\frac{(1+b_4)^2}{2n/\sigma^2}$. I will next obtain the exact variance of $\hat{\sigma}$ when sampling a normal universe. Again, it is well known in the field of QC that only for a normal universe V(S) = $(1 - c_4^2)\sigma^2$. Using this fact, we obtain V($\hat{\sigma}$) = V (S$\sqrt{\frac{n-1}{n}}$) = $\frac{n-1}{n}$ V(S) = $\frac{n-1}{n}(1 - c_4^2)\sigma^2$. If my glb[V($\hat{\sigma}$)] = $\frac{(1+b_4)^2}{2n/\sigma^2}$ = $\frac{(1+b_4)^2}{2n}\sigma^2 = (\sigma/n)^2\left[\frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}\right]^2 = a_4\sigma^2$ is indeed correct, then we must have V($\hat{\sigma}$) = $\frac{n-1}{n}(1 - c_4^2)\,\sigma^2 = k_4\sigma^2 \ge \sigma^2\left[\frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}\right]^2/n^2$. Table 6 below lists the values of $k_4 = \frac{n-1}{n}$

$(1-c_4^2)$ and $a_4 = \frac{(1+b_4)^2}{2n} = \left[\frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}\right]^2/n^2$ for n = 5(5)20(10)90. For n = 100, k_4 = 0.004987 and a_4 = 0.004925; at n = 150, k_4 = 0.0033277 and a_4 = 0.003300. Elsayed mentions that $V(\hat{\sigma}) = \sigma^2/2n = I^{-1}(\sigma)$ atop his page 246; this is somewhat conservative because $\sigma^2/2n$ exceeds the $V(\hat{\sigma})$ for all n. However, as $n \to \infty$, then $V(\hat{\sigma}) \to \sigma^2/2n$. Note that $1/2n \geq k_4 \geq a_4$ for all n, where $k_4\sigma^2$ is the exact variance of $\hat{\sigma}$ and $(a_4\sigma^2)$ is the CROGLB for the $V(\hat{\sigma})$. For the normal likelihood estimators, we already found that $I_{11} = I(\mu) = n/\sigma^2$ and $I_{22} = I(\sigma) = 2n/\sigma^2$. In order to obtain the off-diagonal components of $\mathbf{I}$, we need to compute the $E[-\partial^2\mathbf{L}(x; \mu, \sigma)/\partial\mu\partial\sigma]$. Since, $\partial\mathbf{L}(\mu, \sigma)/\partial\sigma = -n/\sigma + \sum_{i=1}^{n}(x_i-\mu)^2\sigma^{-3}$, then $\partial^2\mathbf{L}(x; \mu, \sigma)/\partial\mu\partial\sigma = -2\sum_{i=1}^{n}(x_i-\mu)\sigma^{-3} = \frac{-2}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu) \to -\partial^2\mathbf{L}(x; \mu, \sigma)/\partial\mu\partial\sigma = \frac{2}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu) \to E[-\partial^2\mathbf{L}(x; \mu, \sigma)/\partial\mu\partial\sigma] = \frac{2}{\sigma^3}\sum_{i=1}^{n}E(x_i-\mu) = 0$; therefore, the off-diagonal elements of the information matrix $\mathbf{I}$ are zero, and as a result $\mathbf{I} = \begin{bmatrix} n/\sigma^2 & 0 \\ 0 & 2n/\sigma^2 \end{bmatrix}$. The inverse of $\mathbf{I}$, $\mathbf{I}^{-1} = \begin{bmatrix} \sigma^2/n & 0 \\ 0 & \sigma^2/2n \end{bmatrix}$, gives the asymptotic covariance matrix for the vector $[\bar{x} \quad \hat{\sigma}]'$. The information matrix, $\mathbf{I}$, verifies a very well and widely known results from statistical theory that the sample mean and variance from a normal universe are stochastically independent.

We now apply the above developments to the data of Example 15.8 atop p. 402 of Ebeling. Minitab computations give $\hat{\mu} = \bar{x} = 137.3240$ and $\hat{\sigma} = \sqrt{\frac{1}{25}\sum_{i=1}^{25}(t_i-\bar{x})^2}$ = 92.355705, and cv(X) = 67.254%. The estimate of the inverse information matrix is given by $\hat{\mathbf{I}}^{-1} = \begin{bmatrix} \hat{\sigma}^2/n & 0 \\ 0 & \hat{\sigma}^2/2n \end{bmatrix} = \begin{bmatrix} 341.183 & 0 \\ 0 & 170.5915 \end{bmatrix}$. Note that this covariance matrix shows that when sampling a normal universe, the statistics $\overline{x}$ and $\hat{\sigma}$ are independent, i.e., the sample mean and standard deviation have zero covariance. The sample standard deviation, $\hat{\sigma}$, is not an efficient estimator of σ because its variance V($\hat{\sigma}$) = $\frac{n-1}{n}(1-c_4^2)\sigma^2$ does not attain the glb of the Cramer-Rao inequality given by glb[V($\hat{\sigma}$)] = $\frac{(1+b_4)^2}{2n/\sigma^2}$, which was illustrated in Table 10. Therefore, the relative efficiency of $\hat{\sigma}$ as an estimator of σ for a normal universe is given by

$$\text{Rel-Eff}(\hat{\sigma}) = \frac{\text{glb}[V(\hat{\sigma})]}{V(\hat{\sigma})} = \frac{(1+b_4)^2\sigma^2/2n}{(n-1)(1-c_4^2)\sigma^2/n} = \frac{(1+b_4)^2}{2(n-1)(1-c_4^2)} \qquad (136)$$

Table 10

n	5	10	15	20	30	40	50	60	70	80	90
1/(2n)	0.1000	0.05	0.03333	0.025	0.16667	0.01250	0.0100	0.00833	0.007143	0.00625	0.00556
k_4	0.09314	0.04854	0.03272	0.02466	0.01652	0.01242	0.00995	0.00830	0.00712	0.00623	0.00554
a_4	0.07069	0.04257	0.03002	0.02313	0.01584	0.01203	0.00970	0.00813	0.00699	0.00613	0.00546

Table 11 lists the Rel-Eff of $\hat{\sigma}$ as a point MLE of σ for n = 4(1)10(5)30(10)100(50)300 in percent. Table 11 clearly shows that $\hat{\sigma}$ is an asymptotic efficient estimator of σ, i.e., V($\hat{\sigma}$) attains the CROGLB as $n \to \infty$. Further, Rel-Eff(S) = $\frac{(c_4)^2}{2n(1-c_4^2)}$ = Rel-Eff($\hat{\sigma}$) for all n from a normal universe, while both S and $\hat{\sigma}$ are biased estimators of σ, where $|B(S)| < |B(\hat{\sigma})|$ but

Table 11

n	4	5	6	7	8	9	10	15	20	25	30
Rel-Eff	70.19%	75.89	79.77	82.58	84.70	86.37	87.71	91.76	93.80	95.03	95.85
n	40	50	60	70	80	90	100	150	200	250	300
Rel-Eff	96.89%	97.51	97.92	98.22	98.44	98.61	98.75	99.17	99.38	99.50	99.58

V(S) > V($\hat{\sigma}$). It can be verified that for a normal universe the MSE(S) = V(S) + $B^2(S)$ = 2(1 – c_4) σ^2, and MSE($\hat{\sigma}$) = V($\hat{\sigma}$) + $B^2(\hat{\sigma})$ =2(1 – $c_4 \sqrt{\frac{n-1}{n}} - \frac{1}{2n}$)$\sigma^2$. As a result, the Relative Efficiency of $\hat{\sigma}$ to S is given by MSE(S)/ MSE($\hat{\sigma}$) = (1 – c_4)/ (1 – $c_4 \sqrt{\frac{n-1}{n}} - \frac{1}{2n}$) ≥ 100%. This relative efficiency approaches 100% very rapidly. For n = 30, its value is 100.03487%, while at n = 50 its value is 100.012532%.

MLE for lognormal parameters is quite similar to that of the base-line normal distribution. For the repair-data of Example 15.16 on p. 410 of Ebeling, Minitab shows that the data is not at all normally distributed, but that the ln(t_i) is normally distributed with a P-value of 0.855 for the Anderson-Darling statistic. Then, I created a column in Minitab as ln(t_i) and computed the pertinent statistic as $\hat{\mu} = \overline{\ln(t_i)}$ = 4.75313 and $\hat{\sigma}_{\ln(t)}$ = 0.56391. These last two values match those Ebeling's on his 410.

Chapter 15 Summary, where r = n

1. In order to obtain MLE of Weibull β and θ, solve the two Eqs.

$$\hat{\theta} = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{\hat{\beta}}\right]^{1/\hat{\beta}} \qquad \textbf{(125a)}$$

and $\xi(\beta) = \partial L(\theta, \beta)/\partial\beta = \frac{n}{\beta} - n\ln(\theta) + \sum_{i=1}^{n}\ln(t_i) - [\sum_{i=1}^{n}\frac{\partial}{\partial\beta}(t_i/\theta)^{\beta}]$

$$= \frac{n}{\beta} - n\ln(\theta) + \sum_{i=1}^{n}\ln(t_i) - \sum_{i=1}^{n}\left[(t_i/\theta)^{\beta}\times\ln(t_i/\theta)\right] \xrightarrow{\text{Set}} 0 \qquad \textbf{(125b)}$$

simultaneously in order to obtain $\hat{\beta}$ and $\hat{\theta}$. Then, the point estimate of RE at time t is given by $\hat{R}(t) = e^{-[(t/\hat{\theta})^{\hat{\beta}}]}$.

2. Approximate the *se*'s of the above three estimates from $se(\hat{\beta}) = \hat{\beta}\sqrt{c_{22}/n}$, $se(\hat{\theta}) = (\hat{\theta}/\hat{\beta})\sqrt{c_{11}/n}$, and $se[\hat{R}(t)] \cong \hat{R}(t) \times$ [ln(1/$\hat{R}(t)$]×

$\sqrt{\{1.70 - 0.70\ \ln(-\ln(\hat{R})) + 0.70[\ln(-\ln(\hat{R}\ ))]^2\}/n}$ → Normal Approximation to the

$\text{CIs} \approx \text{MLE}\ \pm 1.96\times se(\text{Estimate})$

The reader should be cognizant of the fact the above 95% CI is generally fairly conservative.

3. Better Approximations for CI limits on β and θ for n > 2.

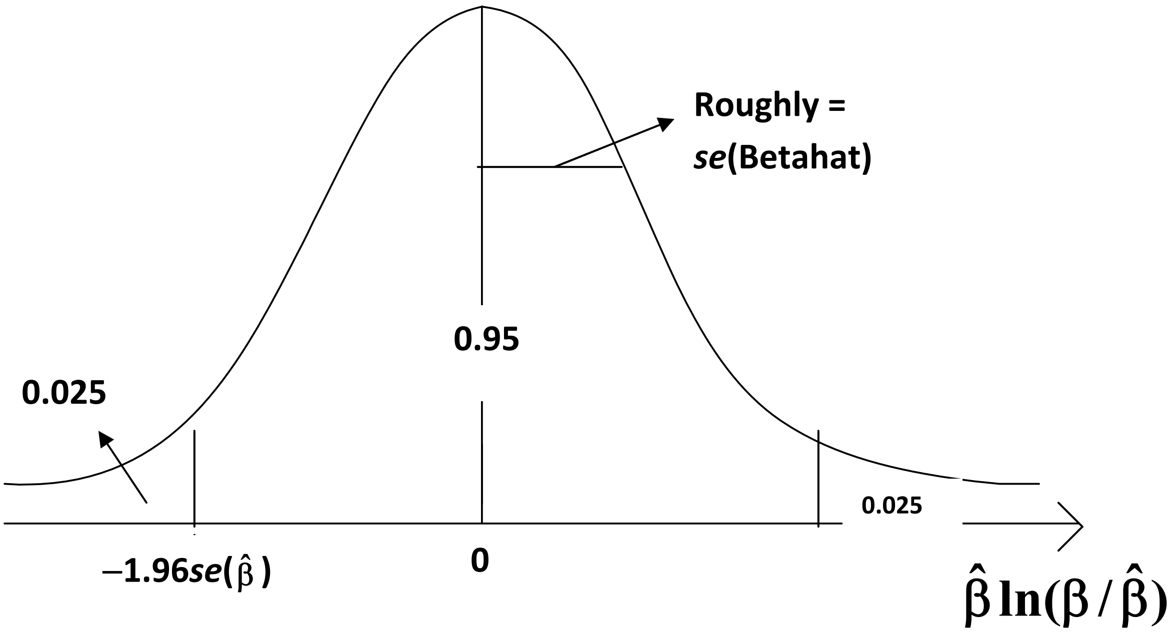
From Minitab's help menu, I obtained the more accurate 95% CI formulas $\beta_L = \hat{\beta}e^{-Z_{0.025}\times cv(\hat{\beta})}$ and $\beta_u = \hat{\beta}e^{Z_{0.025}\times cv(\hat{\beta})}$. I inverted Minitab's formulas to ascertain that the SMD of $\hat{\beta}\times\ln(\beta/\hat{\beta})$ must be approximately normal with mean zero and Stdev of roughly equal to

$se(\hat{\beta})$, as depicted below. The Figure shows that the $P[-1.96\times se(\hat{\beta}) \le \hat{\beta}\ln(\beta/\hat{\beta}) \le$

$1.96\times se(\hat{\beta})\ ] \cong 0.95.$ → $P[-1.96\times cv(\hat{\beta}) \le \ln(\beta/\hat{\beta}) \le 1.96\times cv(\hat{\beta})\ ] \cong 0.95$

$$P\left[e^{-1.96\times cv(\hat{\beta})} \le \beta/\hat{\beta} \le e^{1.96\times cv(\hat{\beta})}\right] \cong 0.95$$

$$P[\hat{\beta}e^{-1.96\times cv(\hat{\beta})} \le \beta \le \hat{\beta}e^{1.96\times cv(\hat{\beta})}] \cong 0.95 \ \rightarrow$$



$$\beta_L = \hat{\beta}e^{-1.96\times cv(\hat{\beta})} \text{ and } \beta_u = \hat{\beta}e^{1.96\times cv(\hat{\beta})} ;$$

similarly $\theta_L = \hat{\theta}e^{-Z_{0.025}\times cv(\hat{\theta})}$, $\theta_U = \hat{\theta}e^{Z_{0.025}\times cv(\hat{\theta})}$, and estimates of *SE*'s are obtained by first inverting Fisher's local Information Matrix $\hat{\mathbf{F}} = \hat{\mathbf{I}} = \begin{bmatrix} \hat{\mathbf{I}}_{11} & \hat{\mathbf{I}}_{12} \\ \hat{\mathbf{I}}_{12} & \hat{\mathbf{I}}_{22} \end{bmatrix}$ and then taking the sqrt of diagonal elements of $\hat{\mathbf{F}}^{-1} = \mathbf{I}^{-1}$, where $\hat{\mathbf{I}}_{11} = \frac{n\hat{\beta}^2}{\hat{\theta}^2}$, $\hat{\mathbf{I}}_{22} = n/\hat{\beta}^2$ + $\sum_{i=1}^{n}\left[(t_i/\hat{\theta})^{\hat{\beta}}\times[\ln(t_i/\hat{\theta})]^2\right]$, and $\hat{\mathbf{I}}_{12} = n(\hat{\beta}/\hat{\theta})\ln(\hat{\theta}) - (\hat{\beta}/\hat{\theta})\times\sum_{i=1}^{n}\left[(t_i/\hat{\theta})^{\hat{\beta}}\times\ln(t_i)\right]$. Note that Minitab uses this same procedure to obtain the *se*($\hat{\beta}$) and *se*($\hat{\theta}$). The 95% lower one-sided confidence limit on θ is also given by

$$\theta_L = \hat{\theta}e^{-1.64885363CV(\hat{\theta})}$$

where CV ($\hat{\theta}$) = *se*($\hat{\theta}$)/$\hat{\theta}$ and *se*($\hat{\theta}$) is the sqrt of element in the 1st row and column of $\hat{\mathbf{F}}^{-1} = \mathbf{I}^{-1} = \begin{bmatrix} 218.53855 & 4.465382 \\ 4.465382 & 0.901351 \end{bmatrix}$. For the Example 15.5 of Ebeling, the 95% lower confidence bound is given by $\theta_L^{0.95} = 82.9054e^{-1.64885363\times 0.17831}$ = 61.83086.

For the Example 15.5 of Ebeling, to obtain the glb on R(40 hours), we use the approximation

V[$\hat{R}(t)$] $\cong \hat{R}^2(t)\times$[ln(1/$\hat{R}(t)$]^2 ×{1.70 – 0.70 ln(ln(1/$\hat{R}(t)$))+ 0.70[ln(ln(1/$\hat{R}(t)$))]^2 }/n; because n =5, the following computation is a very rough estimate.

$\hat{R}(40) = e^{-[(40/82.9054)^{2.64551}]}$ = 0.864656 → V[$\hat{R}(40)$] = 0.031379141 →

se[$\hat{R}(40)$] = 0.1771416 → $R_{0.95}^{L}(40)$ = 0.864656 – 1.645×0.1771416 = 0.5733. The WGP provides a rough 95% glb on R(40) as 0.49.