The Moving Average Control Charts

Suppose that a QCH, X, has a Laplace-Gaussian distribution according to $N(\mu, \sigma^2)$. We consider two possibilities just like the case of EWMA charts. (1) The CNTL is targeted at $\mu_0$ with known process variance $\sigma^2$. (2) The CNTL has to be estimated from an initial subgroup of size $m$, $\sigma^2$ is unknown and also has to be estimated from the corresponding moving ranges.

(1) The case of targeted CNTL at $\mu_0$, known $\sigma^2$, and $n=1$

As an example, consider the data on the proportion of un-reacted lime (CaO) given on my website, under the name CaO-MAs, that I borrowed from the text by E. L. Grant & R. S. Leavenworth (1998, 6e, pp. 318-323, McGraw-Hill, ISBN:0-07-024117-1) for $m = 30$ individual subgroups, and the authors used the most-common moving ranges and averages of span (or width) $W = 3$, while I have also added spans $W = 2$ & 5. Further, the authors state on their page 319 that the targeted $\mu_0 = 0.170$ and $\bar{R} = 0.065$ were obtained from the previous two months (July and August) of continuous daily operations, and they list the September and part of October data in their Table 9-5, pp. 320-321 to set up trial control limits, I surmise, for the month of November. Therefore, for the CaO-MAs Example listed on my website, the value of $\mu_0 = 0.170$, and because $W = 3$, $\sigma_0 = 0.065/d_2 = 0.065/1.693 = 0.0383934$. Note that because $\sigma_0 = 0.0383934$ is the target, then it will be used as the known value of $\sigma$ even if $W \neq 3$.

To better understand moving averages, we compute their values using the CaO data at days 7 and 8. The MA of span (or width) $W = 4$ at time $t = 7$ is defined as $\text{MA}_7(W = 4) = \frac{x_7 + x_6 + x_5 + x_4}{4} = 0.16250$, while $\text{MA}_8(W = 4) = \frac{x_8 + x_7 + x_6 + x_5}{4} = 0.15750$; clearly, these two consecutive MA’s are not independent. I will show how to compute their Covariance on the following page.

In general, a moving average of span $W$ at time $t$, for $t \geq W$, is defined as
MA_t(W) = \frac{x_t + x_{t-1} + \ldots + x_{t+W}}{W} = \frac{1}{W} \sum_{i=t+1-W}^{t} x_i \quad \text{(22)}

Note that, unlike Shewhart’s 3-sigma charts, for \( t \geq W \), the points \( MA_t, MA_{t-1}, \ldots, \) and \( MA_{t-W+1} \) on moving range and average charts are correlated, and hence runs of length \( L \), denoted \( RL \), do not have the same statistical significance as they do on 3-Sigma Shewhart charts. For example, using the CaO data, the covariance between \( MA_9 \) and \( MA_6 \) at the span \( W = 5 \) is computed as follows:

\[
\text{COV}(MA_9, MA_6) = \text{COV}\left(\frac{1}{5} \sum_{i=9+1-5}^{9} x_i, \frac{1}{5} \sum_{i=2}^{6} x_i\right) = \text{COV}\left(\frac{1}{5} \sum_{i=5}^{9} x_i, \frac{1}{5} \sum_{i=2}^{6} x_i\right) = 2\sigma^2/25 = 0.00011792416,
\]

while the \( \text{COV}[MA_{11}(4), MA_8(4)] = \text{COV}\left[\frac{x_{11} + x_{10} + x_9 + x_8}{4}, \frac{x_8 + x_7 + x_6 + x_5}{4}\right] = \sigma^2/16 = 0.00009212825 \), where \( \sigma_0^2 = (0.0383934)^2 = 0.001474052 \).

When \( \mu \) is targeted at \( \mu_0 \) and \( \sigma \) at \( \sigma_0 \), then for any span \( W \), the CNTL is set at \( \mu_0 \), and to obtain the 3-Sigma control limits, we apply the Variance-Operator to Eq. (22).

\[
V[MA_t(W)] = V\left(\frac{1}{W} \sum_{i=t+1-W}^{t} x_i\right) = \frac{1}{W^2} \sum_{i=t+1-W}^{t} V(x_i) = \frac{1}{W^2} \left( \sum_{i=t+1-W}^{t} \sigma_X^2 \right) = \frac{1}{W^2} (W\sigma_X^2) = \sigma^2 / W \quad \rightarrow \text{SE}[MA_t(W)] = \sigma / \sqrt{W} \quad \text{(23)}
\]

Using Eq. (23), the value of the correlation coefficient between \( MA_9(5) \) and \( MA_6(5) \) of CaO data at \( W = 5 \) is given by

\[
\rho = \frac{2\sigma^2 / 25}{\sigma^2 / W} = 10/25 = 0.40.
\]

Eq. (23) shows that for a targeted MA control chart of any span \( W \), the lower and upper control limits, for \( t \geq W \), are given by

\[
LCL_{MA}(W) = LCL_{MA} = \mu_0 - 3\times \sigma_0 / \sqrt{W}, \quad \text{and} \quad UCL_{MA} = \mu_0 + 3\times \sigma_0 / \sqrt{W} \quad \text{(24)}
\]

For the CaO data on my website, I have calculated the process SE’s and the control limits for all 3 spans \( W = 2, 3 \) and \( 5 \) in the indicated columns of the Excel file. At \( W = 3 \) and \( t \geq 3 \), the targeted SE is \( \sigma_0 / \sqrt{W} = 0.0383934/\sqrt{3} = 0.022166431 \), which results in

\[
LCL_{MA} = 0.170 - 3 \times 0.022166431 = 0.170 - 0.0664993 = 0.103501, \quad \text{and} \quad \text{UCL}_{MA} = \]

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$0.170 + 3 \times \sigma_0 / \sqrt{W} = 0.2364993$, which are consistent with those of the authors’ Figure 9-4 and those of Minitab’s. Further, at spans $W = 2$ and $5$, I have also assumed that the process standard deviation is known and still targeted at $\sigma_0 = 0.038393$, even if this was obtained at $W = 3$.

Minitab also provides moving average control limits for $1 \leq t < W$, whose standard errors are given by $SE[MA_t(t < W)] = \sigma / \sqrt{t}$. For example, at time $t = 2$, the control limits at span three are $LCL_{MA}(t = 2) = 0.170 - 3 \times 0.0383934 / \sqrt{2} = 0.0885553$, while the $UCL_{MA}(t = 2) = 0.170 + 3 \times 0.0383934 / \sqrt{2} = 0.25144467$. These are in precise agreement with Minitab’s output, also posted on my website.

(2) The case of Estimated CNTL at $\bar{X}$, Estimated $\sigma^2$, and $n = 1$

Again as an example, consider the data on proportion of un-reacted lime (CaO) given on my website under CaO that I borrowed from the book by E. L. Grant & R. S. Leavenworth (1998, 6e, pp. 318-323, McGraw-Hill, ISBN:0-07-024117-1) for $m = 30$ subgroups, where the authors provide only the targeted MA-chart of span (or width) $W = 3$, while now I will also obtain the control limits for the CaO data when the CNTL is set at $\bar{X}$ and $\sigma$ is estimated from $\frac{\overline{MR}}{d_2}$. From Eq. (23), the estimate of the SE of $MA_t$ at span $W$ is given by

$$se[MA_t(W)] = \frac{\hat{\sigma}}{\sqrt{W}} = \frac{\overline{MR}}{d_2 \sqrt{W}}$$

At the span $W = 5$, my spreadsheet shows that the estimated SE is given by $se[MA_t(5)] = 0.137308 / (2.326 \times 5^{1/2}) = 0.0590317 / \sqrt{5} = 0.0263998$, for all $t \geq 5$.

Recall that the $d_2$ values for the span $W = 2$, $3$, $4$, and $5$ are given by $1.128$, $1.693$, $2.059$, and $2.326$, respectively. The MA control chart at $W = 3$ and $5$ from Minitab are provided on the next page.
Moving Average Chart of \( X_t \) at span \( W = 3 \)

\[ \bar{X} = 0.1663 \]

UCL = 0.2672

LCL = 0.0655

Moving Average Chart of \( X_t \) of span 5

\[ \bar{X} = 0.1663 \]

UCL = 0.2455

LCL = 0.0871