

Review of Fractional Factorials For Base-2 &-3 Designs

The notation 2^3 implies a factorial experiment involving 3 factors each at 2 levels (low = 0 or -1, high = +1). Since $2^3 = 8$, then one full replicate of a 2^3 factorial requires 8 experimental runs. The 8 FLCs in alphabetical order are (1), a, b, ab, c, ac, bc, and abc. For example, the FLC 'ac' means that factors A, and C are at +1 (high level) while factor B is at -1 (the low level), i.e., $ac = (1, 0, 1)$. To generate the 8 FLCs in an OA format, write 3 arbitrarily columns of zeros and ones (or minus ones and plus ones) as shown below.

C B A FLC

0	0	0	(1)	<p>Note that the OA on the LHS has 8 rows, and hence, it provides 7 df for studying effects. If we wished to generate 8 more FLCs in order to obtain a 2^4 design, then we would simply multiply $d = (0, 0, 0, 1)$ by the 8 FLCs (1), a, b, ab, ..., abc. Similarly, we would use "e" and all the 16 FLCs of a 2^4 factorial to generate all the 32 FLCs of</p>
0	0	1	a	
0	1	0	b	
0	1	1	ab	
1	0	0	c	
1	0	1	ac	
1	1	0	bc	
1	1	1	abc	

a 2^5 factorial design by multiplying e by (1), a, b, ab, c, ac, ..., abcd.

A full replicate of a 2^8 factorial design (with $n = 1$) requires 256 experimental runs, but only 36 ($= 8 + {}_8C_2$) df of the 255 total df are absorbed by the 8 main factors A, B, ..., H and the 28 two-way interactions $A \times B$, $A \times C$, ..., $G \times H$. This implies that such a design uses only 14.12% of the total df to study the most important effects (main factors and their 2-way interactions), and therefore, is somewhat wasteful of resources. When our resources are limited, then we must resort to running only a fraction of all possible FLCs. Such experiments are called fractional factorial designs (FFDs, or fractional replicates), and their application in industrial settings is extremely helpful in separating the vital few effects from

trivial many. Once the influential effects are identified, their levels can then be adjusted in such a manner as to reduce waste and to improve outgoing quality.

As an example, suppose that through a brainstorming session we have identified 8 factors (or inputs) that may impact our output y . However, we cannot afford conducting $256 = 2^8$ experiments but have sufficient resources to conduct

32 experiments. This leads to a $32 = 2^5 = 2^{8-3} = \frac{1}{2^3} 2^8 = (1/8)^{\text{th}}$ fraction of a 2^8

factorial. Similarly, a one-fourth fraction of a 2^5 factorial leads to a $\frac{1}{4}(2^5) =$

$\frac{1}{2^2}(2^5) = 2^{-2}(2^5) = 2^{5-2}$ FFD. In designing a FF from scratch, the experimenter

must exercise extreme care in order not to lose the effects of main factors and, if possible, must reserve sufficient room (or df) to study the 2-way (or 1st – order) interactions of interest.

Example 8. Consider an injection molding process, where 5 factors ‘A = Mold Temperature, B = Holding Pressure, C = Gate Size, D = Feed Rate, and E = Hold Time’ may have an impact on the strength, y , of the molded piece (an LTB QCH). Further, the decision has been made that each factor has 2 possible levels of interest (Low and High) so that one full replicate will need $32 = 2^5$ experiments, but we can afford conducting only 8 experiments. This leads to a $(1/4)^{\text{th}}$ fraction of a 2^5 factorial, or a 2^{5-2} FFD. Further, the process engineer is also interested in studying the interactions $A \times D$ and $A \times E$.

To design such an experiment from scratch, we will herein state rules and patterns that will apply to all FFDs in all prime bases (2, 3, 5, 7, etc). (1) Since $2^{5-2} = 8$, then we must have 8 rows (or FLCs) in our design matrix. (2) Because the exponent $5-2=3$, then we will be able to write 3 columns arbitrarily. (3) Generating the other 4 columns is the one that requires extreme care! That is, we must generate them in such a manner that all 5 main factors occupy separate columns and are also separate from AD and AE interactions. The use of the above 3 rules leads to the design matrix below with the corresponding response values y . Note that none of the 3-way interactions ABD, ACD, ADE, ABE, ACE

can be a design generator because in that case either AD or AE interaction will become aliased with one of the main factors, but our objective is to keep AD and AE interactions separate from the 5 main factors. Therefore, we must select our 2 independent generators from the remaining 5 three-way interactions ABC, BCD, BCE, BDE, and CDE in such a manner that they involve all 5 main factors, and also the 3rd generator (i.e., the generalized interaction) has at least 3 letters. This leads only to 2 possible set of generators.

Table 1. (Two independent generators ABC, BDE; $g_3 = ACDE$)

A	B	D	C = A×B	E = B×D	AD	AE	FLC	Y (LTB)
-	-	-	+	+	+	-	ce	14 psi
-	-	+	+	-	-	+	cd	35
-	+	-	-	-	+	+	b	32
-	+	+	-	+	-	-	bde	12
+	-	-	-	+	-	+	ae	9
+	-	+	-	-	+	-	ad	18
+	+	-	+	-	-	-	abc	5
+	+	+	+	+	+	+	abcde	7

Since $C = AB$, then one generator of the above design is $g_1 = ABC$, and similarly, the 2nd independent generator is $g_2 = BDE$. Thus, the 3rd generator of this FFD is the generalized interaction of g_1 and g_2 given by $g_3 = g_1 \times g_2 = ACDE$. Note that this pattern will persist for all FFDs, i.e., because we have divided our original 32 FLCs into 4 blocks of 8 FLCs each (only one of which is studied and shown in the above table), and the 4 blocks each with 8 FLCs carry 3 df amongst them, then all of our generators must also have 3 df. Since we have 3 generators, then each effect will have 3 aliases, i.e., for each block of FLCs that is left unstudied, we create exactly one alias for each effect!

The contrast functions (cfs) for the above $(1/4)^{th}$ FFD are $\xi_1 = x_1 + x_2 + x_3$ and $\xi_2 = x_2 + x_4 + x_5$. Note that since we have only 2 independent generators, we need only 2 contrast functions; further, for base-2 designs the values of cfs can be only 0 or 1. For base-3 designs, the values of cfs can be 0, 1, or 2; for base-5 designs, ξ can take on only the values 0, 1, 2, 3, and 4, etc.

To determine which block the FFD of the Example 8 is, we simply insert one of the FLC of the generated block of Table 1 into our cfs: $\xi_1(\text{ce}) = 0 + 0 + 1 + 0 + 0 = 1$, $\xi_2(\text{ce}) = 0 + 0 + 0 + 0 + 1 = 1$. Similarly, $\xi_1(\text{bde}) = 1$ and $\xi_2(\text{bde}) = 3 = 1 \pmod{2}$. You should verify that the remaining 6 FLCs also have $\xi_1 = \xi_2 = 1$, i.e., the 8 FLCs [ce, cd, b, bde, ae, ad, abc, abcde] belong to the $(\xi_1 = 1, \xi_2 = 1)$ block. The principal block (PB) for all FFDs of prime bases is the one for which the values of all cfs are zero, i.e., the PB always contains the FLC 00000.... The response table (RT) for the data of Table 1 is provided in Table 2 below.

Table 2. (The RT for the data of Table 1)

Effects	A	B	C	D	E	AD = CE	AE = CD
L ₀	93	76	71	60	90	61	49
L ₁	39	56	61	72	42	71	83
Contrast	-54	-20	-10	12	-48	10	34
R _i	1	4	6.5	5	2	6.5	3

A×E

A \ E	0	1
0	67*	26
1	23	16

C×D

C \ D	0	1
0	41*	30
1	19	42*

Exercise 10. Use the same 3 arbitrary columns of Table 1 to generate the PB by letting $C = -A \times B$ and $E = -B \times D$. (b) Use the 8 FLCs in the PB to generate the other 2 blocks, giving the cf values of each block. (c) Use the procedure as in Table 1 to generate the $(\xi_1 = 0, \xi_2 = 1)$ block by using the proper signs in front of $A \times B$ and $B \times D$ columns. (d) In Table 1, suppose we wish to keep the AD and AE interactions separate from the main 5 factors. Generate another design, similar to Table 1, where the 3 arbitrary columns are A, B, and E.

Note that in Table 1 the only 4-letter generators that will allow us to protect AD and AE are $g_1 = ABDE$, or $g_1 = ACDE$ (both of which contain all 3 factors in AD and AE interactions). As a result the only 3 arbitrary columns that can be written are any 3 factors out of one of the two generators ABDE, or ACDE. Thus, there are a total of 8 choices for the 3 arbitrary columns. Table 2 clearly shows that the strongest effect is A, the 2nd most influential effect is E, the 3rd is AE = CD, and B has only a moderate impact on the response y. Based only on the RT, it seems that the preliminary optimal process condition is $X_0 = A_0B_0C_?D_?E_0$. To finalize our optimal condition X_0 , we have to analyze our significant interactions by crossing A with E and cross factor C with D. The two interaction tables on page 19 clearly show that the optimal levels of A and E are $A_0 E_0$ and those of C and D are either C_0D_0 or C_1D_1 . Hence, $X_0 = (1)$, or $X_0 = cd$.

Exercise 11. Obtain the ANOVA table for the data of Table 1. (b) Verify the value of $SS(AE) = SS(CD)$ by using the formula for a contrast SS given by

$$SS(\text{Contrast}) = (\text{Contrast})^2 / \sum_{i=1}^N n_i c_i^2, \text{ where } N = 8 \text{ for this example.}$$

(c) Verify your answers using the orthogonality of the design matrix. Answer: $SS(AE) = 144.50$.

Exercise 12. Carefully study Chapter V (pp. 39 – 48) of my Manual and rework the Example 5.1 using the same data as on page 45. However, column (1) will consist of 8 minuses followed by 8 pluses, column (2) will be 4 minuses, 4 pluses, and so on. The independent generators are $g_1 = ABCE$ and $g_2 = BCDF$. Assign A to column (1), B to (2), and C to column (3). Obtain X_0 assuming that y is an STB type QCH. (b) Obtain all the aliases of only effects C and AD. (c) Work Exercise 5.2 on page 48 of my manual.

Fractional Factorials For Base-3 Designs

The notation 3^4 implies a factorial experiment involving 4 factors, and the quantitative factors must be at 3 equi-spaced levels 0, 1, 2, and hence one full replicate (with $n = 1$) will require running 81 experiments at 81 different FLCs of

the factors A, B, C, D. The superscript 4 in the notation 3^4 will signal that we may write 4 arbitrary columns of 0's, 1's and 2's in order to generate the 81 FLCs of this CR (completely randomized) 3^4 factorial design. I will show the writing of these columns during the discussions in class. The ANOVA Outline is summarized below. The ANOVA outline clearly indicates that only $(8+24)/80 =$

ANOVA Outline

Source of Variation	df
Total	80
Main factors (A, B, C, D)	8
2-way Interactions	24
3-way Interactions	32
A×B×C×D	16

40% of the total df of the design will belong to effects through the 2nd order (or interactions through the 1st order); the other 60% of the df are absorbed by high-order interactions (HOIs).

This situation worsens for a 3^5 factorial (with $n = 1$), where only $50/242 = 20.66\%$ of the total df are absorbed by effects through the 2nd order. In other words, as design base increases from 2 to 3, to 5, and to 7, full factorials use more and more resources of experimenters and become more and more wasteful in the use of df if studying only the main factors A, B, C, ..., and their 1st –order interaction A×B, A×C, B×C, ... were of interest.

Example 9. The life (or time-to-failure = TTF) , y, of a cutting tool in a numerically controlled machine is thought to be affected by the tool angle (A) and the cutting speed (B). Table 3 shows the layout of the design and the resulting coded data with $n = 2$ observations per cell taken in a CR (completely randomized) experiment. As always in any factorial experiment the USS

$$= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^2 y_{ijk}^2 = 156.00 \text{ (with 18 df), the CF} = 24^2/18 = 32 \text{ (with 1 df)} \rightarrow SS_T =$$

156 – 32 = 124.00 (with 17 df). Since the model deals only with variation amongst the 9 cells (or FLCs), then SS(Model with 8 df) =

$$\frac{(-3)^2 + (-3)^2 + 5^2 + 2^2 + 4^2 + 10^2 + (-1)^2 + 11^2 + (-1)^2}{2} - CF = 111.00 \rightarrow$$

SS(Experimental Error) = SS_T – SS(Model) = 124 – 111 = 13.00 (with 9 df).

Table 3

A \ B	125 (in/min)	150	175	y _{i..}
15 degrees	- 2, - 1 (y _{11.} = -3)	- 3, 0 (-3)	2, 3 (y _{13.} = 5)	- 1
20	0, 2 (2)	1, 3 (4)	4, 6 (10)	16
25	-1, 0 (- 1)	5, 6 (11)	0, -1 (- 1)	9
y _{.j.}	- 2	12	14	y _{... = 24}

We may also verify this SS_{Error} = SS_E from within the 9 cell variations, each cell contributing exactly 1 df to the Error SS. Therefore, F₀(Model) = MS(Model)/MS_E = 13.875/1.4444 = 9.6058. Since F_{0.05,8,9} = 3.23, then the impact of the model (A, B, A×B) on the mean response (MTTF) is definitely significant at the 5% level. The *P-value* of this test is given by $\hat{\alpha} = P(F_{8,9} \geq 9.6058) = 0.001337$, i.e., we may easily declare that the overall impact of the 3 model terms on the mean response is significant at a level as small as 0.00134. The % contribution of the model to the variation in the mean response is 89.516%. We next decompose SS(Model) into 4 orthogonal (i.e., additive) SS's each with 2 df.

To this end, we define a contrast function for any effect in base 3 as $\xi = a_1 x_1 + a_2 x_2 + \dots$. Since the elements of base 3 are 0, 1, and 2 (the base -10 number 3 does not exist in base 3 and has to be represented by 10 in base-3), then 3 = 0 (modulus 3), 4 = 1 (mod 3), 5 = 2 (mod 3), 6 = 0 (mod 3), 7 = 1 (mod 3), etc. Further, ξ and the constants a_1, a_2, \dots in a base-3 design can take on only the values of 0, 1, and 2. For example, for the effect A, the contrast function (cf) is $\xi(A) = x_1$. The corresponding values of $A_0 = \xi_0 = -3 -3 +5 = -1$, $\xi_1 = A_1 = 2 + 4 + 10$

= 16, and $\xi_2 = A_2 = -1 + 11 - 1 = 9$. This yields, $SS_A = [(-1)^2 + 16^2 + 9^2] / 6 - CF = 24.33\bar{3}$. Similarly, $SS(B) = 25.33\bar{3}$. Because $SS(\text{Model})$ has 8 df and $SS(A) + SS(B)$ account for only 4 df, then the remaining 4 df must pertain to $SS(A \times B)$, i.e., $SS(A \times B) = 111 - 49.66\bar{6} = 61.33\bar{3}$. However, there is absolutely no way that we could write a cf for $A \times B$ because every cf in base-3 design takes on only 3 values of 0, 1, and 2 and hence can have only 2 df. Therefore, we have to decompose $A \times B$ into 2 orthogonal components each with 2 df. These 2 orthogonal components of $A \times B$ are AB and AB^2 . (Note that $AB = A^2B^2$ and $A^2B = AB^2$.) The OA in table 4 illustrates how to generate the columns for A, B, AB, and AB^2 .

Table 4		AB	AB^2		
A	B	$x_1 + x_2$	$x_1 + 2x_2$	FLC	y_{ijk}
0	0	0	0	00	-2, -1
0	1	1	2	01	-3, 0
0	2	2	1	02	2, 3
1	0	1	1	10	0, 2
1	1	2	0	11	1, 3
1	2	0	2	12	4, 6
2	0	2	2	20	-1, 0
2	1	0	1	21	5, 6
2	2	1	0	22	0, -1
$y_{...} =$					24.00

Table 4 clearly shows that $SS(AB) = \frac{(-3+10+11)^2 + (-3+2-1)^2 + (5+4-1)^2}{6} - CF = 33.33\bar{3}$ (with 2 df) and similarly, $SS(AB^2) = (0^2 + 18^2 + 6^2) / 6 - CF = 28.00$ (with 2 df). Note that AB and AB^2 are 2 orthogonal components of the $A \times B$ interaction because $SS(AB) + SS(AB^2) = 33.33\bar{3} + 28 = 61.33\bar{3} = SS(A \times B)$. Further, note that unlike base-2 designs, all columns in base-3 designs must have 2 df. Since the design matrix in Table 4 has 9 rows (therefore 8 df for all effects in the model), then we can write only 4 independent columns, namely A, B, AB, and AB^2 .

(each with 2 df). Again, this implies that $A^2B^2 = AB$ and $A^2B = AB^2$. The statistical convention is that the 1st letter should never be squared and thus the notations AB and AB^2 are conventional while A^2B^2 and A^2B are correct but not common! Later we will see that Taguchi sometimes uses the unconventional components A^2B^2 and A^2B .

Before discussing the FFD for base-3, we must assert that unlike base-2 designs the question “about the magnitude of the effect of factor A ” in base-3 is ambiguous unless we specify whether we are asking for the linear or the quadratic impact of factor A. When factor levels in base-3 designs are equi-spaced and quantitative, there are two orthogonal effects of a factor: Linear and Quadratic. The linear contrast for factor A of the Example 9 is $A_L = -1 \times (-1) + 0 \times 16 + 1 \times 9 = 10.00$, and the quadratic effect is $A_Q = 1 \times (-1) - 2(16) + 1 \times 9 = -24.00$.

Using the definition of $SS(\text{Contrast})$, we can compute the $SS(A_L) = 10^2 / (6 \times 2) = 8.333\bar{3}$ and $SS(A_Q) = (-24)^2 / (6 \times 6) = 16.00 \rightarrow SS(A_L) + SS(A_Q) = 24.333\bar{3} = SS(A)$, as expected because the linear contrast vector $[-1 \quad 0 \quad 1]'$ and the quadratic contrast vector $[1 \quad -2 \quad 1]'$ are orthogonal.

Exercise 13. For the Example 9, compute $SS(B_L)$, $SS(B_Q)$ and verify that these 2 components are orthogonal. (b) Compute the four orthogonal components of $SS(A \times B)$.

The 3³ Factorial Design

Consider an experiment involving the effects of three factors, each at three levels, on the response variable $y = \text{yield}$. The factors that may impact yield are factor A = Temperature at levels 90, 105, 120 °F; solvent concentration at 0.50, 1.0, 2.0, and factor C which represents 3 operators (O_0 , O_1 and O_2). Note that although the levels of factor B (0.50, 1.0, 2.0) are not equi-spaced, their natural logarithms (-0.69315 , 0 , 0.69315) are equi-spaced so that we can still use mod 3 algebra. Thus, we have an experiment involving 3 factors each at 3 levels with $n = 2$ observations per FLC taken in a completely random order. In order to write the OA for this design, we can write 3 arbitrary columns as depicted in Table 5 on the next

page. The design matrix in Table 5 provides 26 df for studying effects and hence it has 13 independent (or orthogonal) columns each with 2 df. The coded subtotals (by subtracting 20) are given in the last column. I generated the orthogonal columns 4 through 13 using the corresponding cf with mod 3 algebra. As an example, I generated the 13th column under AB^2C^2 by using its contrast function $\xi(AB^2C^2) = x_1 + 2x_2 + 2x_3$, which shows that column 13 must be generated by adding (using mod 3 algebra) the column under A, twice the column under B, and twice the column under C. Or even simpler is to add the AB column with the BC^2 column (mod 3). To compute the $SS(AB^2C^2)$, we must 1st add all observations in Table 5

A	B	C	AB	AB ²	AC	AC ²	BC	BC ²	ABC	AB ² C	ABC ²	AB ² C ²	Y _{ijk} .
0	0	0	0	0	0	0	0	0	0	0	0	0	0.20
0	0	1	0	0	1	2	1	2	1	1	2	2	5.8
0	0	2	0	0	2	1	2	1	2	2	1	1	13.6
0	1	0	1	2	0	0	1	1	1	2	1	2	1.0
0	1	1	1	2	1	2	2	0	2	0	0	1	7.0
0	1	2	1	2	2	1	0	2	0	1	2	0	12.2
0	2	0	2	1	0	0	2	2	2	1	2	1	2.6
0	2	1	2	1	1	2	0	1	0	2	1	0	9.2
0	2	2	2	1	2	1	1	0	1	0	0	2	14.4
1	0	0	1	1	1	1	0	0	1	1	1	1	0.8
1	0	1	1	1	2	0	1	2	2	2	0	0	7.2
1	0	2	1	1	0	2	2	1	0	0	2	2	12.4
1	1	0	2	0	1	1	1	1	2	0	2	0	1.0
1	1	1	2	0	2	0	2	0	0	1	1	2	6.8
1	1	2	2	0	0	2	0	2	1	2	0	1	11.6
1	2	0	0	2	1	1	2	2	0	2	0	2	1.2
1	2	1	0	2	2	0	0	1	1	0	2	1	8.0
1	2	2	0	2	0	2	1	0	2	1	1	0	13.6
2	0	0	2	2	2	2	0	0	2	2	2	2	3.0
2	0	1	2	2	0	1	1	2	0	0	1	1	9.6
2	0	2	2	2	1	0	2	1	1	1	0	0	14.4
2	1	0	0	1	2	2	1	1	0	1	0	1	1.5
2	1	1	0	1	0	1	2	0	1	2	2	0	8.6
2	1	2	0	1	1	0	0	2	2	0	1	2	13.6
2	2	0	1	0	2	2	2	2	1	0	1	0	1.8
2	2	1	1	0	0	1	0	1	2	1	0	2	7.6
2	2	2	1	0	1	0	1	0	0	2	2	1	12.6

the FLCs (000, 012, 021, 101, 110, 122, 202, 211, 220); this will yield $(AB^2C^2)_0 = 68.2$. Note that these 9 FLCs all have $\xi(AB^2C^2) = x_1 + 2x_2 + 2x_3 = 0$. Second, we must add

all y_{ijkl} values obtained under the FLCs (002, 011, 020, 100, 112, 121, 201, 210, 222); this will yield $(AB^2C^2)_1 = 67.3$. The reader should verify that $\xi(AB^2C^2) = x_1 + 2x_2 + 2x_3 = 1$ for these last 9 FLCs. Similarly, $(AB^2C^2)_2 = 65.8$. Hence,

$$SS(AB^2C^2) = \frac{68.2^2 + 67.3^2 + 65.8^2}{18} - \frac{201.3^2}{54} = 0.1633\bar{3}.$$

Exercise14. (a) For the design matrix of the above example, use the above procedure to compute $SS(AB^2C)$, $SS(ABC^2)$, and $SS(ABC)$. Then verify your answer by using the orthogonality of these 4 components of $A \times B \times C$. (b) Use the same procedure to compute $SS(AC^2)$. Given that the $SS_T = 331.7883333$ and $SS(\text{Error}) = 10.125$, obtain the ANOVA Table and identify the vital effects using $\alpha = 0.05$.

Fractional Factorials for Base-3 Designs

If the experimenter's resources are limited and therefore cannot conduct a full 3^k ($k \geq 3$) factorial, then s/he must settle for running a $(1/3)^{\text{rd}}$, $(1/9)^{\text{th}}$, $(1/27)^{\text{th}}$, etc FFD with all quantitative factors at 3 equi-spaced levels. For example, a $(1/3) 3^3$ FFD consists of dividing the 27 FLCs of the 3^3 design into 3 blocks of 9 FLCs each, selecting one of the 3 blocks at random, and then running the 9 experiments in the chosen block in a completely random order. Such a design is

called a $\frac{1}{3} 3^3 = 3^{3-1}$ FF, which will need one generator to be selected from ABC , AB^2C , ABC^2 , and AB^2C^2 in order to maximize resolution. Because there are only 3 letters (A, B, C) in a 3^3 design, it is impossible for this case to generate a FFD with $R > III$.

As in the case of FFs in base-2 designs, it will be easiest to obtain a 3^{3-1} FFD by using an (L_9) OA and generating the PB first. Suppose we let $g = AB^2C^2$; for the PB the value of contrast function $\xi = x_1 + 2x_2 + 2x_3$ must be set equal to zero. As before, 2 columns can be written arbitrarily, but the column pertaining to factor C must be obtained from $x_1 + 2x_2 + 2x_3 = 0$. This yields $x_1 + 2x_2 = -2x_3 = (-2 + 3)x_3 = x_3$, i.e., the levels of factor C must be obtained from $A + 2B \pmod{3}$. The PB is given in Table 6.

Table 6. (The PB Block with $\xi = x_1 + 2 x_2 + 2 x_3 = 0$)

A	B	$x_3 = x_1 + 2 x_2 = C$
0	0	0
0	1	2
0	2	1
1	0	1
1	1	0
1	2	2
2	0	2
2	1	1
2	2	0

Once the PB is obtained, the ($\xi = 1$) and ($\xi = 2$) blocks can be generated by simply identifying only one FLC in these 2 blocks and then adding the identified FLC to those of the PB to generate all the FLCs of ($\xi = 1$) and ($\xi = 2$) blocks.

The alias structure of the above FFD is given by $A = BC = ABC$ (with 2 df), $B = AC^2 = ABC^2$ (with 2 df), $C = AB^2 = AB^2C$ (with 2 df), and $AB = AC = BC^2$ (2 df).

Exercise 15. A company manufacturing engines was concerned about emission characteristics of the engines. To try to minimize various undesirable emission variables, the company proceeded to build and operate experimental engines varying five controllable factors: throat diameter at 3 levels 0, 1, 2; ignition system at 3 levels 0, 1, 2; temperature at 3 levels 0, 1, 2; velocity of the jet stream at 3 levels, and timing system at 3 levels. Since it was cost-prohibitive to build $3^5 = 243$ experimental engines, it was decided to conduct a $(1/9)^{\text{th}}$ fraction of all the 243 runs. (a) Comment on the deficiency of the FFD with the generators $g_1 = AD^2E$ and $g_2 = AB^2CD^2E$. (b) Obtain the PB using the generators $g_1 = AB^2C$ and $g_2 = BCD^2E^2$. Then use this PB to generate the ($\xi_1 = 2, \xi_2 = 1$) block. (c) Give the outline of the ANOVA table listing only the effects through the 1st order. (d) Given the (STB) coded responses for the PB starting with (0, 0, 0, 0, 0) \rightarrow 15, 19, 30, 7, 14, 2, 8, 15, 20, 23, 35, 10, 14, 3, 11, 9, 5, 13, 19, 28, 20, 25, 15, 33, 16, 14, 22, obtain the RT and estimate the optimal setting X_0 . The response $y = 15$ was obtained under the FLC 00000, and the last response, 22, was obtained from the FLC 22020.

Exercise 16. Consider a 3^{6-3} FFD ; describe the exact nature of this design, i.e., determine the values of k, p, how many independent generators, what fractional replicate, how many blocks and the number of FLCs in each block. (b) Give a set of generators to maximize R. (c) Determine all the aliases of factor E. (d) Generate the PB.